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Breakdown of a 2D Heteroclinic Connection in the Hopf-Zero Singularity (I)

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Abstract In this paper we study a beyond all orders phenomenon which appears in the analytic unfoldings of the Hopf-zero singularity. It consists in the breakdown of a two-dimensional heteroclinic surface which exists in the truncated normal form of this singularity at any order. The results in this paper are twofold: on the one hand, we give results for generic unfoldings which lead to sharp exponentially small upper bounds of the difference between these manifolds. On the other hand, we provide asymptotic formulas for this difference by means of the Melnikov function for some non-generic unfoldings.

Keywords Exponentially small splitting · Hopf-zero bifurcation · Melnikov function · Borel transform

Mathematics Subject Classification 34E10 · E4E15 · 37C29 · 37G99

1 Introduction

The Hopf-zero singularity (also called central singularity, Gavrilov–Guckenheimer or fold-Hopf singularity) is any vector field $X^* : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, having the origin as an equilibrium point, and such that the linearization of X^* at this point, $DX^*(0)$, has eigenvalues $0, \pm i\alpha^*$, for some $\alpha^* \neq 0$. Equivalently, $X^*(0) = 0$ and $DX^*(0)$ is conjugated to

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$$\begin{pmatrix} 0 & \alpha^* & 0 \\ -\alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha^* > 0. \tag{1}$$

This singularity has codimension two in the sense that it can be met by a generic family of vector fields depending on at least two parameters. However, since $DX^*(0)$ has zero trace, when one considers X^* in the context of divergence-free vector fields, it has codimension one. Following (Broer and Vegter 1984) we will refer to the divergence-free case as the conservative case.

The generic families which meet the singularity X^* for some value of the parameters, which we assume is $(0, 0)$, are called the versal unfoldings of the Hopf-zero singularity. That is, they are families $X_{\mu, \nu}$ of vector fields on \mathbb{R}^3 depending on two parameters $(\mu, \nu) \in \mathbb{R}^2$, such that $X_{0,0} = X^*$, the vector field described above.

The unfoldings of this singularity, and the different behavior these families can present, have been broadly studied (Takens 1973a, b, 1974; Guckenheimer 1981; Broer and Vegter 1984; Freire et al. 2002; Dumortier and Ibáñez 1998; Champneys and Kirk 2004; Lamb et al. 2004; Gavrilov 1978, 1985; Gavrilov and Roshchin 1983; Guckenheimer and Holmes 1983; Dumortier et al. 2013; Kuznetsov 2004). The standard way to proceed in the study of these unfoldings is as follows: first, one uses normal form theory and, performing changes of variables, writes the vector field in the simplest possible form up to some order. Then, one studies the effects of the non-symmetric (higher-order terms) in the dynamics. In our case, following (Guckenheimer 1981), we consider $X_{\mu, \nu}$ a family of vector fields in \mathbb{R}^3 such that $X_{0,0}$ has the origin as an equilibrium point with linear part (1). After the normal form procedure up to order two, the vector field $X_{\mu, \nu}$ in the new coordinates $(\bar{x}, \bar{y}, \bar{z})$ takes the form

$$\begin{aligned} \frac{d\bar{x}}{d\bar{t}} &= \bar{x} (\beta_0\nu - \beta_1\bar{z}) + \bar{y} (\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{y}}{d\bar{t}} &= -\bar{x} (\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y} (\beta_0\nu - \beta_1\bar{z}) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{z}}{d\bar{t}} &= -\gamma_0\mu + \gamma_1\bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \end{aligned}$$

Note that the coefficients $\beta_1, \gamma_1, \gamma_2$ depend exclusively on the chosen singularity X^* , in fact only on its degree two jet. We also observe that the conservative setting is obtained taking $\nu = 0$ and $\beta_1 = \gamma_1$ and imposing that the higher-order terms are divergence free.

In Takens (1974) (see also Dumortier et al. 2013) it is seen that the generic conditions $\beta_1 \neq 0, \gamma_1 \neq 0$, and $\gamma_2 \neq 0$, characterize a stratum of codimension two (dimension one in the conservative case) in the space of germs of singularities of vector fields on \mathbb{R}^3 . We will assume them from now on. Moreover, the scaling $\bar{z} \rightarrow \gamma_1\bar{z}$, allows us to assume that $\gamma_1 = 1$. In Takens (1974), Dumortier and Ibáñez (1998) the authors see that there are six topological types of singularities of codimension two depending of the choice of the parameters $\beta_1 \neq 0$ and $\gamma_2 \neq 0$ (see Figure 8.16 in Kuznetsov 2004). In this paper we will deal with the Hopf-zero singularity corresponding to

$$\beta_1 > 0, \quad \gamma_2 > 0, \tag{2}$$

that we will denote, following (Dumortier et al. 2013), by HZ^* .

We will also assume the generic conditions $\beta_0 \neq 0, \gamma_0 \neq 0$ for the unfoldings considered. Under these conditions, redefining the parameters, one can assume $\beta_0 = \gamma_0 = 1$, obtaining:

$$\begin{aligned} \frac{d\bar{x}}{d\bar{t}} &= \bar{x} (\nu - \beta_1 \bar{z}) + \bar{y} (\alpha^* + \alpha_1 \nu + \alpha_2 \mu + \alpha_3 \bar{z}) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{y}}{d\bar{t}} &= -\bar{x} (\alpha^* + \alpha_1 \nu + \alpha_2 \mu + \alpha_3 \bar{z}) + \bar{y} (\nu - \beta_1 \bar{z}) + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \frac{d\bar{z}}{d\bar{t}} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + \gamma_3 \mu^2 + \gamma_4 \nu^2 + \gamma_5 \mu \nu + \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \end{aligned} \tag{3}$$

and the conservative setting is obtained taking $\beta_1 = 1, \nu = 0$ and imposing that the higher-order terms are divergence free.

The versal unfoldings of the Hopf-zero singularity have been widely studied in the past, see for example (Broer and Vegter 1984; Gavrilov 1978, 1985; Gavrilov and Roshchin 1983; Guckenheimer 1981; Guckenheimer and Holmes 1983; Takens 1973a, 1974; Kuznetsov 2004). In these works, for generic singularities, depending on the region in the parameter space where (μ, ν) belongs to, the qualitative behavior of $X_{\mu, \nu}$ is studied. However, for the HZ^* , there is one open region U in the parameter space (see (5)) in which the behavior of $X_{\mu, \nu}$ is not completely understood. These unfoldings $X_{\mu, \nu}$ are the candidates to possess chaotic behavior. In this work we study these unfoldings and prove, as a direct consequence of our results, the existence of heteroclinic transversal curves between two equilibrium points of $X_{\mu, \nu}$ when (μ, ν) belongs to a subset of this region U .

Let us to explain how this phenomenon is encountered in generic unfoldings $X_{\mu, \nu}$ of HZ^* .

Let us call $X_{\mu, \nu}^2$ the normal form of these unfoldings truncated at order two, that is, system (3) neglecting the terms of degree equal or higher than three. In fact we also neglect the second-order terms that only depend on the parameters μ, ν . This system has a rotational symmetry and if we write it in cylindrical coordinates:

$$\bar{x} = \bar{r} \cos \theta, \quad \bar{y} = \bar{r} \sin \theta, \quad \bar{z} = \bar{z}$$

it has the form:

$$\frac{d\bar{r}}{d\bar{t}} = \bar{r}(\nu - \beta_1 \bar{z}), \quad \frac{d\theta}{d\bar{t}} = \alpha_0 + \alpha_1 \mu + \alpha_2 \nu + \alpha_3 \bar{z}, \quad \frac{d\bar{z}}{d\bar{t}} = -\mu + \bar{z}^2 + \gamma_2 \bar{r}^2, \tag{4}$$

where $\alpha_0 = \alpha^*$.

The bifurcation diagram of system (4) has been studied by Guckenheimer (1981) and Gavrilov (1978) (see also Guckenheimer and Holmes 1983; Dumortier et al. 2013 or Kuznetsov 2004 for a study of the normal form up to order three). We emphasize some facts when β_1, γ_2 satisfy (2).

If $\mu < 0$, the system has no equilibrium points and the dynamics is known. At $\mu = 0$ the system has an equilibrium at the origin which bifurcates, for $\mu > 0$, in two equilibrium points $\bar{S}_{\pm}^2 = (0, 0, \pm\sqrt{\mu})$. For $\mu > 0$, the linearization $DX_{\mu, \nu}^2(0, 0, \pm\sqrt{\mu})$ has eigenvalues:

$$\lambda_1^\pm = \nu \mp \beta_1 \sqrt{\mu} + i(\alpha_0 + \alpha_1 \mu + \alpha_2 \nu \pm \alpha_3 \sqrt{\mu}), \lambda_2^\pm = \overline{\lambda_1^\pm}, \lambda_3^\pm = \pm 2\sqrt{\mu}.$$

Therefore:

- if $\nu > \beta_1 \sqrt{\mu}$, \bar{S}_+^2 is a repeller and \bar{S}_-^2 is a saddle-focus.
- if $\nu < -\beta_1 \sqrt{\mu}$, \bar{S}_+^2 is a saddle-focus and \bar{S}_-^2 is an attractor.
- if $-\beta_1 \sqrt{\mu} < \nu < \beta_1 \sqrt{\mu}$, \bar{S}_+^2 and \bar{S}_-^2 are saddle-focus.

The dynamics of $X_{\mu,\nu}^2$ as well as the one of $X_{\mu,\nu}$ are well known in the first two cases. In this paper we will focus in the last case, which presents the richest dynamics, and we will take $(\mu, \nu) \in U$, being

$$U = \{(\mu, \nu) \in \mathbb{R}^2 : \mu > 0, |\nu| < \beta_1 \sqrt{\mu}\}. \tag{5}$$

The dynamics of $X_{\mu,\nu}$ when $(\mu, \nu) \in U$ is studied in the previous references, but some global phenomena still need to be completely understood. Let us give some details about this problem.

When $(\mu, \nu) \in U$, the truncated vector field $X_{\mu,\nu}^2$ has \bar{S}_\pm^2 as equilibrium points of saddle-focus type connected by the heteroclinic orbit:

$$W_1 = \{\bar{x} = \bar{y} = 0, |\bar{z}| \leq \sqrt{\mu}\} \tag{6}$$

which consists on a branch of the one-dimensional unstable manifold of \bar{S}_+^2 that coincides with a branch of the one-dimensional stable manifold of \bar{S}_-^2 .

When $\nu = 0$, the two-dimensional stable manifold of \bar{S}_+^2 also coincides with the two-dimensional unstable manifold of \bar{S}_-^2 , giving rise to a two-dimensional heteroclinic surface (see Fig. 1):

$$W_2 = \left\{ \bar{z}^2 + \frac{\gamma_2}{\beta_1 + 1} \bar{r}^2 = \mu \right\}. \tag{7}$$

For $\nu = 0$ the system has a first integral in the general (both conservative and non-conservative) case:

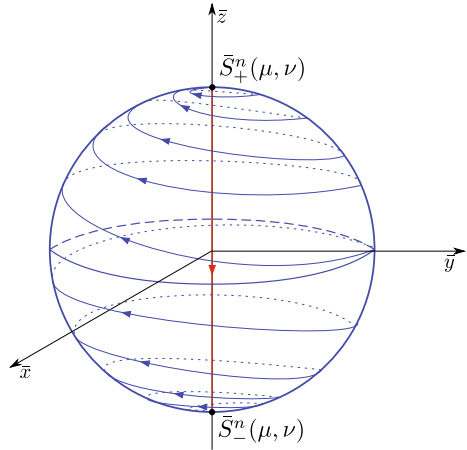
$$H(\bar{r}, \bar{z}) = \bar{r}^{\frac{2}{\beta_1}} \left(-\mu + \bar{z}^2 + \frac{\gamma_2}{\beta_1 + 1} \bar{r}^2 \right).$$

When $\nu \neq 0$, the one-dimensional heteroclinic connection W_1 persists, but this is not the case of the two-dimensional heteroclinic surface. In this case the unstable manifold of \bar{S}_-^2 and the stable manifold of \bar{S}_+^2 do not coincide. More concretely, the intersection of these manifolds with the plane $z = 0$ is two curves C^u, C^s such that C^u is inside the interior of C^s or viceversa depending on the sign of ν . See (Guckenheimer 1981; Kuznetsov 2004; Dumortier et al. 2013).

Let us consider $X_{\mu,\nu}^n$, the truncation of the normal form up to order $n \geq 3$, which is a polynomial of degree n . Then, denoting again the vector field in the new variables by $X_{\mu,\nu}$, one has:

$$X_{\mu,\nu} = X_{\mu,\nu}^n + F_{\mu,\nu}^n, \quad \text{where } F_{\mu,\nu}^n(\bar{x}, \bar{y}, \bar{z}) = \mathcal{O}_{n+1}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu). \tag{8}$$

Fig. 1 Phase portrait of the vector field $X_{\mu,v}^n$ for $(\mu, v) \in \Gamma_n$, for any $n \in \mathbb{N}$. In red and blue, the one- and two-dimensional heteroclinic connections, respectively. The domain bounded by the two-dimensional heteroclinic connection has size $\mathcal{O}(\sqrt{\mu})$ (Color figure online)



If $(\mu, v) \in U$ the truncation of the normal form $X_{\mu,v}^n$ has again two saddle-focus equilibrium points $\bar{S}_{\pm}^n = (0, 0, \mathcal{O}(\sqrt{\mu}))$ connected by a heteroclinic orbit contained in $\bar{r} = 0$.

However, it is known (Guckenheimer 1981; Kuznetsov 2004; Dumortier et al. 2013) that if we fix $\sigma = \frac{v}{\sqrt{\mu}} \neq 0$ and take $\mu \rightarrow 0^+$ the two-dimensional manifolds of \bar{S}_{\pm}^n do not intersect. Furthermore, when $\sigma = \frac{v}{\sqrt{\mu}}$ and μ are small, scaling the system and using classical perturbation methods one can see that the distance between the two-dimensional invariant manifolds measured at their intersection with the plane $z = 0$ is of the form:

$$c_1 v + c_2 \mu + \mathcal{O}\left(v \frac{v}{\sqrt{\mu}}, \mu^{3/2}, v\sqrt{\mu}\right), \quad c_1 \neq 0, \tag{9}$$

where c_1, c_2 depend on the degree three jet of $X_{\mu,v}$. Therefore, if v is not of order μ , this distance cannot be zero and the two-dimensional manifolds of \bar{S}_{\pm}^n do not intersect.

Moreover, if the parameters (v, μ) belong to a curve Γ_n of the form:

$$\Gamma_n = \left\{ (\mu, v) \in U : v = -\frac{c_1}{c_2} \mu + \mathcal{O}(\mu^{3/2}) \right\},$$

one can ensure the existence of a two-dimensional heteroclinic surface (see again Fig. 1) for any finite order n . Let us note here that, in contrast to what happens for $n = 2$, the normal form $X_{\mu,v}^n$ has a first integral only in the conservative case. For this reason the heteroclinic surface exists for any value of the parameter μ in the conservative case. On the contrary, in the general case, if one takes the parameters (μ, v) small enough and away from the curve Γ_n , the two-dimensional stable and unstable manifolds of the points \bar{S}_{\pm}^n do not intersect.

Next, we consider the whole vector field $X_{\mu,v} = X_{\mu,v}^n + F_{\mu,v}^n$. In Dumortier et al. (2013), Kuznetsov (2004), it is shown that:

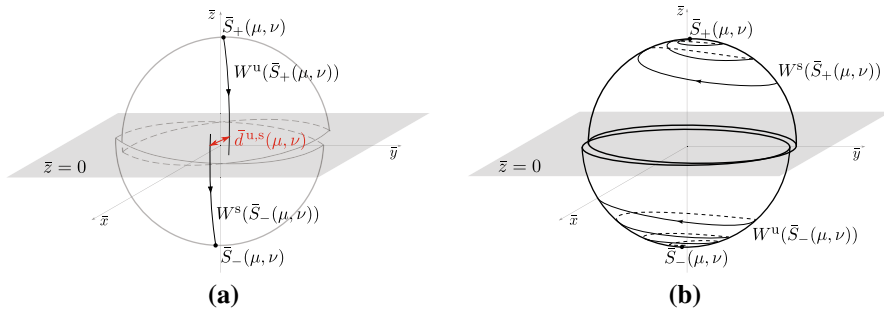


Fig. 2 The distance between the invariant manifolds. **a** The distance $\bar{d}^{u,s}(\mu, \nu)$ between the one-dimensional invariant manifolds of $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$. **b** The two-dimensional invariant manifolds of $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$ until they reach the plane $\bar{z} = 0$

- It also has two equilibrium points $\bar{S}_\pm = (0, 0, \mathcal{O}(\sqrt{\mu}))$, which also are of saddle-focus type if $(\mu, \nu) \in U$ are small enough.
- The one-dimensional invariant manifolds of the equilibrium points $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$ generically split. Indeed, in Baldomá and Seara (2006) and Baldomá et al. (2013) an asymptotic formula for the distance between these manifolds, which turns out to be exponentially small in $\sqrt{\mu}$, was obtained. In the former work the conservative setting for some non-generic unfoldings was considered and the generic case in both the conservative and the general setting was studied in the latter (see Fig. 2a).
- The behavior of the two-dimensional invariant manifolds is more involved. Recall that $X_{\mu,\nu} = X_{\mu,\nu}^n + F_{\mu,\nu}^n$, and the remainder $F_{\mu,\nu}^n$, is of order $\mathcal{O}_{n+1}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu)$. Moreover, as the points and its manifolds until they meet the plane $\bar{z} = 0$ are inside a domain in \mathbb{R}^3 of size $\mathcal{O}(\sqrt{\mu})$, in this region we have: $F_{\mu,\nu}^n(\bar{x}, \bar{y}, \bar{z}) = \mathcal{O}_{n+1}(\sqrt{\mu})$. Therefore, the distance between the manifolds is given again by (9) and consequently, when $|\nu| \neq \mathcal{O}(\mu)$, they do not intersect.

After these considerations the relative position the two-dimensional stable and unstable manifolds of the points \bar{S}_\pm is known, except when $|\nu| = \mathcal{O}(\mu)$. For this reason, we will study the distance between them when

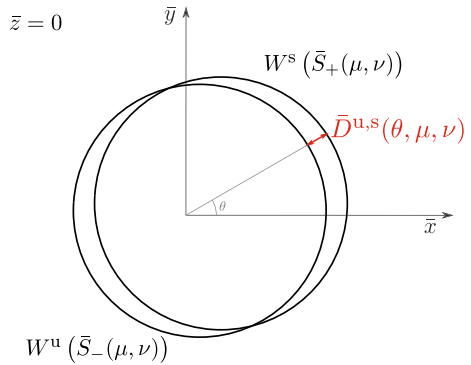
$$|\nu| \leq \sigma^* \mu, \quad (\mu, \nu) \in U \tag{10}$$

for any given constant σ^* .

In particular, when (μ, ν) are close to Γ_n (or for any sufficiently small μ in the conservative case) the heteroclinic connections that exist for $X_{\mu,\nu}^n$ will be generically destroyed in $X_{\mu,\nu}$. Obviously, the breakdown of these heteroclinic connections cannot be detected in the truncation of the normal form at any finite order and therefore, as it is usually called, it is a phenomenon *beyond all orders*.

Since $X_{\mu,\nu} = X_{\mu,\nu}^n + F_{\mu,\nu}^n$, either in the conservative case or in the general case when (μ, ν) are close to Γ_n , the breakdown of the heteroclinic connections must be caused by the remainder $F_{\mu,\nu}^n$, which is of order $F_{\mu,\nu}^n(\bar{x}, \bar{y}, \bar{z}) = \mathcal{O}_{n+1}(\sqrt{\mu})$. As this is valid for all n , the distance between the invariant manifolds should be smaller than

Fig. 3 The intersection between the invariant manifolds and the plane $\bar{z} = 0$, and the distance $\bar{D}^{u,s}(\theta, \mu, \nu)$ between them



any finite power of the perturbation parameter $\sqrt{\mu}$. For this reason, one expects this distance to be exponentially small in $\sqrt{\mu}$ when the analytic case is considered. Note that, in the general case (that is, when two parameters are considered) one expects to find a curve Γ_* such that the distance between the two-dimensional invariant manifolds is exponentially small when $(\mu, \nu) \in \Gamma_*$. In fact, we will prove the existence of a wedge-shaped domain $\mathcal{W}^{u,s}$ (see Fig. 4) around Γ_* such that, when $(\mu, \nu) \in \mathcal{W}^{u,s}$, this distance is exponentially small and the two-dimensional invariant manifolds intersect transversally along two heteroclinic curves (see Theorem 1.1 and Corollary 1.3).

Let $\bar{D}^{u,s}(\theta, \mu, \nu)$ be the distance between the two-dimensional invariant manifolds of the equilibrium points $\bar{S}_\pm(\mu, \nu)$ at the plane $\bar{z} = 0$ (see Figs. 2b, 3). Our final goal is to provide asymptotic formulas for this quantity. However, due to the technical complications to deal with this exponentially small phenomenon, we have split the whole proof in two papers; the present work and (Baldomá et al. 2016). In the former, we provide asymptotic formulas for $\bar{D}^{u,s}(\theta, \mu, \nu)$ when non-generic analytic unfoldings are considered, whereas for generic unfoldings we provide sharp upper bounds. In the latter we give the asymptotic formula in the generic case. It is worth mentioning that all the proofs in this work are also true for the generic case and therefore, in Baldomá et al. (2016), some results derived in this work will be used.

1.1 The Regular Versus the Singular Case: Main Result

As it is well known by experts in the field, in order to obtain asymptotic formulas for the breakdown of the two-dimensional invariant manifolds of $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$, one needs to obtain suitable parameterizations of these invariant manifolds not only on real domains, but also over complex ones. These domains need to be $\mathcal{O}(\sqrt{\mu})$ -close to the singularities of the corresponding heteroclinic connection of the unperturbed system $X_{\mu, \nu}^2$ in (4). We recall that the one-dimensional heteroclinic connection W_1 in (6) exists for any (μ, ν) . However, the two-dimensional one W_2 in (7) only exists when $\nu = 0$. Let us notice that the heteroclinic surface W_2 can be parameterized by (t, θ) (in cylindric coordinates), by:

$$\left. \begin{aligned} \bar{r} &= \bar{r}(t, \theta) = \sqrt{\mu} \frac{\beta_1 + 1}{\gamma_2} \frac{1}{\cosh(\beta_1 t)}, \\ \theta &= \theta, \\ \bar{z} &= \bar{z}(t, \theta) = \sqrt{\mu} \tanh(\beta_1 t), \end{aligned} \right\} t \in \mathbb{R}, \theta \in [0, 2\pi]. \quad (11)$$

Observe that $(\bar{r}(\bar{t}\sqrt{\mu}), \theta(\bar{t}\sqrt{\mu}), \bar{z}(\bar{t}\sqrt{\mu}))$ is solution of (4), with $\theta(t) = \theta_0 + (\alpha_0 + \alpha_1\mu) \frac{t}{\sqrt{\mu}} + \frac{\alpha_3}{\beta_1} \log \cosh(\beta_1 t)$, for arbitrary θ_0 .

Clearly this parameterization has singularities at $\beta_1 t = \pm i\pi/2$. Far from these singularities, we will find parameterizations of the invariant manifolds for $X_{\mu, \nu}$ which are well approximated by the unperturbed heteroclinic connection (11), but this will not be the case close to the singularities. This yields some technical difficulties.

A good way to start the study of the invariant manifolds in these complex domains is considering smaller perturbations of the vector field $X_{\mu, \nu}^2$. Recall that $X_{\mu, \nu} = X_{\mu, \nu}^2 + P_{\mu, \nu}^2 + F_{\mu, \nu}^2$ where $P_{\mu, \nu}^2$ contains the degree two terms in the parameters μ, ν in (3) and $F_{\mu, \nu}^2 = \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu)$. We introduce a new parameter $q \geq 0$ and consider the following (artificial) vector field,

$$X_{\mu, \nu}^{\text{reg}} := X_{\mu, \nu}^2 + (\sqrt{\mu})^q (P_{\mu, \nu}^2 + F_{\mu, \nu}^2). \quad (12)$$

Clearly, for $q = 0$ we recover (8), while for $q > 0$ the perturbation terms are smaller than those in (8). We call the case $q > 0$ the *regular case*, while $q = 0$ is the *singular* one which represents a generic family of unfoldings of HZ^* .

Imposing the condition $q > 0$, one can see that the heteroclinic connections of the unperturbed system $X_{\mu, \nu}^2$ give good approximations of the invariant manifolds, even close to their singularities. The asymptotic formulas measuring the breakdown of the heteroclinic surface in this case consist on suitable versions of the so-called Melnikov integrals (see Guckenheimer and Holmes 1983; Mel'nikov 1963). Thus, one can start studying the regular case to gain some intuition without getting lost with technical problems and, after that, one can proceed with the singular case. This is what we have done in the present paper. Notice that in the general case we take the parameters satisfying $|\nu| \leq \sigma^*(\sqrt{\mu})^{q+2}$, $(\mu, \nu) \in U$ which gives condition (10) when $q = 0$.

More precisely we have proven:

Theorem 1.1 Consider a Hopf-zero singularity HZ^* and $X_{\mu, \nu}^{\text{reg}}$ as in (12) with $q \geq 0$. There exists $\mu_0 > 0$ such that if $0 < \mu < \mu_0$ and $(\mu, \nu) \in U$ defined in (5), the vector field $X_{\mu, \nu}^{\text{reg}}$ has two equilibrium points $\bar{S}_{\pm}(\mu, \nu)$ of saddle-focus type of the form

$$\bar{S}_{\pm}(\mu, \nu) = (0, 0, \pm\sqrt{\mu}) + \mathcal{O}(\mu^2 + \nu^2)^{\frac{q+1}{2}}.$$

In addition, \bar{S}_+ has a two-dimensional stable manifold and \bar{S}_- has a two-dimensional unstable manifold.

For any $u \in \mathbb{R}$ and $\theta \in [0, 2\pi]$, let $\bar{D}^{\text{u},s}(u, \theta, \mu, \nu)$ ($\bar{D}^{\text{u},s}(u, \theta, \mu)$ in the conservative case) be the distance between the two-dimensional unstable manifold of $\bar{S}_-(\mu, \nu)$ and the two-dimensional stable manifold of $\bar{S}_+(\mu, \nu)$ when they meet the plane $\bar{z} = \sqrt{\mu} \tanh(\beta_1 u)$ (see (11)).

Then, there exist constants C_1, C_2 (see their formula in Theorem 2.5) and L_0 (see its formula in Remark 5.7) in such a way that, given $T_0 > 0$, for all $u \in [-T_0, T_0]$ and $\theta \in \mathbb{S}^1$, introducing the function:

$$\bar{\vartheta}(u, \mu) = \frac{\alpha_0 u}{\sqrt{\mu}} + \frac{1}{\beta_1} [\alpha_3 \log \cosh(\beta_1 u) - (\alpha_3 + \alpha_0 L_0 (\sqrt{\mu})^q) \log \sqrt{\mu}],$$

the following holds:

1. In the conservative case, which corresponds to $\beta_1 = 1$ and $v = 0$, as $\mu \rightarrow 0^+$,

$$\begin{aligned} \bar{D}^{u,s}(u, \theta, \mu) = & \sqrt{\frac{\gamma_2}{2}} \frac{e^{-\frac{\alpha_0 \pi}{2\sqrt{\mu}}}}{(\sqrt{\mu})^{3-q}} \cosh^3(u) \left[C_1 \cos(\theta + \bar{\vartheta}(u, \mu)) \right. \\ & \left. + C_2 \sin(\theta + \bar{\vartheta}(u, \mu)) + \mathcal{O}((\sqrt{\mu})^q + (\sqrt{\mu})^3) \right]. \end{aligned}$$

2. In the general case, given $\sigma^* > 0$, for $\mu \rightarrow 0^+$ and $|v| \leq \sigma^* (\sqrt{\mu})^{q+2}$, there exists $C_0 = C_0(\mu, v)$ given by:

$$C_0(\mu, v) = vI + J(\sqrt{\mu})^{q+2} + \mathcal{O}((\sqrt{\mu})^{q+3}) \tag{13}$$

where $J, I \neq 0$ are constants defined in (92) and (90), such that:

$$\begin{aligned} \bar{D}^{u,s}(u, \theta, \mu, v) = & \sqrt{\frac{\gamma_2}{\beta_1 + 1}} \cosh^{1+\frac{2}{\beta_1}}(\beta_1 u) \left\{ C_0 \left(1 + \mathcal{O}((\sqrt{\mu})^{q+1}, \frac{v}{\sqrt{\mu}}) \right) \right. \\ & + \frac{e^{-\frac{\alpha_0 \pi}{2\beta_1 \sqrt{\mu}}}}{(\sqrt{\mu})^{\frac{2}{\beta_1} + 1 - q}} \left[C_1 \cos(\theta + \bar{\vartheta}(u, \mu)) \right. \\ & \left. \left. + C_2 \sin(\theta + \bar{\vartheta}(u, \mu)) + \mathcal{O}((\sqrt{\mu})^q + (\sqrt{\mu})^3) \right] \right\}. \end{aligned}$$

In addition, there exists a curve

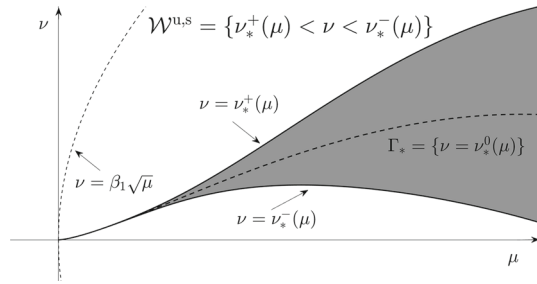
$$\Gamma_* = \left\{ (\mu, v) \in U : v = v_*^0(\sqrt{\mu}) = -\frac{J}{I} \sqrt{\mu}^{q+2} + \mathcal{O}(\sqrt{\mu}^{q+3}) \right\}$$

such that for all $0 < \mu < \mu_0$ one has:

$$C_0 = C_0(\mu, v_*^0(\sqrt{\mu})) = 0.$$

Remark 1.2 Notice that, if we take $q = 0$ in Theorem 1.1, the relative error terms are not small but $\mathcal{O}(1)$ so our result provides sharp upper bounds even in this case. Later, in Sect. 2.5 these upper bounds are proven in a different (and easier) way. To obtain asymptotic formulas when $q = 0$, one needs to deal with the so-called *inner equation* as it is done in Baldomá et al. (2016).

Fig. 4 The curve Γ_* and the wedge-shaped domain $\mathcal{W}^{u,s}$ around it. For values of μ, ν inside this open domain, $X_{\mu,\nu}$ has two transversal heteroclinic orbits, whereas if μ, ν belongs to $\{|\nu| < \beta_1 \sqrt{\mu}\} \setminus \mathcal{W}^{u,s}$ there are not heteroclinic intersections



Corollary 1.3 Take $T_0 > 0$, and $0 < \mu < \mu_0$. Consider the curve Γ_* given in Theorem 1.1. Then, there exists a wedge-shaped domain $\mathcal{W}^{u,s}$ in the parameter plane around this curve (see Fig. 4) such that, for $(\mu, \nu) \in \mathcal{W}^{u,s}$, and for fixed $u \in [-T_0, T_0]$, the function $\bar{D}^{u,s}(u, \theta, \mu)$ is exponentially small and has two simple zeros which give rise to two transversal heteroclinic orbits between the points $\bar{S}_{\pm}(\mu, \nu)$. Moreover, for $(\mu, \nu) \notin \mathcal{W}^{u,s}$, $\bar{D}^{u,s}(u, \theta, \mu)$ has no zeros and therefore the two-dimensional stable and unstable manifolds of $\bar{S}_{\pm}(\mu, \nu)$ do not intersect.

A more accurate description of the wedge-shaped domain $\mathcal{W}^{u,s}$ in terms of the parameters $\delta = \sqrt{\mu}$, $\sigma = \frac{\nu}{\sqrt{\mu}}$ is given in Corollary 2.17.

Proposition 1.4 Consider the set \mathcal{HZ}^* of the Hopf-zero singularities HZ^* , with $\beta_1, \beta_2 > 0$. Then,

1. Given $HZ^* \in \mathcal{HZ}^*$ and $X_{\mu,\nu}^{\text{reg}}$ as in (12) the constant $C := C_1 - iC_2$, where C_1, C_2 are given in Theorem 1.1, only depends on the chosen singularity $HZ^* \in \mathcal{HZ}^*$
2. Let \mathcal{A} be the subset of \mathcal{HZ}^* such that, if $HZ^* \in \mathcal{A}$, then the constant $C \neq 0$. Then \mathcal{A} is open and dense in \mathcal{HZ}^* with the supremum norm.

1.2 Exponentially Small Splitting of Invariant Manifolds

The formulas given in Theorem 1.1 prove that the breakdown of the heteroclinic connection is exponentially small in the perturbation parameter $\sqrt{\mu}$ when $(\mu, \nu) \in \Gamma_*$ in the general case or in the conservative case. Therefore, this work deals with the problem of the so-called exponentially small splitting of separatrices.

This problem was already considered by Poincaré in his famous work (Poincaré 1890). There he studied Hamiltonian systems with two and a half degrees of freedom and realized that this phenomenon was responsible for the creation of chaotic behavior. He considered a model which, after reduction, became the perturbed pendulum:

$$\ddot{y} = 2\mu \sin y + 2\mu\varepsilon \cos y \cos t.$$

Poincaré developed an analytic tool, rediscovered by V.K. Melnikov 70 years later, to prove that the splitting of the separatrices is exponentially small in μ , provided that ε is smaller than some exponentially small quantity. Of course, this latter assumption is enormously restrictive, but many years had to go by until it could be removed.

When studying exponentially small phenomena, one cannot use classical perturbation methods. Over the last decades more sophisticated techniques have been developed mainly for Hamiltonian systems and area-preserving maps. Indeed, this problem was not studied from a rigorous point of view until the end of the 1980s and during the 1990s. Neishtadt (1984) gave upper bounds for the splitting in Hamiltonian systems of one and a half and two degrees of freedom and Lazutkin (2003) (see also Gelfreich 1999) was the first to give an asymptotic expression of the splitting angle between the stable and unstable manifolds for the standard map.

After Lazutkin's paper, some works gave bounds for the splitting for rapidly forced systems and for area-preserving diffeomorphisms close to the identity (Holmes et al. 1988; Fontich and Simó 1990a, b).

Later on, asymptotic formulas for several examples were obtained. The first works were (Kruskal and Segur 1991; Delshams and Seara 1992; Gelfreich 1994). After these pioneering works, partial results for general Hamiltonian systems were given in Delshams and Seara (1997), Gelfreich (1997a), Baldomá and Fontich (2004), Baldomá and Fontich (2005). A new approach that has had much influence in posterior studies of exponentially small splitting was introduced in Sauzin (2001), Lochak et al. (2003). It is important to note that, besides (Lazutkin 2003) and (Kruskal and Segur 1991), all the examples cited above deal with the so-called regular case, in which some artificial condition about the smallness of the perturbation is required. In this case the Melnikov method gives the correct size of the splitting.

In the singular case one often has to study a certain equation independent of parameters, usually called the *inner equation*. There are a few works dealing with this kind of equations in different settings, see (Gelfreich 1997b; Gelfreich and Sauzin 2001; Olivé et al. 2003; Baldomá 2006; Baldomá and Seara 2008; Baldomá and Martín 2012), but, besides the works of Lazutkin and Kruskal and Segur, there are very few works with rigorous proofs in the singular case for Hamiltonian systems (see for instance Treschev 1997; Gelfreich 2000; Guardia et al. 2010; Baldomá et al. 2012; Guardia 2013) or conservative maps (Gelfreich and Brännstrom 2008; Martín et al. 2011). Numerical results about the splitting in the Hamiltonian setting can be found in Benseny and Olivé (1993), Gelfreich (1997b), for two-dimensional symplectic maps in Delshams and Ramírez-Ros (1999), Gelfreich and Simó (2008), Simó and Vieiro (2009), Miguel et al. (2013) and in Gelfreich et al. (2013) the splitting is computed for two-dimensional manifolds in four-dimensional symplectic maps. Besides the previous works on either Hamiltonian systems or symplectic maps, (Lazaro Ochoa 2003) deals with exponentially small splitting of separatrices in the reversible setting and the work (Fontich 1995) gives results for dissipative perturbations of Hamiltonian systems.

The exponentially small phenomenon can be encountered in several problems of bifurcation theory. In Broer and Roussarie (2001), the authors prove that chaotic dynamics near a degenerate fixed point, in quite general families of planar diffeomorphisms, is confined to, at most, an exponentially narrow horn in the parameter space. A particular case is the Bodganov–Takens bifurcation. Numerical studies for this bifurcation are done in Simó et al. (1991), Gelfreich and Naudot (2009), Gelfreich (2003). The latter also provides heuristic arguments to derive an asymptotic formula

for the size of the chaotic zone. A related paper is Gelfreich and Naudot (2008) where the authors study a inner equation associated with the Bodgánov–Takens bifurcation.

For Hamiltonian systems of two degrees of freedom, the analogous singularity to the Hopf-zero singularity is the $0^2i\omega$ singularity, also called Hamiltonian–Hopf bifurcation. In Gaivão and Gelfreich (2011), the authors study this singularity combining numerical and analytic techniques. The same singularity in the reversible case is considered in Lombardi (1999), Lombardi (2000) where the author proves the existence of homoclinic connections for every periodic orbit exponentially close to the origin, except the origin itself. We emphasize the paper (Jézéquel et al. 2016), where the authors prove the same result considering homoclinic connections with several loops, using geometric arguments.

For the Hopf-zero singularity in Baldomá et al. (2013), an asymptotic formula to measure the breakdown of the one-dimensional heteroclinic connection W_1 in (6) is proven in the singular case for any value of (μ, ν) small enough both in the conservative and the general case.

It is worth mentioning that here we do not deal with a Hamiltonian system, namely we study a vector field in \mathbb{R}^3 whose flow might not be volume-preserving (since we consider not only the conservative setting but also the general one) and it is not a reversible system. For this reason, some new ideas had to be used in order to prove the results found in this paper. We stress that, even if the general case may seem more difficult, the freedom to choose the parameters in a suitable curve on the parameter space where a relevant quantity, the constant C_0 in Theorem 1.1, vanishes simplifies the proof of this theorem. The same procedure has been used in Broer and Roussarie (2001), Gelfreich (2003). However, in the conservative case, the volume-preserving property of the vector field must be used to prove that this constant is zero for any unfolding.

1.3 The Shilnikov Bifurcation and the Hopf-Zero Singularity

To finish this introduction, let us mention that we believe that our results can lead to prove the existence of Shilnikov bifurcations, (Šil'nikov 1965), in suitable unfoldings $X_{\mu,\nu}$. Indeed, the existence of such Shilnikov bifurcations for C^∞ unfoldings of the Hopf-zero singularity is studied in Broer and Vegter (1984). Doing the normal form procedure up to order infinity and using Borel–Ritt's theorem, the vector field $X_{\mu,\nu}$ can be decomposed as $X_{\mu,\nu} = X_{\mu,\nu}^\infty + F_{\mu,\nu}^\infty$, where $X_{\mu,\nu}^\infty$ has the same phase portrait as the vector field $X_{\mu,\nu}^n$ described above (Fig. 1) and $F_{\mu,\nu}^\infty = F_{\mu,\nu}^\infty(\bar{x}, \bar{y}, \bar{z})$ is a flat function at the origin. Their strategy consists in constructing suitable perturbations $p_{\mu,\nu}^\infty$, which are also flat functions, such that the heteroclinic connections of the family $X_{\mu,\nu}^\infty$ are destroyed and some homoclinic ones appear giving rise to the so-called Shilnikov bifurcation. Therefore, an existence theorem is given, but the results do not provide conditions to check whether a concrete family $X_{\mu,\nu}$ possesses or not a Shilnikov bifurcation.

The case of real analytic unfoldings of the singularity HZ^* has been open since then. It is possible that the strategy of Broer and Vegter can be adapted to the analytic case. Of course one cannot consider flat perturbations, but suitable perturbations could be constructed (although not straightforwardly) following (Broer and Tangerman 1986)

and (Broer and Takens 1989). However, another strategy must be followed if given *any* unfolding $X_{\mu,v}$ one wants to determine whether it will or will not possess a sequence of Shilnikov bifurcations. The key point, as in the C^∞ case, is to check if the given unfolding $X_{\mu,v}$ does not have the aforementioned heteroclinic connections.

Progress was made recently in Dumortier et al. (2013), where the authors prove a result equivalent to that of Broer and Vegter (1984) in the real analytic context, assuming some upper and lower bounds of the distance between the invariant manifolds of $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$. The authors assume, among other conditions, that the heteroclinic connections are destroyed and quantitative information about the splitting is required. Our work provides asymptotic formulas of the splitting of these invariant manifolds which, as a consequence, allow to check if the corresponding assumptions in Dumortier et al. (2013) are satisfied. We leave the complete proof of the existence of Shilnikov bifurcations for a future work.

1.4 Plan of the Article

The paper is organized as follows. In Sect. 2, we expose the strategy we will follow to prove Theorem 1.1 by enunciating, without proving, the main results we will need, namely, i) existence of suitable parameterizations of the invariant manifolds in complex domains, ii) derivation and computation of the Melnikov function, iii) expression, in complex domains, for the difference between the invariant manifolds and iv) the exponentially small formulas for the difference in real domains. After that, still in Sect. 2, we prove Theorem 1.1 as a consequence of Theorem 2.14. We postpone all the technical proofs to Sects. 3–5.

2 The Regular Case: Heuristics of the Proof

Let us first define the complex norm in \mathbb{C}^n , $\|\zeta\| = \max\{|\zeta_1|, \dots, |\zeta_n|\}$. With this norm, we will denote by $B(r)$, the open ball of \mathbb{C}^n of radius $r > 0$ centered at the origin.

Following the same strategy as the one presented in Baldomá et al. (2013) and also discussed in Sect. 1, which involves normal form changes, scalings and redefinitions of parameters, we can write $X_{\mu,v}^{\text{reg}}$, in (12), in its normal form of order three, namely:

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{x}(\nu - \beta_1\bar{z}) + \bar{y}(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + (\sqrt{\mu})^q \bar{f}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \\ \frac{d\bar{y}}{dt} &= -\bar{x}(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y}(\nu - \beta_1\bar{z}) + (\sqrt{\mu})^q \bar{g}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \quad (14) \\ \frac{d\bar{z}}{dt} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2 + \bar{y}^2) + (\sqrt{\mu})^q \bar{h}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu). \end{aligned}$$

The functions \bar{f} , \bar{g} and \bar{h} are analytic functions in $B(\bar{r}_0)^3 \times B(\bar{\mu}_0) \times B(\bar{\nu}_0) \subset \mathbb{C}^3 \times \mathbb{C}^2$,

$$\begin{aligned} \bar{f}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) &= \bar{x}\bar{A}(\bar{x}^2 + \bar{y}^2, \bar{z}, \mu, \nu) + \bar{y}\bar{B}(\bar{x}^2 + \bar{y}^2, \bar{z}, \mu, \nu) + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \bar{g}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) &= \bar{y}\bar{A}(\bar{x}^2 + \bar{y}^2, \bar{z}, \mu, \nu) - \bar{x}\bar{B}(\bar{x}^2 + \bar{y}^2, \bar{z}, \mu, \nu) + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \\ \bar{h}(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) &= \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \bar{C}(\bar{x}^2 + \bar{y}^2, \bar{z}, \mu, \nu) + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu) \end{aligned}$$

and \bar{A} , \bar{B} and \bar{C} are functions satisfying:

$$\bar{x}\bar{A}, \bar{x}\bar{B}, \bar{y}\bar{A}, \bar{y}\bar{B}, \bar{C} = \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu),$$

when they are evaluated in their arguments.

Remark 2.1 In Baldomá et al. (2013) when the breakdown of the one-dimensional heteroclinic connection was considered, we performed the normal form up to order two. In this work we need to perform an additional step of the normal form procedure for technical reasons which will be explained later on, see Sect. 4.1.1, even though the terms of order three do not appear explicitly neither in our hypotheses nor in our results.

In the remaining part of this section we give the main ideas of the proof of Theorem 1.1. The rest of the paper is devoted to prove the results stated in this section. We now summarize the subsections that can be found in this Section, each one consisting in one step of the proof of Theorem 1.1 and Corollary 1.3. The first step, explained in detail in Sect. 2.1, consists in scaling variables and introducing the new parameters $\delta = \sqrt{\mu}$, $\sigma = \delta^{-1}\nu$ and calling $p = q - 2$ as in Baldomá et al. (2013). Still in this preliminary section, we give a parameterization of the heteroclinic connection of the unperturbed system which corresponds to the normal form of order two. In Sect. 2.2 we give parameterizations of the two-dimensional invariant manifolds adequate to our purposes. In Sect. 2.3, we introduce and study the Melnikov function adapted to this problem. This Melnikov function will be the dominant term in the difference between the invariant manifolds. After that, in Sect. 2.4, we give some properties of this difference which allows us, in Sect. 2.5, to give a sharp upper bound of this difference. Finally, in Sect. 2.6, we state and prove Theorem 2.14 which is equivalent to Theorem 1.1 and Corollary 1.3.

2.1 Preliminary Considerations

This subsection is mainly devoted to fix notation and perform straightforward changes of variables to put the vector field $X_{\mu,\nu}^{\text{reg}}$, in (12), in a suitable way to work with. Moreover, we also study what we call the unperturbed system.

2.1.1 Notation, Scalings and Set Up

We scale system (14) as in Baldomá et al. (2013), namely we define the new parameters p , δ , σ and rename the coefficients $\gamma_2, \alpha_3, \beta_1$ as:

$$p = q - 2, \quad \delta = \sqrt{\mu}, \quad \sigma = \delta^{-1}\nu, \quad b = \gamma_2, \quad c = \alpha_3, \quad d = \beta_1.$$

We also introduce the constant h_3 from \bar{h} given by

$$\bar{h}(0, 0, \bar{z}, 0) = h_3\bar{z}^3 + \mathcal{O}(\bar{z}^4).$$

In the new variables

$$x = \delta^{-1}\bar{x}, \quad y = \delta^{-1}\bar{y}, \quad z = \delta^{-1}\bar{z} + \delta^{p+3}h_3/2, \quad t = \delta\bar{t},$$

system (14) becomes:

$$\begin{aligned} \frac{dx}{dt} &= x(\sigma - dz) + \left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right)y + \delta^p f(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\ \frac{dy}{dt} &= -\left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right)x + y(\sigma - dz) + \delta^p g(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \delta^p h(\delta x, \delta y, \delta z, \delta, \delta\sigma), \end{aligned} \tag{15}$$

where $\alpha(\delta^2, \delta\sigma) = \alpha_0 + \alpha_1\delta\sigma + \alpha_2\delta^2$ with $\alpha_0 \neq 0$ and f, g and h are the corresponding ones to \bar{f}, \bar{g} and \bar{h} . To shorten the notation, we write system (15) as

$$\frac{d\zeta}{dt} = X(\zeta, \delta, \sigma) = X_0(\zeta, \delta, \sigma) + \delta^p X_1(\delta\zeta, \delta, \delta\sigma), \quad \zeta = (x, y, z). \tag{16}$$

From now on, we will omit the dependence of α with respect to δ and σ .

Remark 2.2 Recall that $b > 0, d > 0$, The parameter $\delta > 0$ is small and $|\sigma| < d$. Without loss of generality, we assume that α_0 and c are positive constants. In particular, for δ small enough, $\alpha(\delta^2, \delta\sigma)$ will be also positive.

Since the functions \bar{f}, \bar{g} and \bar{h} are real analytic, the same happens for X_1 . We call $B^3(r_0) \times B(\delta_0) \times B(\sigma_0) \subset \mathbb{C}^3 \times \mathbb{C}^2$ its analyticity domain.

2.1.2 Unperturbed System: $\sigma = 0, X_1 \equiv 0$

Consider system (15) with $\sigma = 0, f = g = h = 0$. It is clear that it has rotational symmetry. For our purposes it will be very useful to consider “symplectic” cylindric coordinates:

$$x = \sqrt{2r} \cos \theta, \quad y = \sqrt{2r} \sin \theta, \quad z = z. \tag{17}$$

The main reason is that this change of variables is volume-preserving. Therefore, in the conservative case, after this change of variables the new vector field will be conservative too. The unperturbed system writes out as:

$$\frac{dr}{dt} = -2drz, \quad \frac{d\theta}{dt} = -\frac{\alpha}{\delta} - cz, \quad \frac{dz}{dt} = -1 + 2br + z^2. \tag{18}$$

Since $b > 0$, the unperturbed system (18) has a two-dimensional heteroclinic manifold connecting $S_+(\delta, 0) = (0, 0, 1)$ and $S_-(\delta, 0) = (0, 0, -1)$ given by:

$$\left\{ (r, z) \in \mathbb{R}^2 : -1 + \frac{2br}{d+1} + z^2 = 0 \right\}.$$

This manifold can be parameterized for $t \in \mathbb{R}$ by the solutions of the unperturbed system starting at time $t = 0$ on the plane $z = 0$ and with angular variable $\theta = \theta_0 \in [0, 2\pi)$ by:

$$r = R_0(t) := \frac{(d + 1)}{2b} \frac{1}{\cosh^2(dt)}, \tag{19}$$

$$\theta = \Theta_0(t, \theta_0) := \theta_0 - \frac{\alpha}{\delta}t - \frac{c}{d} \log \cosh(dt),$$

$$z = Z_0(t) := \tanh(dt). \tag{20}$$

Remark 2.3 For bounded $\|\xi\|$, with $\xi = (x, y, z, 1, \sigma)$, $f(\delta\xi), g(\delta\xi), h(\delta\xi) = \mathcal{O}(\delta^3)$. Thus, using classical perturbation methods, one can easily see that the difference between the two-dimensional invariant manifolds is of order $\mathcal{O}(\sigma) + \mathcal{O}(\delta^{p+3})$. Therefore, if σ is not of order δ^{p+3} , this difference is not exponentially small in δ . For this reason, in the rest of the paper we assume that $|\sigma| \leq \sigma^* \delta^{p+3}$, for some constant σ^* , since the exponentially small case is the only one where the Shilnikov phenomenon can occur, see (Dumortier et al. 2013).

Analogous considerations were done in the introduction for the generic case $p = -2$ (see (10)).

From now on, we will omit the dependence of α with respect to δ and σ .

2.2 Local Parameterizations of the Invariant Manifolds

System (16) has two equilibrium points $S_{\pm}(\delta, \sigma)$ of saddle-focus type, see (Baldomá et al. 2013) for instance, and also Lemma 4.1. The goal in this subsection is to provide good parameterizations for the two-dimensional invariant manifolds associated with $S_{\pm}(\delta, \sigma)$.

It is useful to write system (16) in symplectic cylindrical coordinates (17):

$$\begin{aligned} \frac{dr}{dt} &= 2r(\sigma - dz) + \delta^p \mathbf{F}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p \mathbf{G}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p \mathbf{H}(\delta r, \theta, \delta z, \delta, \delta\sigma), \end{aligned} \tag{21}$$

where $\mathbf{X}_1 = (\mathbf{F}, \mathbf{G}, \mathbf{H})$ is defined as

$$\mathbf{X}_1(\delta r, \theta, \delta z, \delta, \delta\sigma) = \begin{pmatrix} \sqrt{2r} \cos \theta & \sqrt{2r} \sin \theta & 0 \\ -\frac{1}{\sqrt{2r}} \sin \theta & \frac{1}{\sqrt{2r}} \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} X_1(\delta\zeta, \delta, \delta\sigma) \tag{22}$$

being $\zeta = (\sqrt{2r} \cos \theta, \sqrt{2r} \sin \theta, z)$.

Let us explain how we construct good parameterizations of the invariant manifolds which will be solutions of the same equation. The experts in the field know this is a key

point to prove the exponential smallness of their difference. Due to the geometry of the unperturbed system, it seems natural to write them as graphs over z and the angular variable θ (see Fig. 2). However, we will not do exactly that, but instead we will introduce a new variable u defined by: $u = Z_0^{-1}(z) = d^{-1} \operatorname{atanh}(z)$, or equivalently $z = Z_0(u)$ (recall that Z_0 was defined in (20)). The invariant manifolds in symplectic polar coordinates will be parameterized by:

$$r = r^{u,s}(u, \theta), \quad z = Z_0(u), \tag{23}$$

or in Cartesian coordinates

$$x = \sqrt{2r^{u,s}(u, \theta)} \cos \theta, \quad y = \sqrt{2r^{u,s}(u, \theta)} \sin \theta, \quad z = Z_0(u).$$

This method, being very useful for our purposes, has some drawbacks. For example, it is obvious that $z = Z_0(u) \rightarrow \pm 1$ as $u \rightarrow \pm\infty$. Thus, if the z -component of the equilibrium points $S_{\pm}(\delta, \sigma)$ is not equal to ± 1 , respectively, these parameterizations will not work for large values of $|u|$. Nevertheless, we will prove that these parameterizations exist for bounded values of u .

Now we give the invariance equation that the parameterizations $r^{u,s}$ satisfy. To simplify the notation, we introduce

$$\tilde{X}_1(r)(u, \theta) = \mathbf{X}_1(\delta(R_0(u) + r(u, \theta)), \theta, \delta Z_0(u), \delta, \delta\sigma), \quad \tilde{X}_1 = (F, G, H) \tag{24}$$

for a given function $r(u, \theta)$. To avoid cumbersome notations, if there is no danger of confusion, we will omit the dependence on variables (u, θ) . Using this notation, the parameterizations $r^{u,s}$ have to satisfy the following PDE:

$$\frac{d\theta}{dt} \partial_{\theta} r + \frac{du}{dt} \partial_u r = 2(\sigma - dZ_0(u))r + \delta^p F(r - R_0(u)),$$

and, using Eq. (21) and that $\frac{du}{dt} = d^{-1}(1 - Z_0^2(u))^{-1} \frac{dz}{dt}$:

$$\begin{aligned} \left(-\frac{\alpha}{\delta} - cZ_0(u) + \delta^p G(r - R_0(u))\right) \partial_{\theta} r + \left(\frac{-1 + 2br + Z_0^2(u) + \delta^p H(r - R_0(u))}{d(1 - Z_0^2(u))}\right) \partial_u r \\ = 2(\sigma - dZ_0(u))r + \delta^p F(r - R_0(u)). \end{aligned} \tag{25}$$

Since it is reasonable to consider system (21) as a perturbation of the unperturbed system (18) ($\sigma = 0$ and $\mathbf{X}_1 = 0$) studied in Sect. 2.1.2, we impose that $r^{u,s}(u, \theta) = R_0(u) + r_1^{u,s}(u, \theta)$, where R_0 is given in (19). Using the relations

$$R_0'(u) = -2dR_0(u)Z_0(u), \quad -1 + 2bR_0(u) + Z_0^2(u) = d(1 - Z_0^2(u)),$$

and putting all terms which are either small or nonlinear in $r_1^{u,s}$ in the right-hand side of the equality and the remaining terms in the left, Eq. (25) writes out as

$$\mathcal{L}(r_1) = \mathcal{F}(r_1), \tag{26}$$

where \mathcal{L} and \mathcal{F} are the differential operators defined by:

$$\mathcal{L}(r) := \left(-\delta^{-1}\alpha - cZ_0(v)\right) \partial_\theta r + \partial_u r - 2Z_0(u)r, \tag{27}$$

$$\begin{aligned} \mathcal{F}(r) := & 2\sigma(R_0(u) + r) + \delta^p F(r) + \delta^p \frac{d+1}{b} Z_0(u)H(r) \\ & - \delta^p G(r)\partial_\theta r - \left(\frac{2br + \delta^p H(r)}{d(1 - Z_0^2(u))}\right) \partial_u r. \end{aligned} \tag{28}$$

We now define the complex domains in which $r_1^{u,s}$ (and therefore $r^{u,s}$) will be defined. We first deal with the unstable case. We want these domains to be close to the singularities of the heteroclinic connection of the unperturbed system (see (19)–(20)) closest to the real line. These are $\pm \frac{i\pi}{2d}$. Moreover, it will be convenient that these domains have a triangular shape. To this aim, let $0 < \beta < \pi/2$ and $\kappa^* > 0$ be two constants independent of δ and σ . Take $\kappa = \kappa(\delta)$ any function satisfying that for $0 < \delta < 1$:

$$\kappa^* \delta \leq \kappa \delta < \frac{\pi}{8d}. \tag{29}$$

Then we define the domain (see Fig. 5a):

$$D_{\kappa,\beta}^u = \left\{ v \in \mathbb{C} : |\operatorname{Im} v| \leq \frac{\pi}{2d} - \kappa \delta - \tan \beta \operatorname{Re} v \right\}. \tag{30}$$

We will split the domain $D_{\kappa,\beta}^u$ in two subsets. Let $T > 0$ be any constant independent of β, κ^*, δ and σ . We introduce (see Fig. 5a):

$$D_{\kappa,\beta,\infty}^u = \left\{ v \in D_{\kappa,\beta}^u : \operatorname{Re} v \leq -T \right\}, \quad D_{\kappa,\beta,T}^u = \left\{ v \in D_{\kappa,\beta}^u : \operatorname{Re} v \geq -T \right\}.$$

Analogously, for the stable case we define (see Fig. 5b):

$$D_{\kappa,\beta}^s = -D_{\kappa,\beta}^u, \quad D_{\kappa,\beta,\infty}^s = -D_{\kappa,\beta,\infty}^u, \quad D_{\kappa,\beta,T}^s = -D_{\kappa,\beta,T}^u.$$

For any fixed real $\omega > 0$, we also define the complex domains:

$$\mathbb{T}_\omega := \{ \theta \in \mathbb{C}/(2\pi\mathbb{Z}) : |\operatorname{Im} \theta| \leq \omega \}. \tag{31}$$

The next result gives the main properties of the functions $r_1^{u,s}$.

Theorem 2.4 *Let $p \geq -2, \sigma^* > 0$ and $0 < \beta < \pi/2$ be any constants. There exist $\kappa^* \geq 1, \delta^* > 0$, such that for all $0 < \delta \leq \delta^*$, if $\kappa = \kappa(\delta)$ satisfies condition (29) and*

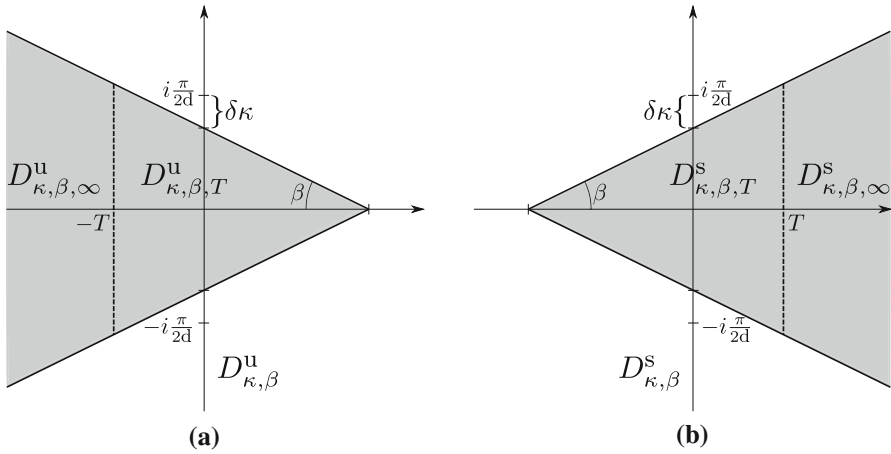


Fig. 5 The outer domains $D_{\kappa,\beta}^u$ and $D_{\kappa,\beta}^s$. **a** Outer domain $D_{\kappa,\beta}^u$ for the unstable case with subdomains $D_{\kappa,\beta,T}^u$ and $D_{\kappa,\beta,\infty}^u$. **b** Outer domain $D_{\kappa,\beta}^s$ for the stable case with subdomains $D_{\kappa,\beta,T}^s$ and $D_{\kappa,\beta,\infty}^s$

$|\sigma| \leq \sigma^* \delta^{p+3}$, the unstable manifold of $S_-(\delta, \sigma)$ and the stable manifold of $S_+(\delta, \sigma)$ are given, respectively, by:

$$\zeta^{u,s}(u, \theta) = (\sqrt{2r^{u,s}(u, \theta)} \cos \theta, \sqrt{2r^{u,s}(u, \theta)} \sin \theta, Z_0(u)), \quad (u, \theta) \in D_{\kappa,\beta,T}^{u,s} \times \mathbb{T}_\omega$$

with $r^{u,s}(u, \theta) = R_0(u) + r_1^{u,s}(u, \theta)$ and $r_1^{u,s}$ satisfying Eq. (26).

Let us introduce

$$\eta^\mp(w) = \alpha w \mp \delta(cw \mp cd^{-1} \log(1 + e^{\pm 2dw})).$$

We decompose $r_1^{u,s}$ into $r_1^{u,s} = r_{10}^{u,s} + r_{11}^{u,s}$ being

$$r_{10}^{u,s}(u, \theta) = \cosh^{\frac{2}{d}}(du) \int_{\mp\infty}^u \frac{\mathcal{F}(0)(w, \theta - \delta^{-1}(\eta^\mp(w) - \eta^\mp(u)))}{\cosh^{\frac{2}{d}}(dw)} dw,$$

with \mathcal{F} in (28) and we take $-$ in the unstable case and $+$ in the stable one.

Then, there exists $M > 0$ such that for all $(u, \theta) \in D_{\kappa,\beta,T}^{u,s} \times \mathbb{T}_\omega$:

$$\begin{aligned} |r_{10}^{u,s}(u, \theta)| &\leq M \delta^{p+3} |\cosh(du)|^{-3} \\ |r_{11}^{u,s}(u, \theta)| &\leq M \left(\delta^{2p+6} |\cosh(du)|^{-4} + \delta^{p+4} |\cosh(du)|^{-1} \right), \end{aligned}$$

and:

$$|\partial_u r_1^{u,s}(u, \theta)| \leq M \delta^{p+3} |\cosh(du)|^{-4}, \quad |\partial_\theta r_1^{u,s}(u, \theta)| \leq M \delta^{p+4} |\cosh(du)|^{-4}.$$

In addition, the function $r_{10}^{u,s}$ is defined in the full domain $D_{\kappa,\beta}^{u,s} \times \mathbb{T}_\omega$.

The proof of this Theorem can be found in Sect. 4.2. We stress that this result is also valid in the singular case $p = -2$.

2.3 The Melnikov Function

Our final aim is to find an asymptotic formula of the difference $\Delta = r^u - r^s = r_1^u - r_1^s$. Recall that by Theorem 2.4 we have:

$$\Delta(u, \theta) = r_{10}^u(u, \theta) - r_{10}^s(u, \theta) + r_{11}^u(u, \theta) - r_{11}^s(u, \theta).$$

Theorem 2.4 suggests that r_{10}^u and r_{10}^s are larger than r_{11}^u and r_{11}^s . Hence, it is natural to expect that the first order of the difference is given by the difference of these dominant terms. That is, we expect that:

$$\Delta(u, \theta) = r_{10}^u(u, \theta) - r_{10}^s(u, \theta) + \text{h.o.t.}$$

We will see that this approach is valid for $p > -2$, that is, for non-generic unfoldings. We postpone the study of the case $p = -2$ to Baldomá et al. (2016), where we will see that this assumption is not true.

Let us consider the difference $r_{10}^u - r_{10}^s$, which is 2π -periodic in θ , so that we can write its Fourier series:

$$M(u, \theta) := r_{10}^u(u, \theta) - r_{10}^s(u, \theta) = \sum_{l \in \mathbb{Z}} M^{[l]}(u) e^{il\theta}. \tag{32}$$

We introduce $\eta(w) = \alpha w + \delta c d^{-1} \log(\cosh dw)$. We observe that, for real values of $u, w, \eta(w) - \eta(u) = \eta^\mp(w) - \eta^\mp(u)$, with η^\mp introduced in Theorem 2.4. Therefore, using the formula for $r_{10}^{u,s}$ in the mentioned result, we have that

$$M(u, \theta) = \cosh^{\frac{2}{d}}(du) \int_{-\infty}^{+\infty} \frac{\mathcal{F}(0)(w, \theta - \delta^{-1}(\eta(w) - \eta(u)))}{\cosh^{\frac{2}{d}}(dw)} dw, \tag{33}$$

which is the Melnikov function adapted to this problem. As $\mathcal{F}(0)(u, \theta)$ is periodic in θ , the coefficients $M^{[l]}(u)$ for $u \in \mathbb{R}$ are:

$$M^{[l]}(u) = \cosh^{\frac{2}{d}}(du) e^{il\delta^{-1}\eta(u)} \int_{-\infty}^{+\infty} \frac{e^{-il\delta^{-1}\eta(w)} \mathcal{F}^{[l]}(0)(w)}{\cosh^{\frac{2}{d}}(dw)} dw. \tag{34}$$

Moreover, from (34) it is clear that we can write series (32) as:

$$M(u, \theta) = \cosh^{\frac{2}{d}}(du) \sum_{l \in \mathbb{Z}} \Upsilon_0^{[l]} e^{il(\theta + \delta^{-1}\alpha u + cd^{-1} \log \cosh(du))}, \tag{35}$$

where $\Upsilon_0^{[l]}$ are the constants:

$$\Upsilon_0^{[l]} = \int_{-\infty}^{+\infty} \frac{e^{-il(\delta^{-1}\alpha w + cd^{-1} \log \cosh(dw))} \mathcal{F}^{[l]}(0)(w)}{\cosh^{\frac{2}{d}}(dw)} dw. \tag{36}$$

In addition $M^{[l]}(u) = \cosh^{\frac{2}{d}}(du) e^{il(\delta^{-1}\alpha u + cd^{-1} \log \cosh(du))} \Upsilon_0^{[l]}$.

In the following theorem we provide upper bounds for $\Upsilon_0^{[l]}$ for $|l| \geq 2$ and closed formulas for $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$ in terms of the Borel transform of some functions depending on the perturbation terms. We also prove that (besides the average $\Upsilon_0^{[0]}$) they are the dominant coefficients of M . To this purpose, we recall that given a function $m(w, \theta)$ such that for some $k \in \mathbb{R}$ can be written of the form $m(w, \theta) = \sum_{n \geq 0} m_n(\theta) w^{n+1+ik}$, with m_n , periodic in θ , we define its Borel transform $\hat{m}(\zeta, \theta)$ as:

$$\hat{m}(\zeta, \theta) = \sum_{n \geq 0} m_n(\theta) \frac{\zeta^{n+ik}}{\Gamma(n+1+ik)}. \tag{37}$$

To avoid a cumbersome notation, we introduce

$$\mathbf{w}(w, \theta) = \left(\sqrt{\frac{d+1}{b}} w \cos \theta, \sqrt{\frac{d+1}{b}} w \sin \theta, -iw, 0, 0 \right)$$

and $\tilde{F}(w, \theta) = \cos \theta f(\mathbf{w}(w, \theta)) + \sin \theta g(\mathbf{w}(w, \theta))$ with f and g the perturbation terms in system (15).

Theorem 2.5 Consider the 2π -periodic in θ function

$$m(w, \theta) = \sqrt{\frac{d+1}{b}} w^{1+\frac{2}{d}+i\frac{c}{d}} \left(\tilde{F}(w, \theta) - i \sqrt{\frac{d+1}{b}} h(\mathbf{w}(w, \theta)) \right). \tag{38}$$

Let $\hat{m}(\zeta, \theta)$ be its Borel transform as defined in (37) and $\hat{m}^{[1]}$ its first Fourier coefficient. Then, writing $\mathcal{C} = \mathcal{C}_1 - i\mathcal{C}_2 = \frac{4\pi}{d} \hat{m}^{[1]} \left(\frac{\alpha}{d} \right)$, $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}$,

$$\Upsilon_0^{[1]} = \overline{\Upsilon_0^{[-1]}} = \delta^{p-\frac{2}{d}-i\frac{c}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \left(\frac{\mathcal{C}}{2} + \mathcal{O}(\delta) \right).$$

Moreover, there exists a constant K such that:

$$\left| \Upsilon_0^{[l]} \right| \leq K \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta} \frac{3|l|}{4}}, \quad |l| \geq 2. \tag{39}$$

In conclusion, defining $\vartheta(u, \delta) = \delta^{-1}\alpha u + c d^{-1} [\log \cosh(du) - \log \delta]$ for $u \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$ one has that:

$$M(u, \theta) = \cosh^{\frac{2}{d}}(du) \left[\Upsilon_0^{[0]} + \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \left(\mathcal{C}_1 \cos(\theta + \vartheta(u, \delta)) + \mathcal{C}_2 \sin(\theta + \vartheta(u, \delta)) + \mathcal{O}(\delta) \right) \right]. \tag{40}$$

The proof of this result can be found in Sect. 3.1.

Due to the exponential smallness of $\Upsilon_0^{[l]}$, $|l| \geq 1$, the dominant term of the Melnikov function for real values of u is its average $\Upsilon_0^{[0]}$. We will give more details about this coefficient in Sect. 2.6, Theorem 2.9.

Remark 2.6 An immediate corollary of Theorem 2.5 is Proposition 1.4. Indeed, on the one hand, by its definition, the constant \mathcal{C} depends only on the singularity HZ^* . On the other hand, $\mathcal{C} \neq 0$ if and only if $\hat{m}^{[1]}(\frac{\alpha}{d}) \neq 0$, which is a generic condition.

2.4 The Difference

In this section we study the difference $\Delta(u, \theta) = r_1^u(u, \theta) - r_1^s(u, \theta)$. We give only the main result and some intuitive ideas of the proof. For all the details we refer the reader to Sect. 5.

First, we find an equation for the difference Δ . To this aim, we subtract the PDEs (26) for r_1^u and r_1^s , and then using the mean value theorem, we obtain an equation of the following form:

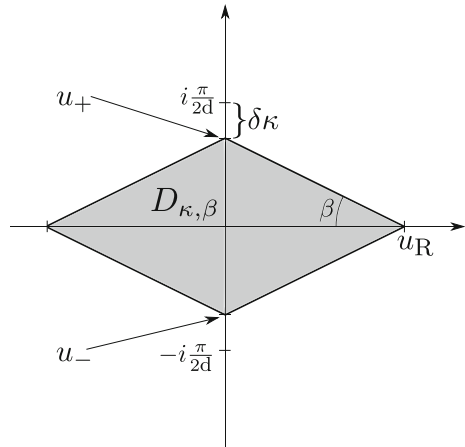
$$\begin{aligned} & (-\delta^{-1}\alpha - cZ_0(u))\partial_\theta \Delta + \partial_u \Delta - 2Z_0(u)\Delta \\ & = (2\sigma + l_1(u, \theta))\Delta + l_2(u, \theta)\partial_u \Delta + l_3(u, \theta)\partial_\theta \Delta. \end{aligned} \tag{41}$$

Here the functions l_1, l_2, l_3 are ‘‘small’’ in the appropriate sense. More precisely, denoting $r_\lambda = (r_1^u + r_1^s)/2 + \lambda(r_1^u - r_1^s)/2$, the functions l_i are:

$$\begin{aligned} l_1(u, \theta) &= \frac{\delta^p}{2} \int_{-1}^1 \partial_r F(r_\lambda) d\lambda + \frac{\delta^p(d+1)}{2b} Z_0(u) \int_{-1}^1 \partial_r H(r_\lambda) d\lambda \\ &\quad - \frac{\delta^p}{2} \int_{-1}^1 \partial_r G(r_\lambda) \partial_\theta r_\lambda d\lambda - \frac{\delta^p}{2d(1-Z_0^2(u))} \int_{-1}^1 \partial_r H(r_\lambda) \partial_u r_\lambda d\lambda \\ &\quad - \frac{b}{d(1-Z_0^2(u))} (\partial_u r_1^u + \partial_u r_1^s), \end{aligned} \tag{42}$$

$$l_2(u, \theta) = -\frac{b}{d(1-Z_0^2(u))} (r_1^u + r_1^s) - \frac{\delta^p}{2d(1-Z_0^2(u))} \int_{-1}^1 H(r_\lambda) d\lambda, \tag{43}$$

Fig. 6 The domain $D_{\kappa,\beta}$



$$l_3(u, \theta) = -\frac{\delta^p}{2} \int_{-1}^1 G(r_\lambda) d\lambda. \tag{44}$$

The precise meaning of “small” will be given in Lemma 5.2.

Recall that r_1^u and r_1^s are defined, respectively, in the domains $D_{\kappa,\beta,T}^u \times \mathbb{T}_\omega$ and $D_{\kappa,\beta,T}^s \times \mathbb{T}_\omega$. Thus, their difference will be defined in the intersection of these two domains. So, from now on we will consider $(u, \theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$, where we define $D_{\kappa,\beta}$ as (see Fig. 6):

$$D_{\kappa,\beta} = D_{\kappa,\beta,T}^u \cap D_{\kappa,\beta,T}^s. \tag{45}$$

Now, we study *all* the solutions of Eq. (41). First we notice that, by the so-called method of variation of constants, every solution Δ of (41) can be written as:

$$\Delta(u, \theta) = P(u, \theta)k(u, \theta), \tag{46}$$

where P is a particular solution of this same equation satisfying $P(u, \theta) \neq 0$, and $k(u, \theta)$ satisfies the associated homogeneous PDE:

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right) \partial_\theta k + \partial_u k = l_2(u, \theta)\partial_u k + l_3(u, \theta)\partial_\theta k. \tag{47}$$

Let us now mention some properties of these functions k and P .

To study the function k we shall rely on the form of Eq. (47). One of its main features is that if ξ is a particular solution of (47) such that $(\xi(u, \theta), \theta)$ is injective in $D_{\kappa,\beta} \times \mathbb{T}_\omega$, then *any* solution k of (47) can be written as:

$$k(u, \theta) = \tilde{k}(\xi(u, \theta)),$$

for some function $\tilde{k}(\tau)$. As a consequence of (46) and of the above equality

$$\Delta(u, \theta) = P(u, \theta)\tilde{k}(\xi(u, \theta)) \tag{48}$$

with P a particular solution of (41) and ξ a particular solution of (47).

Since the functions l_i are “small,” Eq. (47) is a perturbation of:

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right) \partial_\theta k + \partial_u k = 0.$$

A solution of this equation is given by $\xi_0(u, \theta) = \theta + \delta^{-1}\alpha u + cd^{-1} \log \cosh(du)$. Then, we look for a solution of (47) of the form:

$$\xi(u, \theta) = \theta + \delta^{-1}\alpha u + cd^{-1} \log \cosh(du) + C(u, \theta), \tag{49}$$

where, as expected, C will be a “small” function.

Notice that, if the existence of ξ of the form (49) can be proven then the function $\tilde{k}(\tau)$ has to be 2π -periodic in its argument. Indeed, since k is 2π -periodic in θ one has that $\tilde{k}(\xi(u, \theta + 2\pi)) = \tilde{k}(\xi(u, \theta))$. The claim follows from the fact that $\xi(u, \theta + 2\pi) = \xi(u, \theta) + 2\pi$.

To study the particular solution P of (41) we note that, being $\sigma = \mathcal{O}(\delta^{p+3})$ and l_i “small,” Eq. (41) is a perturbation of:

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right) \partial_\theta \Delta + \partial_u \Delta - 2Z_0(u)\Delta = 0.$$

A solution of this equation is given by $P_0(u) = \cosh^{2/d}(du)$. Therefore, we look for a particular solution of (41) of the form:

$$P(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)),$$

where $P_1(u, \theta)$ will be “small.”

As a conclusion of all the previous considerations, one obtains the following result, which characterizes the form of the difference Δ as well as the sizes of the functions P_1 and C described above.

Theorem 2.7 *Let $p \geq -2$ and $|\sigma| \leq \delta^{p+3}\sigma^*$. The difference Δ can be written as:*

$$\Delta(u, \theta) = P(u, \theta)\tilde{k}(\xi(u, \theta)) = \cosh^{2/d}(du)(1 + P_1(u, \theta))\tilde{k}(\xi(u, \theta)), \tag{50}$$

where $\tilde{k}(\tau)$ is a 2π -periodic function, the function ξ is a solution of (47) defined as:

$$\xi(u, \theta) = \theta + \delta^{-1}\alpha u + cd^{-1} \log \cosh(du) + C(u, \theta), \tag{51}$$

and it is such that $(\xi(u, \theta), \theta)$ is injective in $D_{\kappa, \beta} \times \mathbb{T}_\omega$. In addition, P is a solution of (41) and P_1 and C are real analytic functions, defined in $D_{\kappa, \beta} \times \mathbb{T}_\omega$ such that:

1. There exist $L_0 \in \mathbb{R}$ and functions $L(u)$ and $\chi(u, \theta)$ such that

$$C(u, \theta) = \delta^{p+2}d^{-1}\alpha L_0 \log \cosh(du) + \alpha L(u) + \chi(u, \theta), \tag{52}$$

where, for all $(u, \theta) \in D_{\kappa, \beta} \times \mathbb{T}_\omega$:

$$|L(u)| \leq M\delta^{p+2}, \quad |L'(u)| \leq M\delta^{p+2}, \quad |\chi(u, \theta)| \leq \frac{M\delta^{p+3}}{|\cosh(du)|}, \quad (53)$$

for some constant M . L_0 and $L(u)$ are determined by a finite number of Taylor coefficients of the functions f , g and h appearing in (15). Formulas for L_0 and $L(u)$ are given in Remark 5.7. Moreover, $L(0) = 0$, $L(u)$ can be analytically extended to $D_{0, \beta}$ and it is well defined on the limit $u \rightarrow \pm i\pi/(2d)$, $u \in D_{0, \beta}$.

2. There exists a constant M such that for all $(u, \theta) \in D_{\kappa, \beta} \times \mathbb{T}_\omega$:

$$|P_1(u, \theta)| \leq \frac{M\delta^{p+3}}{|\cosh(du)|}. \quad (54)$$

Moreover, in the conservative case P_1 can be chosen as:

$$P_1(u, \theta) = \frac{\partial_u C(u, \theta) - l_3(u, \theta)}{\delta^{-1}\alpha + cZ_0(u) + l_3(u, \theta)},$$

where $l_3(u, \theta)$ is given by (44).

The proof of this result can be found in Sect. 5.

Remark 2.8 Notice that, if $p = -2$, the logarithmic term in the function C (see (52)) has the same size as the corresponding one in definition (51) of ξ . However, when $p > -2$, the function C is indeed a perturbation term of order $\mathcal{O}(\delta^{p+2}|\log \delta|)$ over complex values of u .

In fact, when $p > -2$, we do not need the exact form (52) of C , we only need to know that $|C(u, \theta)| \leq K\delta^{p+2}$ when $u \in \mathbb{R}$ which is easier to check. However, it is mandatory in the generic case $p = -2$.

2.5 Sharp Upper Bound

Even though this is not the final goal of this work, which deals with asymptotic expressions, in Proposition 2.12 of this section, we provide an upper bound for $\Delta(u, \theta)$ when $u, \theta \in \mathbb{R}$. On the one hand, we will gain some intuition about the main problems we will have to overcome, and on the other hand, some of the results proven in this section will be used in the proof of Theorem 1.1.

A straightforward consequence of Theorem 2.7 is that:

$$\Delta(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)) \sum_{l \in \mathbb{Z}} \Upsilon^{[l]} e^{il\xi(u, \theta)}, \quad (55)$$

where P_1 and ξ are given in Theorem 2.7 and $\Upsilon^{[l]}$, the Fourier coefficients of the function $\tilde{k}(\tau)$, are unknown. They depend on δ and σ although we do not write it explicitly.

Now we are going to study separately the average $\Upsilon^{[0]}$ in Theorem 2.9 (in fact this result also deals with the average $\Upsilon_0^{[0]}$ of the Melnikov function) and the rest of the Fourier coefficients $\Upsilon^{[l]}$, $l \neq 0$ in Lemma 2.11. The sharp upper bound for $\Delta(u, \theta)$ is a straightforward consequence of these results.

Theorem 2.9 *Let $p \geq -2$. Let $\Upsilon^{[0]}$ be the average of the function $\tilde{k}(\tau)$, given in Theorem 2.7, and $\Upsilon_0^{[0]}$ be the constant defined in (36).*

1. *In the conservative case, for all $0 < \delta < \delta_0$ one has:*

$$\Upsilon^{[0]} = 0, \quad \Upsilon_0^{[0]} = 0.$$

2. *In the general case, for all $0 < \delta < \delta_0$ and $|\sigma| \leq \delta^{p+3}\sigma^*$*

$$\Upsilon^{[0]} = \Upsilon_0^{[0]} + \mathcal{O}(\delta^{p+4}), \quad \Upsilon_0^{[0]} = \sigma I + \delta^{p+3} \bar{J}(\delta, \sigma), \tag{56}$$

being $I \neq 0$ independent of δ and σ and $\bar{J} = J + \mathcal{O}(\delta)$ given in (90) and (91), respectively.

In addition, there exists a curve

$$\sigma = \sigma_*^0(\delta) = -\frac{J}{I} \delta^{p+3} + \mathcal{O}(\delta^{p+4})$$

such that for all $0 \leq \delta \leq \delta_0$ one has:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta, \sigma_*^0(\delta)) = 0.$$

Moreover, given constants $a_1, a_2 \in \mathbb{R}$ and $a_3 > 0$, there exists a curve

$$\sigma = \sigma_*(\delta) = \sigma_*^0(\delta) + \mathcal{O}(\delta^{a_2} e^{-\frac{a_3\pi}{2\delta}}) \tag{57}$$

such that for all $0 \leq \delta \leq \delta_0$ one has:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta, \sigma_*(\delta)) = a_1 \delta^{a_2} e^{-\frac{a_3\pi}{2\delta}}.$$

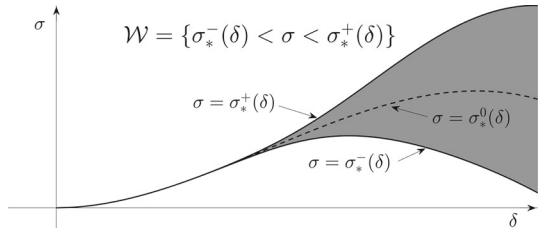
Along these curves one has:

$$\Upsilon_0^{[0]} = \Upsilon_0^{[0]}(\delta, \sigma_*(\delta)) = \mathcal{O}(\delta^{p+4}).$$

For the proof of this theorem we refer the reader to Sect. 3.2. We stress that the item 1 is standard in the usual scenarios of Hamiltonian or reversible vector fields and symplectic maps. However, in our setting, the proof involves delicate arguments.

Remark 2.10 In the general case, let us now fix some constants $a_1^+ > 0$ and $a_1^- < 0$ as in the previous theorem. Fix also $a_2^+, a_2^- \in \mathbb{R}$ and $a_3^+, a_3^- > 0$. Define $\sigma_*^+(\delta)$ as the curve of Theorem 2.9 corresponding to the constants a_1^+, a_2^+ and a_3^+ , and $\sigma_*^-(\delta)$

Fig. 7 The curve $\sigma = \sigma_*^0(\delta)$ and a wedge-shaped domain \mathcal{W} around it. Inside this domain, the coefficient $\Upsilon^{[0]}$ is exponentially small



as the curve in of Theorem 2.9 corresponding to the constants a_1^-, a_2^- and a_3^- . By (57) one has that $\sigma_*^-(\delta) \leq \sigma_*^0(\delta) \leq \sigma_*^+(\delta)$ for δ sufficiently small. Define the domain:

$$\mathcal{W} := \{(\delta, \sigma) \in \mathbb{R}^2 : \sigma_*^-(\delta) < \sigma < \sigma_*^+(\delta)\}$$

in the parameter plane. This domain is a wedge-shaped domain around $\sigma_*^0(\delta)$ (see Fig. 7). Moreover, there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and $(\delta, \sigma) \in \mathcal{W}$, by (56), the coefficient $\Upsilon^{[0]}(\delta, \sigma)$ is exponentially small. More precisely, let us denote $\bar{a}_3 = \min\{a_3^+, a_3^-\}$. Define $\bar{a}_1 = a_1^+$ and $\bar{a}_2 = a_2^+$ if the minimum is achieved in a_3^+ , otherwise we take $\bar{a}_1 = a_1^-$ and $\bar{a}_2 = a_2^-$. Then:

$$|\Upsilon^{[0]}(\delta, \sigma)| \leq |\bar{a}_1| \delta^{\bar{a}_2} e^{-\frac{\bar{a}_3 \pi}{2d\delta}}, \quad \text{if } 0 < \delta < \delta_0, \quad (\delta, \sigma) \in \mathcal{W}.$$

Now we are going to deal with the Fourier coefficients $\Upsilon^{[l]}$, with $l \neq 0$. From expression (55) of Δ , one can see that $\Upsilon^{[l]}$ are exponentially small with respect to δ . Indeed, as a first exploration, we can consider the case $P_1 = \chi \equiv 0$. This case can give some insight since, as one can see from the bounds of P_1 and χ given in (54) and (53), respectively, they are “small” functions when we take large κ . If we make this simplification, using expression (55) of Δ :

$$\Delta(u, \theta) = \cosh^{2/d}(du) \sum_{l \in \mathbb{Z}} \Upsilon^{[l]} e^{il(\theta + \tilde{\xi}(u))}$$

with $\tilde{\xi}(u) = \delta^{-1} \alpha u + d^{-1}(c + \delta^{p+2} \alpha L_0) \log \cosh(du) + \alpha L(u)$ (see formula (51) of ξ and take $\chi \equiv 0$). Then we have that $\Upsilon^{[l]} e^{il\tilde{\xi}(u)}$ are the Fourier coefficients of the function $\Delta(u, \theta) \cosh^{-2/d}(du)$. In other words:

$$\begin{aligned} |\Upsilon^{[l]}| &= \left| \frac{e^{-il\tilde{\xi}(u)}}{2\pi} \int_0^{2\pi} \frac{\Delta(u, \theta) e^{-il\theta}}{\cosh^{2/d}(du)} d\theta \right| \\ &\leq \left| e^{-il\tilde{\xi}(u)} \right| \sup_{\theta \in [0, 2\pi]} \left| \frac{\Delta(u, \theta)}{\cosh^{2/d}(du)} \right|. \end{aligned}$$

We note that this inequality is valid for all $u \in D_{\kappa, \beta} = D_{\kappa, \beta, T}^u \cap D_{\kappa, \beta, T}^s$. In particular, taking $u = u_+ := i(\pi/(2d) - \kappa\delta)$ for $l < 0$ and $u = u_- := -u_+$ for $l > 0$ and

using that the constant $L_0 \in \mathbb{R}$ and that $\text{Im} \log \cosh(du_{\pm}) = \arg(\cosh(du_{\pm})) = 0$, one obtains:

$$\left| \Upsilon^{[l]} \right| \leq K(\kappa\delta)^{-2/d} e^{-\left(\frac{\alpha\pi}{2\delta} - \alpha\kappa - \alpha|\text{Im} L(u_{\pm})|\right)|l|} \sup_{\theta \in [0, 2\pi]} |\Delta(u_{\pm}, \theta)|.$$

Recalling that $\Delta = r_1^u - r_1^s$ and using that $|r_1^{u,s}(u_{\pm}, \theta)| \leq M\delta^p \kappa^{-3}$ by Theorem 2.4, we obtain readily (renaming K):

$$\left| \Upsilon^{[l]} \right| \leq K \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\left(\frac{\alpha\pi}{2\delta} - \alpha\kappa - \alpha|\text{Im} L(u_{\pm})|\right)|l|}.$$

In particular, there exists a constant K , independent of l such that:

$$\left| \Upsilon^{[\pm 1]} \right| \leq K \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\frac{\alpha\pi}{2\delta} + \alpha\kappa},$$

and for $|l| \geq 2$:

$$\left| \Upsilon^{[l]} \right| \leq K \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\frac{\alpha\pi}{2\delta} \frac{3|l|}{4}},$$

where we have used that $\delta|\text{Im} L(u_{\pm})|$ is arbitrarily small by bound (53) and condition (29) on κ .

The following result, whose proof is postponed to Sect. 3, states that the same exponentially small bounds hold when $P_1(u, \theta) \neq 0$ and $\chi(u, \theta) \neq 0$.

Lemma 2.11 *Let $\Upsilon^{[l]}$, $l \in \mathbb{Z}$, $l \neq 0$, be the coefficients appearing in expression (55) of Δ . Take κ as in Theorem 2.4. There exists a constant M , independent of κ such that:*

$$\left| \Upsilon^{[\pm 1]} \right| \leq M \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\frac{\alpha\pi}{2\delta} + \alpha\kappa}, \quad \left| \Upsilon^{[l]} \right| \leq M \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\frac{\alpha\pi}{2\delta} \frac{3|l|}{4}}, \quad |l| \geq 2$$

As a consequence of this result we obtain the sharp upper bound:

Proposition 2.12 *Let $p \geq -2$, κ be as in Theorem 2.4 and $|\delta| \leq \delta_0$. In the general case we take $|\sigma| \leq \delta^{p+3}\sigma^*$. Let $\Upsilon^{[0]} = \Upsilon^{[0]}(\sigma, \delta)$ be the constant provided by Theorem 2.9. In the conservative case $\Upsilon^{[0]} = 0$. Then, for real values of u and θ :*

$$|\Delta(u, \theta)| \leq \cosh^{2/d}(du) \left(|\Upsilon^{[0]}| + M \frac{\delta^{p-2/d}}{\kappa^{3+2/d}} e^{-\frac{\alpha\pi}{2\delta} + \alpha\kappa} \right).$$

for some constant M .

Moreover, in the conservative case, or in the general case if we take the parameters (δ, σ) in a wedge-shaped domain \mathcal{W} around $\sigma_*^0(\delta)$ as in Remark 2.10, the distance

between the manifolds is exponentially small. In particular, if we choose the parameters $a_3^\pm > \alpha$, $a_2^\pm > p - 2/d$, for $(\delta, \sigma) \in \mathcal{W}$ one has:

$$|\Delta(u, \theta)| \leq \tilde{M} \cosh^{2/d}(du) \delta^{p-2/d} e^{-\frac{\alpha\pi}{2d\delta}},$$

for some constant \tilde{M} . The bound is also valid in the conservative case taking $d = 1$.

2.6 First Order of the Difference: End of the Proof of Theorem 1.1

In this section we provide Theorem 2.14 and Corollary 2.17. The former gives a first order of the difference $\Delta(u, \theta)$ also studied in Theorem 2.7 and the latter a measure of the domain where heteroclinic transversal intersections occur. Moreover, we will give the proof of Theorem 1.1 and Corollary 1.3 as corollaries of Theorem 2.14 and Corollary 2.17, respectively.

Recall that $\Upsilon^{[0]}$ is the average of the function $\tilde{k}(\tau)$ of Theorem 2.7. Since we want to obtain (non-sharp) results also in the case $p = -2$, we define:

$$\tilde{k}_0(\tau) := \Upsilon^{[0]} + \sum_{l \neq 0} \hat{\Upsilon}_0^{[l]} e^{il\tau}, \quad \hat{\Upsilon}_0^{[l]} = \Upsilon_0^{[l]} e^{-il\alpha d^{-1} L_0 \delta^{p+2} \log \delta}, \quad (58)$$

where $L_0 \in \mathbb{R}$ is given in Theorem 2.7 and $\Upsilon_0^{[l]}$ are the constants appearing in the Fourier coefficients of the Melnikov function, defined in (36). Our candidate to be the first order of the difference is:

$$\Delta_0(u, \theta) = \cosh^{2/d}(du) (1 + P_1(u, \theta)) \tilde{k}_0(\xi(u, \theta)), \quad (59)$$

with ξ defined in (51). We note that we have not chosen the average of \tilde{k}_0 to be the coefficient $\Upsilon_0^{[0]}$ appearing in the average of the Melnikov function (as one might expect) but $\Upsilon^{[0]}$, the average of $\tilde{k}(\tau)$ in Theorem 2.7.

The next result shows that $\Upsilon^{[\pm 1]}$, the Fourier coefficients of \tilde{k} , are well approximated by $\hat{\Upsilon}^{[\pm 1]}$, the Fourier coefficients of \tilde{k}_0 defined in (58). The proof of this Proposition is done in Sect. 3.4.

Proposition 2.13 *Let $p \geq -2$ and κ a sufficiently large constant. Let $\Upsilon^{[\pm 1]}$ be the Fourier coefficients of order ± 1 of $\tilde{k}(\tau)$, in Theorem 2.7, and $\hat{\Upsilon}_0^{[\pm 1]}$ the ones given in (58). Then there exists a constant M such that:*

$$\left| \Upsilon^{[\pm 1]} - \hat{\Upsilon}_0^{[\pm 1]} \right| \leq M \left(\frac{|\log \kappa|}{\kappa^{4+\frac{2}{d}}} \delta^{2(p+1-\frac{1}{d})} + \frac{\delta^{p+3-\frac{2}{d}}}{\kappa^{1+\frac{2}{d}}} \right) e^{-\frac{\alpha}{8}(\frac{\pi}{2d}-\kappa\delta)},$$

where we assume that, in the general case, $|\sigma| \leq \delta^{p+3}\sigma_*$. Recall that $d = 1$ in the conservative case

Now we can state the theorem which gives the asymptotic for the difference of the invariant manifolds $\Delta = r_1^u - r_1^s$.

Theorem 2.14 Let $p \geq -2$, and $0 < \delta < \delta_0$. Consider the functions $m(w, \theta)$, $\vartheta(u, \delta)$ and the constants C_1, C_2 defined in Theorem 2.5. In the general case take $|\sigma| \leq \delta^{p+3}\sigma^*$. Let $\Upsilon^{[0]} = \Upsilon^{[0]}(\sigma, \delta)$ be the constant provided by Theorem 2.9. In the conservative case recall that $\Upsilon^{[0]} = 0$.

There exists $T_0 > 0$ such that for all $u \in [-T_0, T_0]$ and $\theta \in \mathbb{S}^1$

$$\begin{aligned} \Delta(u, \theta) = & \cosh^{\frac{2}{d}}(du)\Upsilon^{[0]}\left(1 + \mathcal{O}\left(\delta^{p+3}\right)\right) \\ & + \delta^{p-\frac{2}{d}} \cosh^{\frac{2}{d}}(du)e^{-\frac{\alpha\pi}{2d\delta}} \left[C_1 \cos\left(\theta + \vartheta(u, \delta) - \alpha d^{-1}L_0\delta^{p+2} \log \delta\right) \right. \\ & \left. + C_2 \sin\left(\theta + \vartheta(u, \delta) - \alpha d^{-1}L_0\delta^{p+2} \log \delta\right) + \mathcal{O}\left(\delta^{p+2} + \delta^3\right) \right], \end{aligned}$$

where we recall that $d = 1$ in the conservative case.

Remark 2.15 Even though this result is valid for $p \geq -2$, it only provides an asymptotic formula for Δ in the case $p > -2$. However, when $p = -2$, it gives an upper bound which coincides with the one given in Proposition 2.12.

Remark 2.16 Observe that if we take the parameters (σ, δ) belonging to one of the curves given in Theorem 2.9 with $a_3 > 0$ we obtain that the distance $\Delta(u, \theta) = \mathcal{O}(e^{-\frac{a_3\pi}{2d\delta}}, e^{-\frac{\alpha\pi}{2d\delta}})$ and, therefore, it is exponentially small.

Proof Recalling definition (59) of Δ_0 and the form (50) of Δ given in Theorem 2.7, we can write: $\Delta(u, \theta) = \Delta_0(u, \theta) + \Delta_1(u, \theta)$, where

$$\Delta_1(u, \theta) = \cosh^{\frac{2}{d}}(du)(1 + P_1(u, \theta)) \sum_{l \neq 0} \left(\Upsilon^{[l]} - \hat{\Upsilon}_0^{[l]}\right) e^{il\xi(u, \theta)}. \tag{60}$$

First of all we note that since we are taking $u \in [-T_0, T_0]$ and $\theta \in \mathbb{S}^1$ we have that all the functions are real and bounded. Then, using Proposition 2.13 to bound $(\Upsilon^{[\pm 1]} - \hat{\Upsilon}_0^{[\pm 1]})$, Lemma 2.11 to bound $|\Upsilon^{[l]}|$ and Theorem 2.5 to bound $|\hat{\Upsilon}_0^{[l]}|$ for $|l| \geq 2$, we obtain

$$|\Delta_1(u, \theta)| \leq K \left(\delta^{2(p+1-\frac{1}{d})} + \delta^{p+3-\frac{2}{d}} \right) e^{-\frac{\alpha\pi}{2d\delta}}.$$

Again, we omit the explicit dependence on κ . Since Δ_1 has the size of the remainder in the asymptotic expansion for Δ , we only need to deal with Δ_0 .

Recall that by (58), $\hat{\Upsilon}_0^{[l]} = \Upsilon_0^{[l]} e^{-i\alpha d^{-1}L_0\delta^{p+2} \log \delta}$ so that both have the same modulus. Then, by the bounds obtained in Theorem 2.5 for the coefficients $\Upsilon_0^{[l]}, l \neq 0$ and using the expression (59) of Δ_0 one has that

$$\begin{aligned} \Delta_0(u, \theta) = & \cosh^{\frac{2}{d}}(du)(1 + P_1(u, \theta)) \left[\Upsilon^{[0]} + 2\text{Re} \left(\hat{\Upsilon}_0^{[1]} e^{i\xi(u, \theta)} \right) \right. \\ & \left. + \mathcal{O}\left(\delta^{p-\frac{2}{d}} e^{-\frac{3\alpha\pi}{4d\delta}}\right) \right]. \end{aligned}$$

Again, using formula for $\Upsilon_0^{[1]}$ in Theorem 2.5 as well as Theorem 2.7 for ξ one has that

$$\hat{\Upsilon}_0^{[1]} e^{i\xi(u,\theta)} = \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \left(\frac{C}{2} e^{i(\theta+\alpha\delta^{-1}u+\frac{\alpha}{d} \log(\cosh du) - \frac{1}{d}[c+\alpha L_0\delta^{p+2}] \log \delta)} + \mathcal{O}(\delta^{p+2}) \right)$$

and the result follows taking $C_1 = \text{Re}(C)$, and $C_2 = -\text{Im}(C)$, since by Theorem 2.7, $|P_1(u, \theta)| \leq K\delta^{p+3}$ for $u \in \mathbb{R}$. \square

Corollary 2.17 Take $T_0 > 0$, and $0 < \delta < \delta_0$. Consider the curve $\sigma_*^0(\delta)$ given in Theorem 2.9. Take the constants $a_1^\pm = \pm\sqrt{C_1^2 + C_2^2} > 0$, $a_2^\pm = p - 2/d$ and $a_3^\pm = \alpha$. Define $\sigma_*^\pm(\delta)$ as the curves in of Theorem 2.9 corresponding to these constants. Consider the wedge-shaped domain around $\sigma_*^0(\delta)$ (see Fig. 7):

$$\mathcal{W}^* := \{(\delta, \sigma) \in \mathbb{R}^2 \mid 0 < \delta < \delta_0 : \sigma_*^-(\delta) < \sigma < \sigma_*^+(\delta)\}$$

in the parameter plane. Then, for $(\delta, \sigma) \in \mathcal{W}^*$, and for fixed $u \in [-T_0, T_0]$, the function $\Delta^*(\theta) = \Delta(u, \theta)$ is exponentially small and has two simple zeros which give rise to two transversal heteroclinic orbits between the points $S_\pm(\delta, \sigma)$. Moreover, for $(\delta, \sigma) \notin \overline{\mathcal{W}^*}$, $\Delta^*(\theta)$ has no zeros and therefore the two-dimensional stable and unstable manifolds of $S_\pm(\mu, \nu)$ do not intersect.

Proof This corollary is an easy consequence of the implicit function theorem applied to formula $\Delta(u, \theta) = 0$ given in Theorem 2.14 and using that, for $(\delta, \sigma) \in \mathcal{W}^*$, one has that

$$|\Upsilon^{[0]}| \leq \sqrt{C_1^2 + C_2^2} \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}}.$$

\square

Theorem 2.14 and Corollary 2.17 easily yield Theorem 1.1 and Corollary 1.3:

End of the proof of Theorem 1.1 and Corollary 1.3 We point out that $\Delta(u, \theta)$ is not the actual distance between the invariant manifolds, since we computed the difference in “symplectic” cylindric coordinates. The actual distance is given by:

$$\begin{aligned} D(u, \theta) &= \sqrt{2(R_0(u) + r_1^u(u, \theta))} - \sqrt{2(R_0(u) + r_1^s(u, \theta))} \\ &= \frac{1}{\sqrt{2R_0(u)}} \Delta(u, \theta) + \mathcal{O}_2(\Delta(u, \theta)). \end{aligned}$$

Using definition (19) of $R_0(u)$ one obtains:

$$D(u, \theta) = \sqrt{\frac{b}{d+1}} \cosh(du) \Delta(u, \theta) + \mathcal{O}_2(\Delta(u, \theta)).$$

And using the results of Theorem 2.14 one obtains:

$$\begin{aligned}
 D(u, \theta) = & \sqrt{\frac{b}{d+1}} \cosh^{\frac{2}{d}+1}(du) \left\{ \Upsilon^{[0]} \left(1 + \mathcal{O}(\sigma, \delta^{p+3}) \right) \right. \\
 & + \delta^{p-\frac{2}{d}} e^{-\frac{\sigma\pi}{2d\delta}} \left[C_1 \cos \left(\theta + \vartheta(u, \delta) - \alpha d^{-1} L_0 \delta^{p+2} \log \delta \right) \right. \\
 & \left. \left. + C_2 \sin \left(\theta + \vartheta(u, \delta) - \alpha d^{-1} L_0 \delta^{p+2} \log \delta \right) + \mathcal{O} \left(\delta^{p+2} + \delta^3 \right) \right] \right\}
 \end{aligned}$$

Where $\Upsilon^{[0]}$ is given in Theorem 2.9.

To obtain the formulas given in Theorem 1.1 and Corollary 1.3 we undo the change of variables in Sect. 2.1, which gives $\bar{D}(u, \theta) = \sqrt{\mu} D(u, \theta)$, we take into account the notation $\delta = \sqrt{\mu}$, $\sigma = \delta^{-1}v = v/\sqrt{\mu}$, $b = \gamma_2$, $d = \beta_1$ and $c = \alpha_3$ (so that the conservative case is proven). The wedge-shaped domain $\mathcal{W}^{u,s}$ is the domain \mathcal{W}^* expressed in terms of (μ, v) . □

The remaining part of this work includes the proofs of the above results. However, these proofs are not exposed in the order provided in this section. We have preferred to postpone the most technical but standard demonstrations to the end of this work and give priority to the ones involving the exponentially small behavior of the difference $\Delta(u, \theta)$ and of the Melnikov function $M(u, \theta)$ when $u, \theta \in \mathbb{R}$, namely, the results in Sects. 2.3 and 2.6. As any expert in exponentially small phenomena knows, the results for real values of u, θ are consequence of the results for complex values. Therefore, we will perform the proofs of the above-mentioned results, assuming that the result about the existence of complex parameterizations (Theorem 2.4) and the general form of the difference $\Delta(u, \theta)$ for complex values of u (Theorem 2.7) hold true. We will do this in Sect. 3. Then, we will proof Theorems 2.4 and 2.7 in Sects. 4 and 5, respectively.

All the constants that appear in the statements of the following results might depend on δ^* , σ^* and κ^* , but never on δ, σ and κ . We assume that δ^* and σ^* are sufficiently small, and κ^* is sufficiently large satisfying condition (29). Finally, to make formulas shorter and avoid keeping track of constants that do not play any role in the proofs, we will use K to denote *any* constant independent of the parameters δ, σ and κ . These conventions are valid for all the sections of this work. We shall not write the proofs that are either for standard results or too technical and that do not provide any interesting insight. For these proofs we refer the reader to Castejón (2015).

3 The Exponentially Small Behavior

We first begin with the results related to the exponentially small behavior for real values of u . That is, Theorems 2.5 and 2.9 and Proposition 2.13.

3.1 The Melnikov Function: Proof of Theorem 2.5

Since for real values of (u, θ) the Melnikov function $M(u, \theta) \in \mathbb{R}$ (see (33) for its definition), one has that $\Upsilon_0^{[-l]} = \overline{\Upsilon_0^{[l]}}$, where the coefficients $\Upsilon_0^{[l]}$ were defined in (36). Hence, we just compute $\Upsilon_0^{[l]}$ with $l > 0$.

For $C \in \mathbb{R}$ and $l, n, Q \in \mathbb{N}$, we define the following integrals:

$$I_{n,Q}^{l,C} = \int_{-\infty}^{+\infty} \frac{e^{-\delta^{-1}\alpha i|l|s} \sinh^n(ds)}{\cosh^{Q+1+iC|l|(ds)}} ds, \quad Q + 1 > n. \tag{61}$$

Let us denote by f_{qkmn} , g_{qkmn} and h_{qkmn} the Taylor coefficients of f , g and h , respectively, namely:

$$f(\delta x, \delta y, \delta z, \delta, \delta\sigma) = \sum_{q=3}^{\infty} \delta^q \sum_{k+m+n \leq q} f_{qkmn}(\sigma) x^k y^m z^n, \tag{62}$$

and analogously for g and h . In the following we shall write f_{qkmn} instead of $f_{qkmn}(\sigma)$, but of course these coefficients still depend on σ . Note that one has $f_{qkmn} = f_{qkmn}(0) + \mathcal{O}(\sigma) = f_{qkmn}(0) + \mathcal{O}(\delta^{p+3})$, since we just consider the case $|\sigma| \leq \sigma^* \delta^{p+3}$.

Denote by $a_{k,m}^{[l]}$ the l -th Fourier coefficient of the function $\cos^k \theta \sin^m \theta$. Recalling definition (28) of \mathcal{F} , notation (24) and definition (22) of \mathbf{F} and \mathbf{H} , it can be seen that, for $l > 0$, $\Upsilon_0^{[l]}$ introduced in (36) writes out as:

$$\Upsilon_0^{[l]} = \Upsilon_{0,f}^{[l]} + \Upsilon_{0,g}^{[l]} + \Upsilon_{0,h}^{[l]} \tag{63}$$

with,

$$\begin{aligned} \Upsilon_{0,f}^{[l]} &= \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n \leq q} \delta^q f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} a_{k+1,m}^{[l]} I_{n,k+m+n+2d-1}^{l,cd^{-1}}, \\ \Upsilon_{0,g}^{[l]} &= \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n \leq q} \delta^q g_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} a_{k,m+1}^{[l]} I_{n,k+m+n+2d-1}^{l,cd^{-1}}, \\ \Upsilon_{0,h}^{[l]} &= \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n \leq q} \delta^q h_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+2} a_{k,m}^{[l]} I_{n+1,k+m+n+2d-1}^{l,cd^{-1}}, \end{aligned} \tag{64}$$

being $I_{n,Q}^{l,C}$ the integrals defined in (61). We are interested in bounding these integrals for $|l| \geq 2$:

Lemma 3.1 *Let C be fixed. For any $M > 0$ there exist $\delta_0, K > 0$ satisfying that for all $0 < \delta < \delta_0, |l| \geq 2, Q \geq 1$ and n such that $Q + 1 > n$, the following hold:*

$$\left| I_{n,Q}^{l,C} \right| \leq K M^Q \delta^{-Q} e^{-\frac{\alpha\pi}{2\delta} \frac{3|l|}{4}}.$$

Proof Take $\rho > 0$. Using Cauchy's theorem, the integration path of the integrals $I_{n,Q}^{l,C}$ can be changed to: $s = s(t) := -\frac{i}{d} \left(\frac{\pi}{2} - \rho\delta \right) + t, t \in (-\infty, \infty)$. Then one obtains:

$$I_{n,Q}^{l,C} = e^{-\frac{\alpha}{d}|l|(\frac{\pi}{2\delta}-\rho)} \int_{-\infty}^{+\infty} \frac{e^{-\delta^{-1}\alpha i|l|t} \sinh^n(ds(t))}{\cosh^{Q+1+iC|l|}(ds(t))} dt. \tag{65}$$

Since for $z \in \mathbb{C}, |z^{Q+1+iC|l|}| \geq |z|^{Q+1}e^{-|C| \arg z}$ and $|\arg \cosh(s(t))| \leq \pi/2$, then

$$|\cosh^{Q+1+iC|l|}(ds(t))| \geq |\cosh^{Q+1}(ds(t))|e^{-|C|l|\frac{\pi}{2}}.$$

Using this bound in expression (65) of $I_{n,Q}^{l,C}$ and standard arguments, one can prove that there exists $K > 0$ (which is also independent of ρ) such that

$$\left| I_{n,Q}^{l,C} \right| \leq K^{Q+1}(\rho\delta)^{-Q} e^{-\frac{\alpha}{d}|l|(\frac{\pi}{2\delta}-\rho-|C|\frac{\pi}{2})}.$$

The proof is finished taking ρ sufficiently large and δ sufficiently small. □

Our goal now will be to find an asymptotic formula for the integrals $I_{n,Q}^{l,C}$ with $l = 1$, which will dominate over the integrals with $|l| \geq 2$. First of all, we give a recurrence that is valid for all $l \neq 0$. The proof follows integrating by parts.

Lemma 3.2 *Let C be fixed. Then, for all $l \neq 0, n \geq 1$ and $Q > 0$ such that $Q+1 > n$, the following recurrence holds:*

$$I_{n,Q}^{l,C} = \frac{-|l|\alpha i}{d\delta(Q+iC|l|)} I_{n-1,Q-1}^{l,C} + \frac{n-1}{Q+iC|l|} I_{n-2,Q-2}^{l,C}.$$

Now we summarize some properties of the Gamma function that will be needed later on.

Lemma 3.3 *Let $z, A \in \mathbb{C}$. Then:*

1. $\Gamma(z)\Gamma(\bar{z}) = |\Gamma(z)|^2$.
2. (Stirling Formula) *If $|\arg z| < \pi$, then:*

$$\Gamma(z) = e^{-z} e^{\left(z-\frac{1}{2}\right)\log z} (2\pi)^{\frac{1}{2}} (1 + \mathcal{O}(z^{-1})).$$

3. *If $z = iy, y \in \mathbb{R}$, then:*

$$|\Gamma(iy)| = \frac{\sqrt{\pi}}{|y \sinh(\pi y)|^{1/2}}.$$

4. *If $|\arg z| < \pi$ and $|A| \leq A^*$ for some constant A^* , then:*

$$\Gamma(z + A) = \Gamma(z)z^A(1 + \mathcal{O}(z^{-1})).$$

5. There exists a constant $M \geq 3/2$ and a function $J(z, A)$ such that for all $z \in \mathbb{C}$ with $|z| \geq 3$, $|\arg z| < \pi$, and all $A \in \mathbb{R}$ with $A \geq 1$, one has:

$$\Gamma(z + A) = \Gamma(z)z^A(1 + z^{-1}J(z, A)),$$

and $|J(z, A)| \leq M\Gamma(A)$.

Proof Every item above, except item 5, are standard facts, see for instance in Abramowitz et al. (1964). To prove item 5, we use previous property for $A^* \geq 3$ and after that we proceed by induction. See the details in Castejón (2015). \square

Finally, we can give an asymptotic formula of $I_{n,Q}^{1,C}$.

Lemma 3.4 *Let C be fixed. Then for all $Q \geq 1$ and $n \geq 0$ such that $Q + 1 > n$ one has:*

$$I_{n,Q}^{1,C} = \frac{2\pi}{d} \left(\frac{\alpha}{d\delta}\right)^{Q+iC} \frac{(-i)^n}{\Gamma(Q+1+iC)} e^{-\frac{\alpha\pi}{2d\delta}} + \mathcal{O}\left(\left(\frac{\alpha}{d\delta}\right)^{Q-1} e^{-\frac{\alpha\pi}{2d\delta}}\right),$$

where the \mathcal{O} means uniformly on n, Q and C .

Proof We first deal with the case $n = 0$ and after that we will proceed by induction. Performing the change of variables $w = \tanh(ds)$, one has that:

$$I_{0,Q}^{1,C} = \frac{1}{d} \int_{-1}^1 (1+w)^{\frac{d(Q-1+iC)-i\delta^{-1}\alpha}{2d}} (1-w)^{\frac{d(Q-1+iC)+i\delta^{-1}\alpha}{2d}} dw.$$

Naming:

$$a = \frac{d(Q+1+iC) + i\delta^{-1}\alpha}{2d}, \quad b = Q+1+iC,$$

we can rewrite the last equation (see for instance Abramowitz et al. 1964) as:

$$I_{0,Q}^{1,C} = \frac{1}{d} \int_{-1}^1 (1+w)^{b-a-1} (1-w)^{a-1} dw = 2^{b-1} d^{-1} \frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)},$$

so that we can write:

$$I_{0,Q}^{1,C} = 2^{Q+iC} d^{-1} \frac{\Gamma_Q^C}{\Gamma(Q+1+iC)}, \quad \Gamma_Q^C := \Gamma(b-a)\Gamma(a). \tag{66}$$

We now shall find an asymptotic expression for Γ_Q^C . Let:

$$A = \frac{Q+1}{2} \geq 1, \quad z_{\pm} = i \frac{dC \pm \delta^{-1}\alpha}{2d},$$

so that $b - a = A + z_-$ and $a = A + z_+$. We note that $|\arg z_{\pm}| = \pi/2 < \pi$ and that for sufficiently small δ one has $|z_{\pm}| \geq 3$. Then, by item 5 of Lemma 3.3 we have $\Gamma_Q^C = \Gamma(A + z_-)\Gamma(A + z_+)$ and consequently:

$$\Gamma_Q^C = z_+^A z_-^A \Gamma(z_-)\Gamma(z_+) \left(1 + \frac{1}{z_+} J(z_+, A)\right) \left(1 + \frac{1}{z_-} J(z_-, A)\right),$$

with $|J(z_{\pm}, A)| \leq M\Gamma(A)$. Now we are going to give the asymptotic behavior of the above expression. We have that:

$$\begin{aligned} z_+^A z_-^A &= \left(\frac{\alpha}{2d\delta}\right)^{Q+1} \left(1 - \frac{d^2 C^2 \delta^2}{\alpha^2}\right)^{\frac{Q+1}{2}}, \\ \Gamma(z_-)\Gamma(z_+) &= 2\pi \left(\frac{\alpha}{2d\delta}\right)^{iC-1} e^{-\frac{\pi\alpha}{2d\delta}} (1 + \mathcal{O}(\delta)), \\ \left(1 + \frac{1}{z_+} J(z_+, A)\right) \left(1 + \frac{1}{z_-} J(z_-, A)\right) &= 1 + |\Gamma(Q + 1 + iC)| e^{\frac{\pi|C|}{2}} \mathcal{O}(\delta). \end{aligned} \tag{67}$$

The first equality is straightforward from definition. The second one has to be proven by using items 3 and 2 of Lemma 3.3. The third one is the most involved. Taking into account that $|J(z_{\pm}, A)| \leq M|\Gamma(A)|$, $A = (Q + 1)/2$ and that $Q \geq 1$, one checks that

$$\left| \left(1 + \frac{1}{z_+} J(z_+, A)\right) \left(1 + \frac{1}{z_-} J(z_-, A)\right) - 1 \right| \leq K\delta\Gamma(Q + 1). \tag{68}$$

On the one hand, for $C = 0$ it is clear that (68) yields (67). On the other hand, for $C \neq 0$, we have that $|\Gamma(Q + 1 + iC)| \geq \Gamma(Q + 1)|C\Gamma(iC)|$. Thus, using item 3 of Lemma 3.3 we obtain:

$$\Gamma(Q + 1) \leq \frac{|\Gamma(Q + 1 + iC)| |\sinh(\pi C)|^{1/2}}{(\pi|C|)^{1/2}} \leq K\Gamma(Q + 1 + iC) e^{\frac{\pi|C|}{2}}. \tag{69}$$

Equations (68) and (69) yield the last equality in (67).

Substituting the equalities in (67) in expression (66) of $I_{0,Q}^{1,C}$ and using that $|\Gamma(Q + 1 + iC)| \geq K > 0$ we obtain the result for $n = 0$.

For $n \geq 1$ and $Q + 1 > n$ we proceed by induction, using the recurrence of Lemma 3.2. The important fact is that, in the recurrence for $I_{n,Q}^{1,C}$, only the term involving $I_{n-1,Q-1}^{1,C}$ contributes to $I_{n,Q}^{1,C}$ being the other one smaller. \square

End of the proof of Theorem 2.5 First we focus on $\Upsilon_0^{[1]}$. We shall study $\Upsilon_{0,f}^{[1]}$ appearing in formula (63) of $\Upsilon_0^{[l]}$ taking $l = 1$, the other two are done analogously. We decompose $\Upsilon_{0,f}^{[1]}$ into

$$\Upsilon_{0,f}^{[1]} = \Upsilon_{0,0}^{[1]} + \Upsilon_{0,1}^{[1]} \tag{70}$$

where, following formula (64) of $\Upsilon_{0,f}^{[1]}$,

$$\begin{aligned} \Upsilon_{0,0}^{[1]} &= \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n=q} \delta^q f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,q+2d}^{1,cd^{-1}} \\ \Upsilon_{0,1}^{[1]} &= \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n < q} \delta^q f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,k+m+n+2d}^{1,cd^{-1}} \end{aligned} \quad (71)$$

On the one hand, using Lemma 3.4 with $C = cd^{-1}$ and $Q = q + 2d^{-1}$:

$$\begin{aligned} \Upsilon_{0,0}^{[1]} &= \frac{2\pi}{d} \delta^{p-\frac{2}{d}-i\frac{c}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \sum_{q=3}^{\infty} \sum_{k+m+n=q} \frac{f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} (-i)^n a_{k+1,m}^{[1]} \alpha^{q+\frac{2}{d}+i\frac{c}{d}}}{d^{q+\frac{2}{d}+i\frac{c}{d}} \Gamma\left(q+1+\frac{2}{d}+i\frac{c}{d}\right)} \\ &+ \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \sum_{q=3}^{\infty} \sum_{k+m+n=q} \frac{f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} a_{k+1,m}^{[1]} \alpha^{q-1+\frac{2}{d}+i\frac{c}{d}}}{d^{q-1+\frac{2}{d}+i\frac{c}{d}}} \mathcal{O}(\delta) \\ &= \frac{2\pi}{d} \delta^{p-\frac{2}{d}-i\frac{c}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \sum_{q=3}^{\infty} \sum_{k+m+n=q} \frac{f_{qkmn} \left(\sqrt{\frac{d+1}{b}} \right)^{k+m+1} (-i)^n a_{k+1,m}^{[1]} \alpha^{q+\frac{2}{d}+i\frac{c}{d}}}{d^{q+\frac{2}{d}+i\frac{c}{d}} \Gamma\left(q+1+\frac{2}{d}+i\frac{c}{d}\right)} \\ &+ \mathcal{O}\left(\delta^{p+1-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}}\right), \end{aligned} \quad (72)$$

where we have used that $|a_{k,m}^{[1]}| \leq 1$ for all k and m (because $a_{k,m}^{[1]}$ are Fourier coefficients of the functions $\cos^k \theta \sin^m \theta$), and we have assumed that the radius of convergence of f is sufficiently large and thus second sum converges. To bound $\Upsilon_{0,1}^{[1]}$, using again Lemma 3.4 with $C = cd^{-1}$ and $Q = k + m + n + 2d^{-1}$ it is easy to see that:

$$\delta^q \left| I_{n,k+m+n+2d}^{1,cd^{-1}} \right| \leq \delta^{q-(k+m+n+\frac{2}{d})} \left(\frac{\alpha}{d} \right)^{k+m+n+\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}}.$$

Then, $\Upsilon_{0,1}^{[1]}$ in (71) can be bounded by:

$$|\Upsilon_{0,1}^{[1]}| \leq K \delta^{p+1-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}}, \quad (73)$$

where again, we have assumed that the radius of convergence of f is sufficiently large.

Remark 3.5 We need to ensure that a point of the form $\alpha(x_0, y_0, z_0, 0, 0)$ is in $B(r_0)$, the ball of analyticity of f . For that, we note that, rescaling $\delta = \epsilon\bar{\delta}$, we can consider $\alpha = \epsilon\bar{\alpha}$ as small as we want.

Using (72) and (73) in (70) we obtain an asymptotic expression for $\Upsilon_{0,f}^{[1]}$ and reasoning analogously for the other sums appearing in formula (63) of $\Upsilon_0^{[1]}$ we obtain:

$$\begin{aligned} \Upsilon_0^{[1]} = & \frac{2\pi}{d} \delta^{p-\frac{2}{d}-i\frac{c}{d}} e^{-\frac{\alpha\pi}{2d\delta}} \left[\sum_{q=3}^{\infty} \sum_{k+m+n=q} \frac{\left(\sqrt{\frac{d+1}{b}}\right)^{k+m+1} (-i)^n \alpha^{q+\frac{2}{d}+i\frac{c}{d}}}{d^{q+\frac{2}{d}+i\frac{c}{d}} \Gamma\left(q+1+\frac{2}{d}+i\frac{c}{d}\right)} \right. \\ & \left. \left(f_{qkmn} a_{k+1,m}^{[1]} + g_{qkmn} a_{k,m+1}^{[1]} - i\sqrt{\frac{d+1}{b}} h_{qkmn} a_{k,m}^{[1]} \right) \right] \\ & + \mathcal{O}\left(\delta^{p+1-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta}}\right) \end{aligned}$$

Substituting f, g and h by their Taylor's expansion (62) and using that taking the two last variables in f equal to zero implies that the second sum in (62) is done only over the terms $k+m+n=q$, we have that the function m defined in (38) in Theorem 2.5 can be written as

$$\begin{aligned} m(w, \theta) = & \sum_{q=3}^{\infty} \sum_{k+m+n=q} \left(\sqrt{\frac{d+1}{b}}\right)^{k+m+1} (-i)^n w^{q+1+\frac{2}{d}+i\frac{c}{d}} \\ & \cos^k \theta \sin^m \theta \left(f_{qkmn} \cos \theta + g_{qkmn} \sin \theta - i\sqrt{\frac{d+1}{b}} h_{qkmn} \right). \end{aligned}$$

Therefore, using definition (37) of the Borel transform, a direct computation shows the asymptotic expression of $\Upsilon_0^{[1]}$ given in Theorem 2.5 and we are done in this case.

To bound $\Upsilon_0^{[l]}$ for $|l| \geq 2$, we use formula (63) and the bound in Lemma 3.1 with $C = cd^{-1}$ and $Q = k+m+n+2d^{-1}$ to obtain:

$$\begin{aligned} |\Upsilon_0^{[l]}| \leq & K \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta} \frac{3|l|}{4}} \left[\sum_{q=3}^{\infty} \sum_{k+m+n \leq q} M^{k+m+n} \delta^{q-(k+m+n)} \left(\sqrt{\frac{d+1}{b}}\right)^{k+m+1} \right. \\ & \left. \left(|f_{qkmn} a_{k+1,m}^{[l]}| + |g_{qkmn} a_{k,m+1}^{[l]}| + \sqrt{\frac{d+1}{b}} |h_{qkmn} a_{k,m}^{[l]}| \right) \right] \\ \leq & K \delta^{p-\frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta} \frac{3|l|}{4}}, \end{aligned}$$

where in the last inequality we have used that $q - (k+m+n) \geq 0$ that f, g and h , are analytic functions and that the constant M in Lemma 3.1 can be taken sufficiently small so that the series is convergent.

Finally, to prove the asymptotic expression (40) of $M(u, \theta)$, we first take definition (35) of the Melnikov function and use bounds (39) of $\Upsilon_0^{[l]}$ with $|l| \geq 2$. Then, for $u \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$, one has that:

$$M(u, \theta) = \cosh^{\frac{2}{d}}(du) \left[\Upsilon_0^{[0]} + \Upsilon_0^{[1]} e^{i(\theta + \delta^{-1}au + cd^{-1} \log \cosh(du))} + \Upsilon_0^{[-1]} e^{-i(\theta + \delta^{-1}au + cd^{-1} \log \cosh(du))} + \mathcal{O} \left(\delta^{p - \frac{2}{d}} e^{-\frac{\alpha\pi}{2d\delta} \frac{3}{2}} \right) \right].$$

Using the asymptotic formulas for $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$ and the fact that $\delta^{-i \frac{c}{d}} = e^{-i \frac{c}{d} \log \delta}$, we obtain directly expression (40). □

3.2 The Average of the Difference: Proof of Theorem 2.9

Note that both, $\Upsilon_0^{[0]}$ and $M^{[0]}(u)$ (in (36) and (34)) are defined by means of an integral involving $\mathcal{F}^{[0]}(0)$ which from (28) turns out to be:

$$\mathcal{F}^{[0]}(0)(u) = 2\sigma R_0(u) + \delta^p (F(0))^{[0]} + \delta^p \frac{d+1}{b} Z_0(u)(H(0))^{[0]}. \tag{74}$$

In addition, from definition of F , G and H in (24),

$$(F(0))^{[0]} = \mathbf{F}^{[0]}(\delta R_0(u), \delta Z_0(u), \delta), \quad (H(0))^{[0]} = \mathbf{H}^{[0]}(\delta R_0(u), \delta Z_0(u), \delta)$$

with $\mathbf{F}^{[0]}$, $\mathbf{G}^{[0]}$ and $\mathbf{H}^{[0]}$ the average of the functions \mathbf{F} , \mathbf{G} and \mathbf{H} defined by (22).

3.2.1 The Conservative Case

In this subsection we shall prove that the coefficients $\Upsilon^{[0]}$ and $\Upsilon_0^{[0]}$ are zero in the conservative case. In this setting we have $d = 1$ and $\sigma = 0$. Whenever we refer to previous formulas and expressions where these parameters appear, we shall substitute them for these values directly.

Proposition 3.6 *If the vector field (16) is conservative, $\Upsilon_0^{[0]} = 0$.*

Proof We consider the system

$$\begin{aligned} \frac{dr}{dt} &= -2rz + \delta^p \mathbf{F}^{[0]}(\delta r, \delta z, \delta), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p \mathbf{G}^{[0]}(\delta r, \delta z, \delta), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p \mathbf{H}^{[0]}(\delta r, \delta z, \delta). \end{aligned} \tag{75}$$

As system (16) is conservative, system (75) is still conservative and one has:

$$\partial_r \mathbf{F}^{[0]}(\delta r, \delta z, \delta) = -\partial_z \mathbf{H}^{[0]}(\delta r, \delta z, \delta). \tag{76}$$

Using (76) one can easily see that system (75) has the following first integral:

$$\mathcal{U}(r, z) = -r + br^2 + rz^2 + \delta^p \int_0^r \mathbf{H}^{[0]}(\delta s, \delta z, \delta) ds.$$

Note that, using definition (36) of $\Upsilon_0^{[0]}$, with $\mathcal{F}^{[0]}(0)$ in (74), and property (76):

$$\Upsilon_0^{[0]} = \int_{-\infty}^{+\infty} \frac{\mathcal{F}^{[0]}(0)(w)}{\cosh^2 w} dw = - \int_{-\infty}^{+\infty} \frac{d}{dw} (\mathcal{U}(R_0(w), Z_0(w))) dw.$$

Then, we have:

$$\Upsilon_0^{[0]} = - \lim_{t \rightarrow \infty} [\mathcal{U}(R_0(t), Z_0(t)) - \mathcal{U}(R_0(-t), Z_0(-t))].$$

Noting that $(R_0(\pm t), Z_0(\pm t)) \rightarrow (0, \pm 1)$ as $t \rightarrow \pm \infty$ and that $\mathcal{U}(0, \pm 1) = 0$, we obtain $\Upsilon_0^{[0]} = 0$. □

Now we will prove that $\Upsilon^{[0]} = 0$. This proof is more involved and requires some preliminary considerations. We shall use the fact that, in the conservative setting, the two-dimensional invariant manifolds of S_+ and S_- always intersect. This can be seen using standard arguments of volume preservation. Let us introduce some notation concerning this intersection. We fix $\theta_0 \in [0, 2\pi)$ and consider the following plane:

$$\Sigma_{\theta_0} = \{(x, y, z) \in \mathbb{R}^3 : x \sin \theta_0 - y \cos \theta_0 = 0\}.$$

We define p_1 as the first common intersection of the two-dimensional invariant manifolds of S_+ and S_- contained in the section Σ_{θ_0} . This point p_1 is $\mathcal{O}(\delta^{p+3})$ -close to $(\frac{1}{b} \cos \theta_0, \frac{1}{b} \sin \theta_0, 0)$, which is the intersection of the heteroclinic surface with $\Sigma_{\theta_0} \cap \{z = 0\}$ in the unperturbed case. The orbit of p_1 , namely:

$$\Gamma_{p_1} := \{\varphi_t(p_1), t \in \mathbb{R}\}, \tag{77}$$

where φ_t stands for the flow the vector field (15), is a heteroclinic orbit and for small δ it intersects many times the section Σ_{θ_0} . We define:

$$t_2 = \min\{t > 0 : \varphi_t(p_1) \in \Sigma_{\theta_0}\}, \quad p_2 = \varphi_{t_2}(p_1),$$

and:

$$t_3 = \min\{t > t_2 : \varphi_t(p_1) \in \Sigma_{\theta_0}\}, \quad p_3 = \varphi_{t_3}(p_1).$$

Remark 3.7 Note that, $\dot{\theta} < 0$ provided that δ is sufficiently small. Indeed, this can be easily seen since $\dot{\theta} = -\alpha/\delta - cz + \delta^p G(\delta r, \theta, \delta z, \delta, \delta \sigma)$. Then p_2 has angular variable $\theta_0 - \pi$ and p_3 has angular variable $\theta_0 - 2\pi$.

We define z_i and u_i as:

$$z_i = \pi_z(p_i), \quad u_i = Z_0^{-1}(z_i) = \operatorname{atanh}(z_i), \quad i = 1, 2, 3. \tag{78}$$

with π_z the projection on the third component. See Fig. 8a.

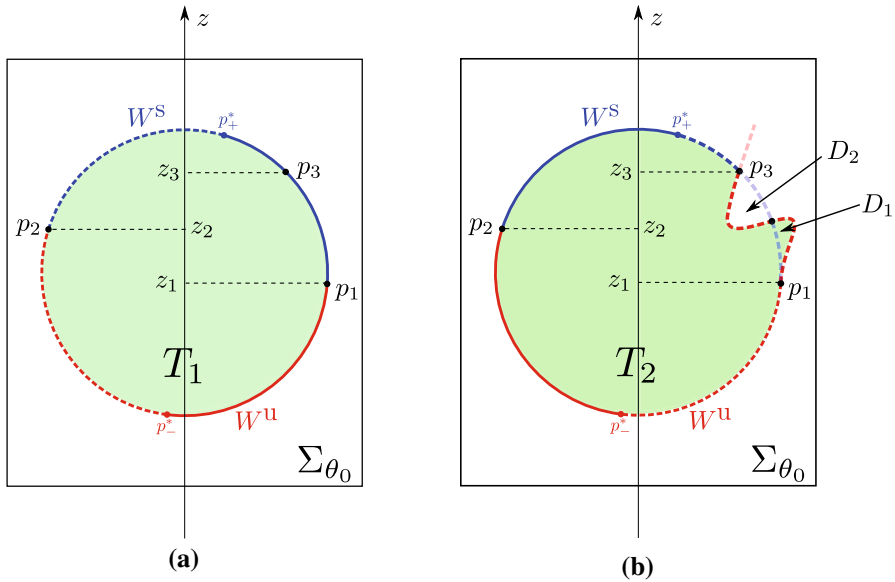


Fig. 8 The domains T_1 and T_2 . In red, the unstable manifold of S_- , and in blue the stable manifold of S_+ . The continuous (respectively, dashed) lines on the left are mapped to the continuous (dashed) lines on the right with the same color via the flow ϕ . **a** The domain T_1 on the section Σ_{θ_0} . **b** The domain T_2 on the section Σ_{θ_0} (Color figure online)

We point out that with this notation we can write:

$$\Delta(u_i, \theta_0) = 0, \quad i = 1, 2, 3,$$

where as usual $\Delta(u, \theta) = r^u(u, \theta) - r^s(u, \theta)$.

Lemma 3.8 Let u_1 and u_3 be defined as in (78). Define:

$$\tau^* = \xi(u_1, \theta_0) = \theta_0 + \delta^{-1} \alpha u_1 + c \log \cosh u_1 + C(u_1, \theta_0),$$

where $\xi(u, \theta)$ and $C(u, \theta)$ are the functions given in Theorem 2.7. Then:

$$\xi(u_3, \theta_0) = \theta_0 + \delta^{-1} \alpha u_3 + c \log \cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi.$$

Proof Let $s_0 > 1$. For any $s \in [-s_0, s_0]$, we define $u = u(s)$ as the (unique) solution of:

$$\xi(u(s), \theta_0 - 2\pi s) = \tau^*. \tag{79}$$

The fact that Eq. (79) has a unique solution for all $s \in [-s_0, s_0]$ if δ is sufficiently small can be seen, for instance, by the implicit function theorem. By definition of τ^* , the unique solution at $s = 0$ is $u(0) = u_1$.

Now, since $\Delta(u_1, \theta_0) = 0$ and $\Delta(u, \theta) = \cosh^2(u)(1 + P_1(u, \theta))\tilde{k}(\xi(u, \theta))$ by Theorem 2.7, using that $\cosh(u_1) \neq 0$ and that P_1 is small we have:

$$0 = \tilde{k}(\xi(u_1, \theta_0)) = \tilde{k}(\tau^*) = \tilde{k}(\xi(u(s), \theta_0 - 2\pi s)).$$

Thus, $\Delta(u(s), \theta_0 - 2\pi s) = 0$. Hence, defining

$$r_h(s) := r^u(u(s), \theta_0 - 2\pi s) = r^s(u(s), \theta_0 - 2\pi s),$$

we have that the curve:

$$\gamma_h(s) := (r_h(s), \theta_0 - 2\pi s, Z_0(u(s))), \quad s \in [-s_0, s_0],$$

is part of a heteroclinic orbit expressed in the symplectic polar coordinates. Since $u(0) = u_1$ and p_1 in these coordinates is $(r^u(u_1, \theta_0), \theta_0, Z_0(u_1)) = \gamma_h(0)$, clearly $\gamma_h(s)$ is a part of the heteroclinic orbit Γ_{p_1} , defined in (77).

Taking $s = 1$, we obtain the point in Γ_{p_1} with angular variable $\theta_0 - 2\pi$. By Remark 3.7, this point is precisely p_3 . This implies that $u(1) = u_3$, and then Eq. (79) yields.

$$\theta_0 - 2\pi + \delta^{-1}\alpha u_3 + c \log \cosh u_3 + C(u_3, \theta_0 - 2\pi) = \tau^*,$$

and since $C(u, \theta)$ is 2π periodic in θ we obtain:

$$\theta_0 + \delta^{-1}\alpha u_3 + c \log \cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi.$$

□

Lemma 3.9 *Let u_1 and u_3 be the u -coordinate of the heteroclinic points $p_1, p_3 \in \Sigma_{\theta_0}$, respectively, defined in (78). Let $\Upsilon^{[0]}$ be the average of the function $\tilde{k}(\tau)$ given in Theorem 2.7. Then one has:*

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_{u_1}^{u_3} \frac{\Delta(u, \theta_0)}{\cosh^2(u)(1 + P_1(u, \theta_0))} \left(\delta^{-1}\alpha + cZ_0(u) + \partial_u C(u, \theta_0) \right) du.$$

Proof It can be obtained straightforwardly from the fact that:

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_{\tau^*}^{\tau^*+2\pi} \tilde{k}(\tau) d\tau,$$

where $\tau^* = \theta_0 + \delta^{-1}\alpha u_1 + c \log \cosh u_1 + C(u_1, \theta_0)$. Indeed, one just has to perform the change $\tau = \theta_0 + \delta^{-1}\alpha u + c \log \cosh u + C(u, \theta_0)$. Then, recalling that by Theorem 2.7:

$$\Delta(u, \theta_0) = \cosh^2(u)(1 + P_1(u, \theta_0))\tilde{k}(\theta_0 + \delta^{-1}\alpha u + c \log \cosh(u) + C(u, \theta_0)),$$

and that by Lemma 3.8:

$$\theta_0 + \delta^{-1} \alpha u_3 + c \log \cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi,$$

one obtains the claim of the lemma. □

Proposition 3.10 *One has:*

$$\int_{u_1}^{u_3} \frac{\Delta(u, \theta_0)}{\cosh^2(u)} \left(\delta^{-1} \alpha + cZ_0(u) + l_3(u, \theta_0) \right) du = 0 \tag{80}$$

where l_3 is defined in (44).

Proof Let us denote by $\tilde{r}^u(z, \theta) := r^u(Z_0^{-1}(z), \theta)$ the r -component of the unstable manifold of S_- as a function of z and θ , and similarly $\tilde{r}^s(z, \theta)$ for the stable manifold of S_+ . We denote:

$$\tilde{G}(r, z) = \mathbf{G}(\delta r, \theta_0, \delta z, \delta),$$

where \mathbf{G} is the function defined in (22) (recall that in the conservative case there is no dependence on the parameter σ). We shall prove the following:

$$\int_{z_1}^{z_3} \int_{\tilde{r}^s(z, \theta_0)}^{\tilde{r}^u(z, \theta_0)} (\delta^{-1} \alpha + cz - \delta^p \tilde{G}(r, z)) dr dz = 0, \tag{81}$$

with z_1 and z_3 defined in (78). This yields claim (80). Indeed, assume (81) is true. Then we make the change:

$$r = \tilde{r}_\lambda := \frac{1}{2}(\tilde{r}^u(z, \theta_0) + \tilde{r}^s(z, \theta_0)) + \frac{\lambda}{2}(\tilde{r}^u(z, \theta_0) - \tilde{r}^s(z, \theta_0)), \quad \lambda \in [-1, 1],$$

and, denoting $\tilde{\Delta}(z, \theta) = \tilde{r}^u(z, \theta_0) - \tilde{r}^s(z, \theta_0)$, Eq. (81) becomes:

$$\int_{z_1}^{z_3} \left(\delta^{-1} \alpha + cz - \frac{1}{2} \int_{-1}^1 \delta^p \tilde{G}(\tilde{r}_\lambda, z) d\lambda \right) \tilde{\Delta}(z, \theta_0) dz = 0.$$

Then, we perform the change $z = Z_0(u)$ and recalling definition (78) of u_1 and u_3 , and definition (44) of l_3 we obtain (80).

To prove (81) we shall use basically that the system is divergence free, and apply the divergence theorem in a suitable three-dimensional domain. However, we first need to introduce some notation. Consider the intersection of the two-dimensional unstable manifold of S_- and Σ_{θ_0} . The lower part of this intersection is a curve that joins p_1 and p_2 , having a shape close to an arch of ellipse. Similarly, if we consider the intersection of the two-dimensional stable manifold of S_+ and Σ_{θ_0} , its upper part is a curve that also joins p_1 and p_2 , with a similar shape. We define $T_1 \subset \Sigma_{\theta_0}$ as the domain bounded by these two curves (see Fig. 8a).

As in (16) we denote by X the vector field defining our system and X_x, X_y and X_z each of its components. We note that if $p \in \partial T_1$ and:

$$X_x(p) \sin \theta_0 - X_y(p) \cos \theta_0 \neq 0$$

then there exists a unique $\tau(p) > 0$ such that $\varphi_{\tau(p)}(p)$ is the next intersection of the orbit going through p and Σ_{θ_0} . This is clear from the fact that the orbits inside $W^u(S_-)$ are $\mathcal{O}(\delta)$ -close to the orbits of the heteroclinic connection of the unperturbed system for $t \in (-\infty, T]$, for some constant T , and the same happens for the orbits inside $W^s(S_+)$ and $t \in [T, \infty)$. Moreover, there are just two points $p_-^*, p_+^* \in \Sigma_{\theta_0}$ (close to S_- and S_+ , respectively) such that:

$$X_x(p_{\pm}^*) \sin \theta_0 - X_y(p_{\pm}^*) \cos \theta_0 = 0.$$

See Fig. 8a. For such points we can define $\tau(p_{\pm}^*) = 0$. With this definition, the function $\varphi_{\tau(p)}(p)$ is continuous for $p \in \partial T_1$. Then we define $T_2 \subset \Sigma_{\theta_0}$ (see Fig. 8b) as the domain bounded by ∂T_2 , where

$$\partial T_2 = \{\varphi_{\tau(p)}(p) : p \in \partial T_1\}.$$

Finally, we define:

$$T_3 = \{\varphi_t(p) : p \in \partial T_1, t \in (0, \tau(p))\}.$$

We point out that T_3 is tangent to the flow of X . Moreover, T_1, T_2 and T_3 are the boundary of a closed three-dimensional domain. That is, there exists a closed domain $V \subset \mathbb{R}^3$ such that $T_1 \cup T_2 \cup T_3 = \partial V$. Now we use the divergence theorem in this domain V . Since $\operatorname{div} X \equiv 0$ we have:

$$\begin{aligned} 0 &= \iiint_V \operatorname{div} X dV = \iint_{\partial V} X \cdot \vec{n}_{\partial V} dS \\ &= \iint_{T_1} X \cdot \vec{n}_{T_1} dS + \iint_{T_2} X \cdot \vec{n}_{T_2} dS + \iint_{T_3} X \cdot \vec{n}_{T_3} dS, \end{aligned} \tag{82}$$

where $\vec{n}_{\partial V}$ denotes the unitary normal vector to ∂V pointing outside V , and the same with $\vec{n}_{T_i}, i = 1, 2, 3$. Since T_3 is tangent to the flow, $X \cdot \vec{n}_{T_3} = 0$ and moreover, $\vec{n}_{T_1} = (-\sin \theta_0, \cos \theta_0, 0) = -\vec{n}_{T_2}$. Thus, (82) becomes:

$$0 = \iint_{D_1} (X_x \sin \theta_0 - X_y \cos \theta_0) dS - \iint_{D_2} (X_x \sin \theta_0 - X_y \cos \theta_0) dS, \tag{83}$$

where $D_1 = T_2 \setminus T_1$ and $D_2 = T_1 \setminus T_2$ (see Fig. 8b). We take the parameterization:

$$x = \sqrt{2r} \cos \theta_0, \quad y = \sqrt{2r} \sin \theta_0, \quad z = z$$

and we note that

$$\begin{aligned} X_x(\sqrt{2r} \cos \theta_0, \sqrt{2r} \sin \theta_0, z) \sin \theta_0 - X_y(\sqrt{2r} \cos \theta_0, \sqrt{2r} \sin \theta_0, z) \cos \theta_0 \\ = \sqrt{2r} \left(\delta^{-1} \alpha + cz - \delta^p \tilde{G}(r, z) \right). \end{aligned}$$

With this parameterization, equality (83) yields (81). □

End of the proof of Theorem 2.9 (conservative case) Proposition 3.6 says that $\Upsilon_0^{[0]} = 0$. To see that $\Upsilon^{[0]} = 0$ we note that from item 2. in Theorem 2.7, we choose P_1 such that:

$$\frac{\delta^{-1} \alpha + cZ_0(u) + \partial_u C(u, \theta_0)}{1 + P_1(u, \theta_0)} = \delta^{-1} \alpha + cZ_0(u) + l_3(u, \theta_0).$$

Then, substituting this in the equality of Lemma 3.9 we get:

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_{u_1}^{u_3} \frac{\Delta(u, \theta_0)}{\cosh^2(u)} \left(\delta^{-1} \alpha + cZ_0(u) + l_3(u, \theta_0) \right) du.$$

Finally, Proposition 3.10 yields that $\Upsilon^{[0]} = 0$, and the proof is finished. □

3.2.2 The General Case

In this section we will prove the statements about the coefficients $\Upsilon_0^{[0]}$ and $\Upsilon^{[0]}$ in Theorem 2.9. We have:

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{k}(\tau) d\tau. \tag{84}$$

We perform the change $\tau = \theta + C(0, \theta)$ in the previous integral, where $C(u, \theta)$ is the function in Theorem 2.7 and we use that by Theorem 2.7 we have:

$$\tilde{k}(\theta + \delta^{-1} \alpha u + cd^{-1} \log \cosh(du) + C(u, \theta)) = \frac{\Delta(u, \theta)}{\cosh^{2/d}(du)(1 + P_1(u, \theta))}.$$

After this change (84) becomes:

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{\Delta(0, \theta)}{1 + P_1(0, \theta)} (1 + \partial_\theta C(0, \theta)) d\theta, \tag{85}$$

where, using bounds for $C(0, \theta)$ obtained in Theorem 2.7,

$$\theta_1 = 0 + \mathcal{O}(\delta^{p+3}) \quad \theta_2 = 2\pi + \mathcal{O}(\delta^{p+3}). \tag{86}$$

Now, on the one hand, by Theorem 2.4 we have:

$$|\Delta(0, \theta)| \leq |r_1^u(0, \theta)| + |r_1^s(0, \theta)| \leq K \delta^{p+3}. \tag{87}$$

and recalling the notation $M(u, \theta) = r_{10}^u(u, \theta) - r_{10}^s(u, \theta)$:

$$|\Delta(0, \theta) - M(0, \theta)| \leq |r_{11}^u(0, \theta)| + |r_{11}^s(0, \theta)| \leq K \left(\delta^{2p+6} + \delta^{p+4} \right),$$

where we have used the bounds of $r_{11}^{u,s}(u, \theta)$ given in Theorem 2.4. On the other hand, by Theorem 2.7:

$$|\partial_\theta C(0, \theta)| \leq K \delta^{p+3}, \quad \left| \frac{1}{1 + P_1(0, \theta)} - 1 \right| \leq K |P_1(0, \theta)| \leq K \delta^{p+3}. \quad (88)$$

Thus, using bounds (86), (87) and (88) in Eq. (85) we obtain:

$$\Upsilon^{[0]} = \frac{1}{2\pi} \int_0^{2\pi} M(0, \theta) d\theta + \mathcal{O}(\delta^{p+4}) = M^{[0]}(0) + \mathcal{O}(\delta^{p+4}), \quad (89)$$

where we have used that $p \geq -2$.

We introduce the following notation:

$$I = \frac{d+1}{b} \int_{-\infty}^{+\infty} \frac{1}{\cosh^{\frac{2}{d}+2}(dw)} dw, \quad (90)$$

$$\bar{J} = \bar{J}(\delta, \sigma) = \delta^{-3} \int_{-\infty}^{+\infty} \frac{(F(0))^{[0]} + \frac{d+1}{b} Z_0(w)(H(0))^{[0]}}{\cosh^{\frac{2}{d}}(dw)} dw \quad (91)$$

and observe that for all $w \in \mathbb{R}$:

$$|F(0)| = |\mathbf{F}(\delta R_0(w), \theta, \delta Z_0(w), \delta, \delta\sigma)| \leq K \delta^3$$

and also $|H(0)| \leq K \delta^3$, so that \bar{J} is bounded as $\delta \rightarrow 0$. Therefore, we can write

$$\bar{J}(\delta, \sigma) = J + \mathcal{O}(\delta). \quad (92)$$

Now, by formula (34) of $M^{[l]}(u)$ and expression (74) of $\mathcal{F}^{[0]}$, we get:

$$\Upsilon_0^{[0]} = M^{[0]}(0) = \sigma I + \delta^{p+3} \bar{J}.$$

We rewrite (89) as:

$$\Upsilon^{[0]} = \Upsilon_0^{[0]} + \mathcal{O}(\delta^{p+4}) = \sigma I + \delta^{p+3} J + \mathcal{O}(\delta^{p+4}).$$

Then, putting $\sigma = \hat{\sigma} \delta^{p+3}$, we have that $\Upsilon^{[0]} = 0$ if:

$$f(\hat{\sigma}, \delta) := \hat{\sigma} I + J + \mathcal{O}(\delta) = 0.$$

It is clear that $I \neq 0$, and thus,

$$f\left(-\frac{J}{I}, 0\right) = 0, \quad \frac{\partial f}{\partial \hat{\sigma}}\left(-\frac{J}{I}, 0\right) = I \neq 0,$$

Then we can apply the implicit function theorem, so that there exists δ_0 and a curve $\hat{\sigma}_*^0(\delta) = -J/I + \mathcal{O}(\delta)$ such that $f(\hat{\sigma}_*^0(\delta), \delta) = 0$ for all $0 \leq \delta \leq \delta_0$. Finally,

$$\sigma_*^0(\delta) := \hat{\sigma}_*^0(\delta)\delta^{p+3} = -\frac{J}{I}\delta^{p+3} + \mathcal{O}(\delta^{p+4}).$$

To obtain the curves $\sigma_*(\delta)$ we solve the equation:

$$g(\sigma, \beta) := \Upsilon^{[0]} - \beta = \sigma I + \delta^{p+3}J + \mathcal{O}(\delta^{p+4}) - \beta = 0.$$

with $\beta = a_1\delta^{a_2}e^{-\frac{a_3\pi}{2d\delta}}$, where we use that $a_3 > 0$ so that β is small when δ is small. As:

$$g(\sigma_*^0(\delta), 0) = 0, \quad \frac{\partial g}{\partial \sigma}(\sigma_*^0(\delta), 0) = I + \mathcal{O}(\delta^{p+3}) \neq 0,$$

these equations have a solution $\sigma_*(\delta)$ satisfying:

$$\sigma_*(\delta) = \sigma_*^0(\delta) + \mathcal{O}(\delta^{a_2}e^{-\frac{a_3\pi}{2d\delta}}).$$

Clearly, since $\Upsilon_0^{[0]} = \sigma I + \delta^{p+3}J$, one has:

$$\Upsilon_0^{[0]} = \Upsilon_0^{[0]}(\delta, \sigma_*(\delta)) = \Upsilon_0^{[0]}\left(\delta, -\frac{J}{I}\delta^{p+3} + \mathcal{O}(\delta^{p+4})\right) = \mathcal{O}(\delta^{p+4}).$$

3.3 The Exponentially Smallness of $\Upsilon^{[l]}$: Proof of Lemma 2.11

Let us to introduce the function

$$F(u, \theta) = \delta\alpha^{-1}(\xi(u, \theta) - \theta) = u + \delta\alpha^{-1}\left[cd^{-1}\log\cosh(du) + C(u, \theta)\right],$$

where ξ and C are defined in Theorem 2.7. In this result it is proven that $(\xi(u, \theta), \theta)$ is injective in $D_{\kappa, \beta} \times \mathbb{T}_\omega$ then $(F(u, \theta), \theta)$ is also injective in the same domain. In particular, for all $(u, \theta) \in D_{\kappa, \beta} \times \mathbb{S}^1$, the change $(w, \theta) = (F(u, \theta), \theta)$ is a diffeomorphism between $D_{\kappa, \beta} \times \mathbb{S}^1$ and its image $\tilde{D}_{\kappa, \beta} \times \mathbb{S}^1$, with inverse $(u, \theta) = (G(w, \theta), \theta)$. Then, if we define the function:

$$\mathcal{E}(w, \theta) = \sum_{l \in \mathbb{Z}} \Upsilon^{[l]} e^{il(\theta + \delta^{-1}\alpha w)}.$$

one has that $G(w, \theta)$ satisfies:

$$\mathcal{E}(w, \theta) = \frac{\Delta(G(w, \theta), \theta)}{\cosh^{2/d}(dG(w, \theta))(1 + P_1(G(w, \theta), \theta))}. \tag{93}$$

Note that $\mathcal{E}(w, \theta)$ is 2π -periodic in θ , and its l -th Fourier coefficient is:

$$\mathcal{E}^{[l]}(w) = \Upsilon^{[l]} e^{il\delta^{-1}\alpha w}.$$

Hence, we know that for all $w \in \tilde{D}_{\kappa, \beta}$:

$$\left| \Upsilon^{[l]} \right| = \frac{1}{2\pi} \left| e^{-i\delta^{-1}\alpha w l} \int_0^{2\pi} \mathcal{E}(w, \theta) e^{-il\theta} d\theta \right| \leq \left| e^{-i\delta^{-1}\alpha w l} \right| \sup_{\theta \in \mathbb{S}^1} |\mathcal{E}(w, \theta)|. \tag{94}$$

This inequality is valid for all $w \in \tilde{D}_{\kappa, \beta}$. Let us denote $u_{\pm} = \pm i \left(\frac{\pi}{2d} - \kappa \delta \right)$. Then, if in (94) we take $w = w_+ := F(u_+, \theta) \in \tilde{D}_{\kappa, \beta}$ for $l < 0$ and $w = w_- := F(u_-, \theta) \in \tilde{D}_{\kappa, \beta}$ for $l > 0$, one obtains:

$$\left| \Upsilon^{[l]} \right| \leq e^{-\left(\frac{\alpha\pi}{2d\delta} - \alpha\kappa - |\text{Im } C(u_{\pm}, \theta)|\right)|l|} \sup_{\theta \in \mathbb{S}^1} |\mathcal{E}(w_{\pm}, \theta)|. \tag{95}$$

Recall that F is the inverse of G , so that from (93) we obtain:

$$\mathcal{E}(w_{\pm}, \theta) = \frac{\Delta(u_{\pm}, \theta)}{\cosh^{2/d}(du_{\pm})(1 + P_1(u_{\pm}, \theta))}.$$

Thus, using bound (54) for P_1 , that $|\cosh(du_{\pm})| \geq K\delta\kappa$, and taking κ sufficiently large, bound (95) writes out as:

$$\left| \Upsilon^{[l]} \right| \leq \frac{K}{\delta^{\frac{2}{d}} \kappa^{\frac{2}{d}}} e^{-\left(\frac{\alpha\pi}{2d\delta} - \alpha\kappa - |\text{Im } C(u_{\pm}, \theta)|\right)|l|} \sup_{\theta \in \mathbb{S}^1} |\Delta(u_{\pm}, \theta)|. \tag{96}$$

Now, on the one hand, taking into account that the constant L_0 , given in Theorem 2.7, satisfies $L_0 \in \mathbb{R}$, we have:

$$|\text{Im } C(u_{\pm}, \theta)| \leq d^{-1}(c + \alpha L_0) |\text{Im } \log \cosh(du_{\pm})| + \alpha |L(u_{\pm})| + |\chi(u_{\pm}, \theta)|.$$

Since u_{\pm} is purely imaginary, $\text{Im } \log \cosh(du_{\pm}) = \arg(\cosh(du_{\pm})) = 0$. Then, using (53) in Theorem 2.7, we obtain $|\text{Im } C(u_{\pm}, \theta)| \leq K\delta^{p+2}$. Therefore:

$$\left| e^{-\left(\frac{\alpha\pi}{2d\delta} - \alpha\kappa - |\text{Im } C(u_{\pm}, \theta)|\right)|l|} \right| \leq K e^{-\frac{\alpha\pi}{2d\delta} + \alpha\kappa}. \tag{97}$$

Moreover, we take δ sufficiently small so that:

$$1 - \frac{2d\delta}{\alpha\pi} (\alpha\kappa + |\text{Im } C(u_{\pm}, \theta)|) \geq \frac{3}{4},$$

and then, for $|l| \geq 2$, one has:

$$\left| e^{-\left(\frac{\alpha\pi}{2d\delta} - \alpha\kappa - |\operatorname{Im} C(u_{\pm}, \theta)|\right)|l|} \right| \leq e^{-\frac{\alpha\pi}{2d\delta} \frac{3|l|}{4}}. \tag{98}$$

On the other hand, by Theorem 2.4 we have:

$$|\Delta(u_{\pm}, \theta)| \leq |r_1^u(u_{\pm}, \theta)| + |r_1^s(u_{\pm}, \theta)| \leq \frac{K\delta^{p+3}}{|\cosh(du_{\pm})|^3} \leq K \frac{\delta^p}{\kappa^3}. \tag{99}$$

To obtain the claim of the lemma for $|l| = 1$, we use bounds (97) and (99) in Eq. (96). Similarly, for $|l| \geq 2$ we use bounds (98) and (99) in Eq. (96).

3.4 Fourier Coefficients of Δ_1 : Proof of Proposition 2.13

Consider the function $\Delta_1(u, \theta) = \Delta(u, \theta) - \Delta_0(u, \theta)$ defined in (60):

$$\Delta_1(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)) \sum_{l \neq 0} \left(\Upsilon^{|l|} - \hat{\Upsilon}_0^{|l|} \right) e^{il\xi(u, \theta)},$$

with $\xi(u, \theta) = \theta + \delta^{-1}\alpha u + d^{-1} \log \cosh(du) + C(u, \theta)$ defined in (51) and $\Upsilon^{|l|}, \hat{\Upsilon}_0^{|l|}$ the Fourier coefficients of \tilde{k} and \tilde{k}_0 in (50) and (58), respectively. We point out that in order to obtain sharp bounds for $\Upsilon^{[\pm 1]} - \hat{\Upsilon}_0^{[\pm 1]}$ we need to take $u \in D_{\kappa, \beta} \subset \mathbb{C}$ (see (45)), but θ can be taken real. Thus, we will take $\theta \in \mathbb{S}^1$.

Proceeding as in beginning of the previous Sect. 3.3, one can prove the following bound for $|\Upsilon^{[\pm 1]} - \hat{\Upsilon}_0^{[\pm 1]}|$ which is similar to the one for $|\Upsilon^{|l|}$ in (96):

$$\left| \Upsilon^{[\mp 1]} - \hat{\Upsilon}_0^{[\mp 1]} \right| \leq \frac{K}{\delta^{\frac{2}{d}} \kappa^{\frac{2}{d}}} e^{-\left(\frac{\alpha\pi}{2d\delta} - \alpha\kappa - |\operatorname{Im} C(u_{\pm}, \theta)|\right)} \sup_{\theta \in \mathbb{S}^1} |\Delta_1(u_{\pm}, \theta)|,$$

where $u_{\pm} = \pm i \left(\frac{\pi}{2d} - \kappa\delta \right)$. Using bound (97), we obtain:

$$\left| \Upsilon^{[\mp 1]} - \hat{\Upsilon}_0^{[\mp 1]} \right| \leq \frac{K}{\delta^{\frac{2}{d}} \kappa^{\frac{2}{d}}} e^{-\frac{\alpha\pi}{2d\delta} + \alpha\kappa} \sup_{\theta \in \mathbb{S}^1} |\Delta_1(u_{\pm}, \theta)|. \tag{100}$$

We claim that there exists a constant K such that for all $\theta \in \mathbb{S}^1$:

$$|\Delta_1(u_{\pm}, \theta)| \leq K \left(\frac{\delta^{2(p+1)} |\log \kappa|}{\kappa^4} + \frac{\delta^{p+3}}{\kappa} \right). \tag{101}$$

Indeed, first we write $\Delta_1 = \Delta - \Delta_0$ in a more adequate form. We recall that, by definition (32) of the Melnikov function M , $\Delta = M + r_{11}^u - r_{11}^s$. Then,

$$\Delta_1(u, \theta) = M(u, \theta) - \Delta_0(u, \theta) + r_{11}^u(u, \theta) - r_{11}^s(u, \theta).$$

It is clear that, by Theorem 2.4,

$$|r_{11}^u(u_{\pm}, \theta) - r_{11}^s(u_{\pm}, \theta)| \leq K \left(\frac{\delta^{2(p+1)}}{\kappa^4} + \frac{\delta^{p+3}}{\kappa} \right),$$

which is smaller than the upper bound in (101). Therefore, to prove (101), it only remains to study the difference between M and Δ_0 .

We introduce some notation:

$$\begin{aligned} F_0(u) &= u + \delta\alpha^{-1}cd^{-1} \log \cosh(du), & \hat{C}(u, \theta) &= C(u, \theta) - \alpha d^{-1}L_0\delta^{p+2} \log(\delta) \\ \hat{F}(u, \theta) &= F_0(u) + \delta\alpha^{-1}\hat{C}(u, \theta). \end{aligned} \tag{102}$$

Notice that $F_0(u)$ is injective so that it has an inverse. We also introduce the function

$$f(u, \theta) = F_0^{-1}(\hat{F}(u, \theta))$$

and we note that, since by (52) in Theorem 2.7, $|\hat{C}(u_{\pm}, \theta)| \leq K\delta^{p+2}|\log \kappa|$ (see (102))

$$|f(u_{\pm}, \theta) - u_{\pm}| \leq K\delta^{p+3}|\log \kappa|. \tag{103}$$

Now we rewrite $M(u, \theta)$ in (35) as

$$M(u, \theta) = \cosh^{\frac{2}{d}}(du) \sum_{l \in \mathbb{Z}} \Upsilon_0^{[l]} e^{il(\theta + \delta^{-1}\alpha F_0(u))}$$

and we observe that

$$M(f(u, \theta), \theta) = \cosh^{\frac{2}{d}}(df(u, \theta)) \sum_{l \in \mathbb{Z}} \Upsilon_0^{[l]} e^{il(\theta + \delta^{-1}\alpha \hat{F}(u, \theta))}. \tag{104}$$

In addition, the function Δ_0 in (59) is:

$$\Delta_0(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)) \left(\Upsilon^{[0]} + \sum_{l \neq 0} \Upsilon_0^{[l]} e^{il(\theta + \delta^{-1}\alpha \hat{F}(u, \theta))} \right)$$

where we have used that $\hat{\Upsilon}_0^{[l]} = \Upsilon_0^{[l]} e^{-il\alpha d^{-1}L_0\delta^{p+2} \log \delta}$. As a consequence

$$\begin{aligned} M(u, \theta) - \Delta_0(u, \theta) &= M(u, \theta) - M(f(u, \theta), \theta) \\ &+ \cosh^{2/d}(df(u, \theta))\Upsilon_0^{[0]} - \cosh^{2/d}(du)(1 + P_1(u, \theta))\Upsilon^{[0]} \\ &+ \left(\sum_{l \neq 0} \Upsilon_0^{[l]} e^{il(\theta + \delta^{-1}\alpha \hat{F}(u, \theta))} \right) \left(\cosh^{2/d}(df(u, \theta)) - \cosh^{2/d}(du)(1 + P_1(u, \theta)) \right) \end{aligned} \tag{105}$$

We shall prove bound (101) bounding each term in (105), with $u = u_{\pm}$.

Recall that $M = r_{10}^u - r_{10}^s$ so that, by Theorem 2.4 and (103)

$$\begin{aligned}
 |M(u_{\pm}, \theta) - M(f(u_{\pm}, \theta), \theta)| &\leq K \delta^{p+3} |u_{\pm} - f(u_{\pm}, \theta)| \sup_{u \in D_{\kappa, \beta}} \frac{1}{|\cosh(du)|^4} \\
 &\leq |\log \kappa| \frac{\delta^{2p+2}}{\kappa^4} \tag{106}
 \end{aligned}$$

so this term satisfies bound (101).

By Theorem 2.9, the terms involving $\Upsilon_0^{[0]}$ and $\Upsilon^{[0]}$ in (105) are zero in the conservative case. In the general case, since we take $\sigma = \sigma_*(\delta)$ these terms satisfy:

$$|\cosh^{2/d}(du_{\pm})(1 + P_1(u_{\pm}, \theta))\Upsilon^{[0]}| \leq K \delta^{a_2} e^{-\frac{a_3 \pi}{2d\delta}}, \quad \left| \cosh^{2/d}(du_{\pm})\Upsilon_0^{[0]} \right| \leq K \delta^{p+4}$$

where we have used that, from Theorem 2.7, $|P_1(u_{\pm}, \theta)| \leq K \delta^{p+2} \kappa^{-1}$.

Finally, we deal with the last term which (see (104)) we rewrite as

$$\begin{aligned}
 &\Sigma(\theta) \\
 &:= \left(M(f(u_{\pm}, \theta), \theta) - \cosh^{2/d}(du_{\pm})\Upsilon_0^{[0]} \right) \left[1 - \frac{\cosh^{2/d}(du_{\pm})}{\cosh^{2/d}(df(u_{\pm}, \theta))} (1 + P_1(u_{\pm}, \theta)) \right].
 \end{aligned}$$

We first note that, since $M = r_{10}^u - r_{10}^s$, using Theorem 2.4 to bound $r_{10}^{u,s}$, Theorem 2.9 to bound $\Upsilon_0^{[0]}$ and (106) one has that

$$|\Sigma(\theta)| \leq K \frac{\delta^p}{\kappa^3} \left| 1 - \frac{\cosh^{2/d}(du_{\pm})}{\cosh^{2/d}(df(u_{\pm}, \theta))} (1 + P_1(u_{\pm}, \theta)) \right|.$$

Therefore, using bound (103) of $|f(u_{\pm}, \theta) - u_{\pm}|$, one has that

$$\left| \left[\frac{\cosh^{2/d}(du_{\pm})}{\cosh^{2/d}(df(u_{\pm}, \theta))} \right] - 1 \right| \leq K \frac{\delta^{p+2} |\log \kappa|}{\kappa}$$

and, since $|P_1(u_{\pm}, \theta)| \leq K \delta^{p+2} \kappa^{-1}$, we conclude that

$$|\Sigma(\theta)| \leq K \frac{\delta^{2(p+1)} |\log \kappa|}{\kappa^4}$$

and bound (101) for $\Delta_1(u_{\pm}, \theta)$ is proven.

Finally, we use bound (101) in (100) and the proposition is proven.

4 Parameterizations of the Invariant Manifolds

We will prove Theorem 2.4 in Sect. 4.2, as a non-trivial consequence of the existence result Proposition 4.4. This proposition is proven by using the fixed point theorem and its proof is given in Sect. 4.1.

The complete proofs of the results in the present section are extremely technical and can be found with all the details in Castejón (2015). Here we present a summary of the methodology used in the proof.

We will use the notation and properties stated in Sects. 2.1, and 2.2. Moreover, as usual, π_x, π_y, π_z will denote, respectively, the projection over the x, y and z -component of a given vector.

4.1 Existence of Complex Parameterizations

As we explained in Sect. 2.2, on the one hand the parameterizations $r^{u,s}(u, \theta)$ in (23) cannot be extended up to the unbounded domains $u \in D_{\kappa,\beta}^{u,s}$. On the other hand, the characterization of the unstable and stable manifolds is given for $u \rightarrow \pm\infty$. To overcome this disagreement, we deal separately with the unstable or the stable manifold of the equilibrium points $S_-(\delta, \sigma)$ or $S_+(\delta, \sigma)$, respectively, of the vector field X in (16).

4.1.1 Setting and Result of the Existence of Invariant Manifolds

We perform two different linear changes of variables, C^u and C^s , to the vector field X , such that they put

- the equilibrium points $S_{\mp}(\delta, \sigma)$ at $(0, 0, \mp 1)$ and
- the linear part $DX(S_{\mp}(\delta, \sigma), \delta, \delta\sigma)$ in their Jordan form.

Here have taken the sign $-$ for the $-u-$ case and $+$ in the case $-s-$.

Lemma 4.1 *Let $|\sigma| \leq \delta^{p+3}\sigma^*$. We write $\hat{S}_{\mp} = (0, 0, \mp 1)$ and $\zeta = (x, y, z)$. The two equilibrium points $S_{\mp}(\delta, \sigma) = (x_{\mp}(\delta, \sigma), y_{\mp}(\delta, \sigma), z_{\mp}(\delta, \sigma))$ of the vector field X in (16) are of the form:*

$$x_{\mp}(\delta, \sigma) = \mathcal{O}(\delta^{p+5}), \quad y_{\mp}(\delta, \sigma) = \mathcal{O}(\delta^{p+5}), \quad z_{\mp}(\delta, \sigma) = \pm 1 + \mathcal{O}(\delta^{p+4}).$$

There are two linear changes of variables C^u and C^s of the form

$$\hat{\zeta} = C^{u,s}(\zeta, \delta, \sigma) = M_{\mp}(\delta, \sigma)(\zeta + S_{\mp}(\delta, \sigma)) - \hat{S}_{\mp}, \tag{107}$$

where $M_{\mp}(\delta, \sigma) = \text{Id} + \mathcal{O}(\delta^{p+5})$ (and therefore $C^{u,s} = \text{Id} + \mathcal{O}(\delta^{p+4})$), such that

$$\frac{d\hat{\zeta}}{dt} = X^{u,s}(\delta\hat{\zeta}, \delta, \delta\sigma) = X_0(\hat{\zeta}, \delta, \sigma) + \delta^p X_1^{u,s}(\delta\hat{\zeta}, \delta, \delta\sigma), \tag{108}$$

with X_0 the same as in (16) and

1. the vector field $X_1^{u,s}(\delta\hat{\zeta}, \delta, \delta\sigma) = \mathcal{O}_3(\delta\hat{\zeta}, \delta, \delta\sigma)$ and it is real analytic in $B^3(\hat{r}_0) \times B(\hat{\delta}_0) \times B(\hat{\sigma}_0) \subset \mathbb{C}^3 \times \mathbb{C}^2$.
2. $\hat{S}_{\mp} = (0, 0, \mp 1)$ are equilibrium points of $X^{u,s}$, respectively, and the linear part is in real Jordan form, that is

$$X_1^u(\delta\hat{S}_-, \delta, \delta\sigma) = X_1^s(\delta\hat{S}_+, \delta, \delta\sigma) = 0,$$

$$DX_1^{u,s}(\delta\hat{S}_{\mp}, \delta, \delta\sigma) = \begin{pmatrix} \mathcal{O}(\delta^{p+3}) & \mathcal{O}(\delta^{p+3}) & 0 \\ \mathcal{O}(\delta^{p+3}) & \mathcal{O}(\delta^{p+3}) & 0 \\ 0 & 0 & \mathcal{O}(\delta^{p+3}) \end{pmatrix}.$$

Proof The proof of this result can be encountered in Castejón (2015) and uses that the vector field X is written up to its normal form of order three, see Remark 2.1. \square

Now we perform the symplectic cylindric change (17) to system (108):

$$\begin{aligned} \frac{dr}{dt} &= 2r(\sigma - dz) + \delta^p \mathbf{F}^{u,s}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p \mathbf{G}^{u,s}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p \mathbf{H}^{u,s}(\delta r, \theta, \delta z, \delta, \delta\sigma), \end{aligned}$$

where $\mathbf{X}_1^{u,s} = (\mathbf{F}^{u,s}, \mathbf{G}^{u,s}, \mathbf{H}^{u,s})$ is defined as

$$\mathbf{X}_1^{u,s}(\delta r, \theta, \delta z, \delta, \delta\sigma) = \begin{pmatrix} \sqrt{2r} \cos \theta & \sqrt{2r} \sin \theta & 0 \\ -\frac{1}{\sqrt{2r}} \sin \theta & \frac{1}{\sqrt{2r}} \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} X_1^{u,s}(\delta\hat{\zeta}, \delta, \delta\sigma) \quad (109)$$

and $\hat{\zeta}$ denotes $\hat{\zeta} = (\sqrt{2r} \cos \theta, \sqrt{2r} \sin \theta, z)$.

The invariant manifolds associated with $\hat{S}_{\mp} = (0, 0, \mp 1)$ will be parameterized as:

$$r = R^{u,s}(v, \theta), \quad z = Z_0(v), \quad v \in \mathbb{R}, \theta \in \mathbb{S}^1$$

with $R^{u,s}(v, \theta) \rightarrow 0$ as $v \rightarrow \mp\infty$, respectively. As the equilibrium points are \hat{S}_{\mp} , one has

$$(\sqrt{2R^{u,s}(v, \theta)} \cos \theta, \sqrt{2R^{u,s}(v, \theta)} \sin \theta, Z_0(v)) \rightarrow \hat{S}_{\mp} \quad \text{as } v \rightarrow \mp\infty.$$

We look for the parameterizations $R^{u,s}$ of the form $R^{u,s} = R_0 + R_1^{u,s}$. We introduce the analogous notation to the one in (24):

$$\begin{aligned} \bar{X}_1^{u,s}(R)(v, \theta) &= \mathbf{X}_1^{u,s}(\delta(R_0(v) + R(v, \theta)), \theta, \delta Z_0(v), \delta, \delta\sigma), \\ \bar{X}_1^{u,s} &= (F^{u,s}, G^{u,s}, H^{u,s}). \end{aligned} \quad (110)$$

As we did in Sect. 2.2, we will omit the dependence on (v, θ) if there is no danger of confusion. One can easily check that $R_1^{u,s}$ have to satisfy the invariance equation given by $\mathcal{L}(R_1^{u,s}) = \mathcal{F}^{u,s}(R_1^{u,s})$, where \mathcal{L} is the linear differential operator in (27):

$$\mathcal{L}(R) := \left(-\delta^{-1}\alpha - cZ_0(v)\right) \partial_\theta R + \partial_v R - 2Z_0(v)R$$

and $\mathcal{F}^{u,s}$ are:

$$\begin{aligned} \mathcal{F}^{u,s}(R) := & 2\sigma(R_0(v) + R) + \delta^p F^{u,s}(R) + \delta^p \frac{d+1}{b} Z_0(v)H^{u,s}(R) \\ & - \delta^p G^{u,s}(R)\partial_\theta R - \left(\frac{2bR + \delta^p H^{u,s}(R)}{d(1 - Z_0^2(v))}\right) \partial_v R. \end{aligned} \tag{111}$$

The functions $R^{u,s} = R_0 + R_1^{u,s}$ lead to parameterizations of the invariant manifolds if $R_1^{u,s}$ satisfy, respectively:

$$\mathcal{L}(R_1^u) = \mathcal{F}^u(R_1^u), \quad \lim_{v \rightarrow -\infty} R_1^u(v, \theta) = 0, \tag{112}$$

$$\mathcal{L}(R_1^s) = \mathcal{F}^s(R_1^s), \quad \lim_{v \rightarrow +\infty} R_1^s(v, \theta) = 0. \tag{113}$$

Problems (112) and (113) can be written as fixed point equations using suitable right inverses of the operator \mathcal{L} . These right inverses can be found easily solving the ordinary differential equations satisfied by the Fourier coefficients $R^{[l]}(v)$ of any periodic function in θ , $R(v, \theta)$ that is a solution of $\mathcal{L}(R) = \phi$, for a given function ϕ . Indeed, given $\phi(v, \theta)$, we define:

$$\mathcal{G}^*(\phi)(v, \theta) := \sum_{l \in \mathbb{Z}} \mathcal{G}^{*[l]}(\phi)(v) e^{il\theta}, \quad * = u, s,$$

with $\mathcal{G}^{*[l]}$ as:

$$\begin{aligned} \mathcal{G}^{*[l]}(\phi)(v) = & \cosh^{\frac{2}{d}}(dv) \int_{\mp\infty}^0 \frac{e^{-il\delta^{-1}(\eta^\mp(v+s) - \eta^\mp(v))}}{\cosh^{\frac{2}{d}}(d(v+s))} \phi^{[l]}(v+s) ds, \\ & * = u, s, \end{aligned} \tag{114}$$

where, on the one hand, we take $-$ sign in the unstable case and $+$ in the stable one and on the other hand, we have used the notation η^\mp introduced in Theorem 2.4:

$$\eta^\mp(w) = \alpha w \mp \delta(cw \mp cd^{-1} \log(1 + e^{\pm 2dw})).$$

We stress that a compact expression for \mathcal{G}^* is given by:

$$\begin{aligned} \mathcal{G}^*(\phi)(v, \theta) = & \cosh^{\frac{2}{d}}(dv) \int_{\mp\infty}^u \frac{\phi(w, \theta - \delta^{-1}(\eta^\mp(w) - \eta^\mp(u)))}{\cosh^{\frac{2}{d}}(dw)} dw, \\ & * = u, s, \end{aligned} \tag{115}$$

Remark 4.2 Using that, if w is real $\eta^\mp(w) = \alpha w + cd^{-1} \log(2 \cosh w)$ one obtains the more natural expression for the Fourier coefficients:

$$\mathcal{G}^{*[l]}(\phi)(v) = \cosh^{\frac{2}{d}}(dv) \int_{\mp\infty}^0 \frac{e^{-il(\delta^{-1}\alpha s + \frac{c}{d} \log \frac{\cosh(d(v+s))}{\cosh(dv)})}}{\cosh^{\frac{2}{d}}(d(v+s))} \phi^{[l]}(v+s) ds, \tag{116}$$

* = u, s.

However, expressions in (116) are not well defined when we take complex values of v . For this reason we take definitions (114), which for real values of v coincide with the ones in (116), and are well defined when we take $v \in D_{\kappa,\beta}^{u,s}$.

Lemma 4.3 *One has $\mathcal{L} \circ \mathcal{G}^{u,s} = \text{Id}$. Moreover, if we define the operators:*

$$\tilde{\mathcal{F}}^{u,s} := \mathcal{G}^{u,s} \circ \mathcal{F}^{u,s},$$

with $\mathcal{F}^{u,s}$ given in (111), we have that if $R_1^{u,s}$ satisfy the fixed point equations:

$$R_1^u = \tilde{\mathcal{F}}^u(R_1^u), \quad R_1^s = \tilde{\mathcal{F}}^s(R_1^s), \tag{117}$$

then they are solutions of problems (112) and (113), respectively.

Since we will find solutions of the fixed point equation (117) by means of the fixed point theorem, we now set the Banach spaces we work with. For the unstable case we will consider functions $\phi : D_{\kappa,\beta}^u \times \mathbb{T}_\omega \rightarrow \mathbb{C}$, where the domain $D_{\kappa,\beta}^u$ is defined in (30) and \mathbb{T}_ω is defined in (31) (see also Fig. 5). They can be written in their Fourier series: $\phi(v, \theta) = \sum_{l \in \mathbb{Z}} \phi^{[l]}(v) e^{il\theta}$. We define the norms:

$$\begin{aligned} \|\phi^{[l]}\|_{n,m}^u &= \sup_{v \in D_{\kappa,\beta,T}^u} \left| \cosh^n(dv) \phi^{[l]}(v) \right| + \sup_{v \in D_{\kappa,\beta,\infty}^u} \left| \cosh^m(dv) \phi^{[l]}(v) \right|, \\ \|\phi\|_{n,m,\omega}^u &= \sum_{l \in \mathbb{Z}} \|\phi^{[l]}\|_{n,m}^u e^{|l|\omega}, \\ \|\phi\|_{n,m,\omega}^u &= \|\phi\|_{n,m,\omega}^u + \|\partial_v \phi\|_{n+1,m,\omega}^u + \delta^{-1} \|\partial_\theta \phi\|_{n+1,m,\omega}^u, \end{aligned}$$

and we consider the Banach spaces:

$$\begin{aligned} \mathcal{X}_{n,m}^u &:= \left\{ \phi : D_{\kappa,\beta}^u \rightarrow \mathbb{C} : \phi \text{ is analytic and } \|\phi\|_{n,m}^u < +\infty \right\}, \\ \mathcal{X}_{n,m,\omega}^u &:= \left\{ \phi : D_{\kappa,\beta}^u \times \mathbb{T}_\omega \rightarrow \mathbb{C} : \phi \text{ is analytic and } \|\phi\|_{n,m,\omega}^u < +\infty \right\}, \\ \tilde{\mathcal{X}}_{n,m,\omega}^u &:= \left\{ \phi : D_{\kappa,\beta}^u \times \mathbb{T}_\omega \rightarrow \mathbb{C} : \phi \text{ is analytic and } \|\phi\|_{n,m,\omega}^u < +\infty \right\}. \end{aligned}$$

For functions $\Phi = (\phi_1, \phi_2, \phi_3) \in \mathcal{X}_{n,m,\omega}^{u,\times} := \mathcal{X}_{n,m,\omega}^u \times \mathcal{X}_{n,m,\omega}^u \times \mathcal{X}_{n,m,\omega}^u$, belonging to the product space, we will take the norm:

$$\|\Phi\|_{n,m,\omega}^{u,\times} := \max \left\{ \|\phi_1\|_{n,m,\omega}^u, \|\phi_2\|_{n,m,\omega}^u, \|\phi_3\|_{n,m,\omega}^u \right\}.$$

For the stable case, we consider norms and Banach spaces analogously defined in the corresponding domains $D_{\kappa,\beta}^s$ and for $\phi : \mathbb{T}_\omega \rightarrow \mathbb{C}$, we will take the Fourier norm: $\|\phi\|_\omega := \sum_{l \in \mathbb{Z}} |\phi^{[l]}| e^{|l|\omega}$.

Now we can state the result which guarantees the existence of solutions of the fixed point equations (117). We will devote the rest of the section to prove it. During this section we will modify the value of the parameters κ, β, T and ω of the domains $D_{\kappa,\beta}^u \times \mathbb{T}_\omega$ and $D_{\kappa,\beta}^s \times \mathbb{T}_\omega$ a finite number of times. We will abuse notation and use the same letters for the modified values.

Proposition 4.4 *Let $p \geq -2$ and $0 < \beta < \pi/2$ be any constants. There exist $\sigma^*, \delta^* > 0$ and $\kappa^* \geq 1$, such that for all $0 < \delta \leq \delta^*$ if $\kappa = \kappa(\delta)$ satisfies condition (29) and σ satisfies $|\sigma| \leq \sigma^* \delta^{p+3}$, the fixed point equations in (117) have solutions $R_1^{u,s}$ defined, respectively, in $D_{\kappa,\beta}^{u,s} \times \mathbb{T}_\omega$.*

Moreover, they satisfy that $R_1^{u,s} = R_{10}^{u,s} + R_{11}^{u,s}$ with the following properties:

1. $R_{10}^{u,s} = \tilde{\mathcal{F}}^{u,s}(0) \in \tilde{\mathcal{X}}_{3,2,\omega}^{u,s}$ and there exists $M > 0$:

$$\|R_{10}^{u,s}\|_{3,2,\omega}^{u,s} \leq M\delta^{p+3}.$$

2. $R_{11}^{u,s} \in \tilde{\mathcal{X}}_{4,2,\omega}^{u,s}$, and there exists a constant M such that:

$$\|R_{11}^{u,s}\|_{4,2,\omega}^{u,s} \leq M\delta^{p+3} \|R_{10}^{u,s}\|_{3,2,\omega}^{u,s}.$$

This result yields the following corollary:

Corollary 4.5 *Under the same assumptions of Proposition 4.4, the two-dimensional invariant manifolds of system (108) can be parameterized, in symplectic polar coordinates as:*

$$r = R^{u,s}(v, \theta) = R_0(v) + R_1^{u,s}(v, \theta), \quad z = Z_0(v), \quad (v, \theta) \in D_{\kappa,\beta}^{u,s} \times \mathbb{T}_\omega,$$

with $R_1^{u,s}$ satisfying the properties in Proposition 4.4.

In the following we will sketch the proof of Proposition 4.4 in the unstable case.

4.1.2 Solutions of the Fixed Point Equation: Proof of Proposition 4.4

First, in Lemma 4.6, we study the behavior of the linear operator $\mathcal{G}^{u,s}$ acting on functions ϕ belonging to the Banach spaces $\mathcal{X}_{n,m,\omega}^u$. Secondly, in Lemma 4.8, we deal with the independent term $\tilde{\mathcal{F}}^u(0)$. Finally, in Lemma 4.9, we check that the operator $\tilde{\mathcal{F}}^{u,s}$ is a contraction.

Lemma 4.6 *Let $n \geq 0, m \geq 0$ and $\phi \in \mathcal{X}_{n,m,\omega}^u$. There exists a constant M such that for all $l \in \mathbb{Z}$:*

1. If $n \geq 1$, then $\|\mathcal{G}^{u[l]}(\phi)\|_{n-1,m}^u \leq M \|\phi^{[l]}\|_{n,m}^u$.

2. If $l \neq 0$ and $n \geq 0$, then $\|\mathcal{G}^{u[l]}(\phi)\|_{n,m}^u \leq \frac{\delta M \|\phi^{[l]}\|_{n,m}^u}{|l|}$.
3. As a consequence we have that if $n \geq 1$, $\|\mathcal{G}^u(\phi)\|_{n-1,m,\omega}^u \leq M \|\phi\|_{n,m,\omega}^u$. Moreover, if $\phi^{[0]}(v) = 0$, then for all $n \geq 0$:

$$\|\mathcal{G}^u(\phi)\|_{n,m,\omega}^u \leq M\delta \|\phi\|_{n,m,\omega}^u.$$

4. If $n \geq 0$, $\|\partial_\theta \mathcal{G}^u(\phi)\|_{n,m,\omega}^u \leq M\delta \|\phi\|_{n,m,\omega}^u$.
5. If $n \geq 1$, $\|\partial_v \mathcal{G}^u(\phi)\|_{n,m,\omega}^u \leq M \|\phi\|_{n,m,\omega}^u$.

In conclusion, if $\phi \in \mathcal{X}_{n,m,\omega}^u$, $n \geq 1$, then $\mathcal{G}^u(\phi) \in \tilde{\mathcal{X}}_{n-1,m,\omega}^u$ and:

$$\|\mathcal{G}^u(\phi)\|_{n-1,m,\omega}^u \leq M \|\phi\|_{n,m,\omega}^u.$$

Sketch of the proof The main idea to prove this result is to redefine adequately the Fourier coefficients $\mathcal{G}^{u[l]}(\phi)$ changing the path of integration. Take $v \in D_{\kappa,\beta}^u$ fixed and consider $s = s_\pm(t, v)$ defined implicitly by (see Fig. 9):

$$s_\pm - \frac{c\delta}{\alpha} s_\pm + \frac{c\delta}{d\alpha} \left(\log \left(1 + e^{2d(v+s_\pm)} \right) - \log \left(1 + e^{2dv} \right) \right) = -t e^{\pm i \frac{\beta}{2}}.$$

It can be proven that the function $s_\pm(t, v)$ is well defined for all $t \in [0, +\infty)$ and $v \in D_{\kappa,\beta}^u$ and moreover that $v + s_\pm(t, v) \in D_{\kappa,\beta}^u$. Consider the curve (see Fig. 9):

$$\Gamma_\pm^R := \{z \in \mathbb{C} : z = s_\pm(t, v), t \in [0, R]\}.$$

Then, one can prove that, if $m > 0$ and $\phi \in \mathcal{X}_{n,m,\omega}^u$ one has:

$$\mathcal{G}^{u[l]}(\phi)(v) = - \lim_{R \rightarrow +\infty} \int_{\Gamma_\pm^R} \frac{e^{-il\delta^{-1}(\eta^-(v+z) - \eta^-(v))}}{\cosh^{\frac{2}{d}}(d(v+z))} \phi^{[l]}(v+z) dz,$$

where the coefficients $\mathcal{G}^{u[l]}$ were defined in (114), and we take the integral over Γ_+^R for $l \geq 0$ and over Γ_-^R otherwise.

The proof of Lemma 4.6 follows now from standard arguments. □

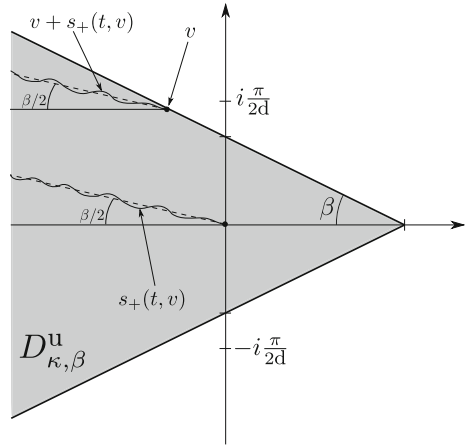
Now we are going to bound the independent term of the fixed point equation (117) which is $\tilde{\mathcal{F}}^u(0) = \mathcal{G}^u \circ \mathcal{F}^u(0)$ (see (111)) with

$$\mathcal{F}^u(0) = 2\sigma R_0(v) + \delta^p F^u(0) + \delta^p \frac{d+1}{b} Z_0(v) H^u(0).$$

Lemma 4.7 *Let C_R be some constant, and R such that $\|R\|_{2,2,\omega} \leq C_R$. There exists a constant M such that:*

$$\|F^u(R - R_0)\|_{4,2,\omega}^u, \|G^u(R - R_0)\|_{2,0,\omega}^u, \|H^u(R - R_0)\|_{3,2,\omega}^u \leq M\delta^3,$$

Fig. 9 The domain $D_{\kappa, \beta}^u$ with an example of the curves $s_+(t, v)$ and $v + s_+(t, v)$. The discontinuous lines are $-te^{i\beta/2}$ and $v - te^{i\beta/2}$, respectively



with F^u, G^u, H^u defined in (110). In particular, this holds for $R = R_0$.

Proof We will use the properties of $X_1^u := (f^u, g^u, h^u)$ stated in Lemma 4.1. We first prove the bound for F^u being the one for G^u analogous. By definitions (110) and (109)

$$\begin{aligned} F^u(R - R_0)(v, \theta) &= \mathbf{F}^u(\delta R(v, \theta), \theta, \delta Z_0(v), \delta, \delta\sigma) \\ &= \sqrt{2R(v, \theta)} \cos \theta f^u(\Phi(v, \theta), \delta, \delta\sigma) + \sqrt{2R(v, \theta)} \sin \theta g^u(\Phi(v, \theta), \delta, \delta\sigma). \end{aligned}$$

with

$$\Phi(v, \theta) = (\delta\sqrt{2R(v, \theta)} \cos \theta, \delta\sqrt{2R(v, \theta)} \sin \theta, \delta Z_0(v)).$$

By Lemma 4.1, f^u is of order three in all their variables and $f^u(0, 0, -\delta, \delta, \delta\sigma) = 0$ for all δ and σ . Therefore, since $\|\sqrt{2R}\|_{1,1,\omega} < +\infty$ and $Z_0(u) = \tanh(du)$, we have that

$$\|f^u(\Phi(v, \theta), \delta, \delta\sigma)\|_{3,1,\omega}^u \leq K\delta^3.$$

Reasoning analogously, we obtain the same bound for g^u , and thus,

$$\|F^u(R - R_0)\|_{4,2,\omega}^u \leq K\delta^3.$$

With respect to H^u , we have:

$$H^u(R - R_0)(v, \theta) = \mathbf{H}^u(\delta R(v, \theta), \theta, \delta Z_0(v), \delta, \delta\sigma) = h^u(\Phi(v, \theta), \delta, \delta\sigma).$$

Again, h^u is of order three in all their variables, $h^u(0, 0, -\delta, \delta, \delta\sigma) = 0$ and moreover $\partial_x h^u(0, 0, -\delta, \delta, \delta\sigma) = \partial_y h^u(0, 0, -\delta, \delta, \delta\sigma) = 0$. The bound for $H^u(R - R_0)$ follows by using the same arguments as above. \square

Now we deal with the independent term $\tilde{\mathcal{F}}^u(0)$.

Lemma 4.8 *Let $|\sigma| \leq \sigma^* \delta^{p+3}$. There exists $M > 0$ such that $\|\tilde{\mathcal{F}}^u(0)\|_{3,2,\omega}^u \leq M \delta^{p+3}$.*

Proof By Lemma 4.6 it is enough to prove that $\|\mathcal{F}^u(0)\|_{4,2,\omega}^u \leq M \delta^{p+3}$ (recall that $\tilde{\mathcal{F}}^u = \mathcal{G}^u \circ \mathcal{F}^u$). This is clear, by Lemma 4.7:

$$\begin{aligned} \|\mathcal{F}^u(0)\|_{4,2,\omega}^u &\leq \|2\sigma R_0\|_{4,2,\omega}^u + \delta^p \|F^u(0)\|_{4,2,\omega}^u + \delta^p \frac{d+1}{b} \|Z_0 \cdot H^u(0)\|_{4,2,\omega}^u \\ &\leq K (\sigma + \delta^p \|F^u(0)\|_{4,2,\omega}^u + \delta^p \|Z_0\|_{1,0}^u \|H^u(0)\|_{3,2,\omega}^u) \\ &\leq K (\sigma + \delta^{p+3}) \leq K \delta^{p+3}. \end{aligned}$$

□

We prove now that the operator $\tilde{\mathcal{F}}^u$ is contractive in an appropriate Banach space. More precisely we prove the following result:

Lemma 4.9 *Let $|\sigma| \leq \sigma^* \delta^{p+3}$. Assume that $\phi_1, \phi_2 \in \tilde{\mathcal{X}}_{3,2,\omega}^u$ satisfy for $C > 0$ and $i = 1, 2$ that $\|\phi_i\|_{3,2,\omega}^u \leq C \delta^{p+3}$. Then there exists $M > 0$ such that:*

$$\|\tilde{\mathcal{F}}^u(\phi_1) - \tilde{\mathcal{F}}^u(\phi_2)\|_{4,2,\omega}^u \leq M \delta^{p+3} \|\phi_1 - \phi_2\|_{3,2,\omega}^u.$$

Proof We skip tedious computations and only give an sketch of the proof. See details in Castejón (2015). Using that \mathcal{G}^u is linear and Lemma 4.6:

$$\|\tilde{\mathcal{F}}^u(\phi_1) - \tilde{\mathcal{F}}^u(\phi_2)\|_{4,2,\omega}^u \leq M \|\mathcal{F}^u(\phi_1) - \mathcal{F}^u(\phi_2)\|_{5,2,\omega}^u.$$

It is only necessary to prove that if $\|\phi_i\|_{3,2,\omega}^u \leq C \delta^{p+3}$, then:

$$\|\mathcal{F}^u(\phi_1) - \mathcal{F}^u(\phi_2)\|_{5,2,\omega}^u \leq K \delta^{p+3} \|\phi_1 - \phi_2\|_{3,2,\omega}^u. \tag{118}$$

We decompose the operator \mathcal{F}^u in (111) into $\mathcal{F}^u = \mathcal{F}_1^u + \mathcal{F}_2^u + \mathcal{F}_3^u + \mathcal{F}_4^u$ with

$$\begin{aligned} \mathcal{F}_1^u(\phi) &= 2\sigma(R_0(v) + \phi) + \delta^p F^u(\phi) + \delta^p \frac{d+1}{b} Z_0(v) H^u(\phi) \\ \mathcal{F}_2^u(\phi) &= -\delta^p G^u(\phi) \partial_\theta \phi \\ \mathcal{F}_3^u(\phi) &= -\delta^p \frac{1}{d(1 - Z_0^2(v))} H^u(\phi) \partial_v \phi \\ \mathcal{F}_4^u(\phi) &= \frac{2b}{d(1 - Z_0^2(v))} \phi \partial_v \phi. \end{aligned}$$

Proceeding as in the proof of Lemma 4.7, one can prove (after tedious but easy computations):

$$\begin{aligned} \|F^u(\phi_1) - F^u(\phi_2)\|_{5,2,\omega}^u &\leq M\delta^3 \|\phi_1 - \phi_2\|_{3,2,\omega}^u, \\ \|G^u(\phi_1) - G^u(\phi_2)\|_{3,0,\omega}, \|H^u(\phi_1) - H^u(\phi_2)\|_{4,2,\omega} &\leq M\delta^3 \|\phi_1 - \phi_2\|_{3,2,\omega}. \end{aligned} \quad (119)$$

As a consequence, the operator \mathcal{F}_1^u satisfies the bound in (118) since $\|Z_0\|_{1,0} \leq K$. With respect to \mathcal{F}_2^u , we write:

$$\mathcal{F}_2^u(\phi_1) - \mathcal{F}_2^u(\phi_2) = (G^u(\phi_1) - G^u(\phi_2))\partial_\theta\phi_1 + G^u(\phi_2)\partial_\theta(\phi_1 - \phi_2).$$

Then, since $\phi_i \in \tilde{\mathcal{X}}_{3,2,\omega}^u$,

$$\begin{aligned} \|\mathcal{F}_2^u(\phi_1) - \mathcal{F}_2^u(\phi_2)\|_{5,2,\omega}^u &\leq \|G^u(\phi_1) - G^u(\phi_2)\|_{3,0,u}^u \|\partial_\theta\phi_1\|_{2,2,\omega}^u \\ &\quad + \|G^u(\phi_2)\|_{2,0,\omega}^u \|\partial_\theta(\phi_1 - \phi_2)\|_{3,2,\omega}^u. \end{aligned} \quad (120)$$

Now we note that:

$$\|\partial_\theta\phi_1\|_{2,2,\omega}^u \leq K \frac{\|\partial_\theta\phi_1\|_{4,2,\omega}^u}{\delta^2\kappa^2} \leq K \frac{\|\phi_1\|_{3,2,\omega}^u}{\delta\kappa^2} \leq K \frac{\delta^{p+2}}{\kappa^2} \leq K. \quad (121)$$

and:

$$\|\partial_\theta(\phi_1 - \phi_2)\|_{3,2,\omega}^u \leq \frac{K}{\delta\kappa} \|\partial_\theta(\phi_1 - \phi_2)\|_{4,2,\omega}^u \leq \frac{K}{\kappa} \|\phi_1 - \phi_2\|_{3,2,\omega}^u. \quad (122)$$

Note that since $\|\phi_i\|_{3,2,\omega}^u \leq C\delta^{p+3}$, then $\|\phi_i\|_{2,2,\omega}^u \leq C\kappa^{-1}\delta^{p+2}$. Therefore, Lemma 4.7 with $R = \phi_2$ applies. Thus, using this lemma and the bounds in (121), (122) and (119) in inequality (120), we also obtain that the operator \mathcal{F}_2^u satisfies bound in (118).

We leave to the reader to check that \mathcal{F}_3^u and \mathcal{F}_4^u also satisfy bound (118) using the fact that $\|(1 - Z_0^2)^{-1}\|_{-2,-2,\omega}^u \leq K$. \square

End of proof of Proposition 4.4 Proposition 4.4 is a corollary of Lemmas 4.8 and 4.9. Indeed, let $p \geq -2$ and $|\sigma| \leq \sigma^*\delta^{p+3}$. Define $\varrho := 2\|\tilde{\mathcal{F}}^u(0)\|_{3,2,\omega}^u$ and $B(\varrho) \subset \tilde{\mathcal{X}}_{3,2,\omega}^u$ the ball of radius ϱ centered at zero.

We claim that $\tilde{\mathcal{F}}^u$ has a unique fixed point in $B(\varrho)$. Indeed, we point out that $\varrho \leq K\delta^{p+3}$ by Lemma 4.8. We first check that $\tilde{\mathcal{F}}^u$ is contractive. By the properties of the norm $\|\cdot\|_{n,m,\omega}^u$ and Lemma 4.9, for $\phi_1, \phi_2 \in B(\varrho)$:

$$\|\tilde{\mathcal{F}}^u(\phi_1) - \tilde{\mathcal{F}}^u(\phi_2)\|_{3,2,\omega}^u \leq \frac{K}{\delta\kappa} \|\tilde{\mathcal{F}}^u(\phi_1) - \tilde{\mathcal{F}}^u(\phi_2)\|_{4,2,\omega}^u \leq \frac{K\delta^{p+2}}{\kappa} \|\phi_1 - \phi_2\|_{3,2,\omega}^u.$$

Clearly, since $p \geq -2$ and κ^* is large enough, $\tilde{\mathcal{F}}^u$ is contractive in $B(\varrho)$.

It remains to check that $\tilde{\mathcal{F}}^u : B(\varrho) \rightarrow B(\varrho)$. If $\phi \in B(\varrho)$, by Lemma 4.9:

$$\|\tilde{\mathcal{F}}^u(\phi)\|_{3,2,\omega}^u \leq \|\tilde{\mathcal{F}}^u(\phi) - \tilde{\mathcal{F}}^u(0)\|_{3,2,\omega}^u + \|\tilde{\mathcal{F}}^u(0)\|_{3,2,\omega}^u \leq K \frac{\delta^{p+2}}{\kappa} \|\phi\|_{3,2,\omega}^u + \frac{1}{2}\varrho.$$

Taking $\kappa^* \leq \kappa$ large enough, $\|\tilde{\mathcal{F}}^u(\phi)\|_{3,2,\omega}^u < \varrho$. That is, $\tilde{\mathcal{F}}^u : B(\varrho) \rightarrow B(\varrho)$. Therefore, by the fixed point theorem $\tilde{\mathcal{F}}$ has a unique fixed point in $B(\varrho)$.

It is clear that $R_1^u \in B(\varrho)$ is the fixed point of $\tilde{\mathcal{F}}^u$ obtained before. Then, item 1 of Proposition 4.4 is a direct consequence of Lemma 4.8. To prove item 2, we just need to note that: $R_{11}^u = R_1^u - R_{10}^u = \tilde{\mathcal{F}}^u(R_1^u) - \tilde{\mathcal{F}}^u(0)$. Using Lemma 4.9 and that $\|R_1^u\|_{3,2,\omega}^u \leq K \|R_{10}^u\|_{3,2,\omega}^u$, we obtain that:

$$\|R_{11}^u\|_{4,2,\omega}^u \leq K \delta^{p+3} \|R_1^u\|_{3,2,\omega}^u \leq K \delta^{p+3} \|R_{10}^u\|_{3,2,\omega}^u.$$

□

4.2 Suitable Complex Parameterizations: Theorem 2.4

In this section we shall prove Theorem 2.4 concerning the functions r_1^u and r_1^s . The fact that R_1^u and R_1^s satisfy different equations is not adequate for our purposes of comparing them. We will now proceed to obtain new parameterizations $r^{u,s} = R_0 + r_1^{u,s}$ of the invariant manifolds which will be solutions of the same functional equation. To obtain such a parameterizations we i) undo the changes (107) until we get a parameterization of system (15) and ii) perform the symplectic polar change of coordinates.

The technical proofs can be encountered in Castejón (2015).

4.2.1 Setting

Let $R_1^{u,s}$ be the functions given by Proposition 4.4. We consider $R^{u,s} = R_0 + R_1^{u,s}$ and we introduce the parameterizations of the invariant manifolds of the equilibrium points $\hat{S}_{\mp} = (0, 0, \mp 1)$ of the vector field $X^{u,s}$ in (108):

$$\hat{\xi}^{u,s}(v, \theta) := (\sqrt{2R^{u,s}(v, \theta)} \cos \theta, \sqrt{2R^{u,s}(v, \theta)} \sin \theta, Z_0(v)). \tag{123}$$

We define:

$$\xi^{u,s}(v, \theta) := (C^{u,s})^{-1} \hat{\xi}^{u,s}(v, \theta) = (x^{u,s}(v, \theta), y^{u,s}(v, \theta), z^{u,s}(v, \theta)) \tag{124}$$

where $C^{u,s}$ are given in (107). These are parameterizations of the two-dimensional unstable (respectively, stable) manifold associated with the equilibrium points $S_{\mp}(\delta, \sigma)$ of the original system (15).

To compare $(x^u(v, \theta), y^u(v, \theta))$ and $(x^s(v, \theta), y^s(v, \theta))$ on the z -plane (or equivalently in the u -plane given by $z = Z_0(u)$), we implicitly define the functions $v^{u,s}$ as:

$$Z_0(u) = z^u(v^u(u, \theta), \theta), \quad Z_0(u) = z^s(v^s(u, \theta), \theta).$$

The result about the existence of functions $v^{u,s}$ is given below. Its proof is an elementary application of the fixed point theorem.

Lemma 4.10 *Let $\kappa = \kappa(\delta)$ given in Proposition 4.4. Fix $m > 0$ a constant independent of δ and σ . Let $\bar{\kappa} = \bar{\kappa}(\delta)$ satisfying condition (29) and such that $\bar{\kappa} > \kappa + m$, and let $z^{u,s}(v, \theta)$ be the functions defined in (124) for $(v, \theta) \in D_{\kappa, \beta, T}^{u,s} \times \mathbb{T}_\omega$. Let \bar{T} be a constant such that $0 < \bar{T} < T$. Then, if δ is sufficiently small, the functions $v^{u,s}$ defined implicitly by:*

$$Z_0(u) = z^{u,s}(v^{u,s}(u, \theta), \theta)$$

are well defined for all $u \in D_{\bar{\kappa}, \beta, \bar{T}}^{u,s}$ and $\theta \in \mathbb{T}_\omega$, and there exists a constant M such that:

$$|v^{u,s}(u, \theta) - u| \leq M\delta^{p+4} |\cosh(du)|^2.$$

We take $m > 0$ fixed and $\kappa, \bar{\kappa}, T, \bar{T}$ and β as in Lemma 4.10.

Note that, as $\zeta^{u,s}(v, \theta)$ in (124), the functions $\zeta^{u,s}(v^{u,s}(u, \theta), \theta)$ are other parameterizations of the unstable and stable manifolds of $S_\mp(\delta, \sigma)$, respectively. We define $r^{u,s}(u, \theta)$ as:

$$r^{u,s}(u, \theta) = \frac{1}{2} \left[(x^{u,s}(v^{u,s}(u, \theta), \theta))^2 + (y^{u,s}(v^{u,s}(u, \theta), \theta))^2 \right]. \tag{125}$$

We claim that there exists $K > 0$ such that for all $(u, \theta) \in D_{\bar{\kappa}, \beta, \bar{T}}^{u,s} \times \mathbb{T}_\omega$,

$$r^{u,s}(u, \theta) = R_0(u) + R_1^{u,s}(u, \theta) + r_2^{u,s}(u, \theta),$$

$$|\cosh(du)r_2^{u,s}(u, \theta)| \leq K\delta^{p+4}, \tag{126}$$

where $R_1^{u,s}$ are given in Proposition 4.4. We deal only with the unstable case being the stable one analogous. We first begin by studying the difference between $\pi_{x,y}\hat{\zeta}^u$ and $\pi_{x,y}\zeta^u$ defined, respectively, in (123) and (124). We note that, from Lemma 4.1, $\pi_{x,y}S_-(\delta, \sigma) = \mathcal{O}(\delta^{p+5})$ and the matrices $M_-(\delta, \sigma)$ in the same lemma have inverse of the form $M_-^{-1}(\delta, \sigma) = \text{Id} + \delta^{p+5}\hat{M}_-^{-1}(\delta, \sigma)$ having $\hat{M}_-^{-1}(\delta, \sigma)$ bounded entries. We denote by $\pi_{x,y}\hat{M}_-^{-1}$ the two first rows of \hat{M}_-^{-1} . Then, using the form of the change C^u in (107):

$$\pi_{x,y}\zeta^u(v, \theta) = \pi_{x,y}\hat{\zeta}^u(v, \theta) + k\delta^{p+5} + \delta^{p+5}\pi_{x,y}\hat{M}_-^{-1}\hat{\zeta}^u(v, \theta)$$

for some bounded coefficients $k := k(\delta, \sigma)$. By Corollary 4.5:

$$|\hat{\zeta}^u(v, \theta)| \leq \frac{K}{|\cosh(dv)|}, \quad (v, \theta) \in D_{\kappa, \beta, T}^u \times \mathbb{T}_\omega \tag{127}$$

and therefore, if $(v, \theta) \in D_{\kappa, \beta, T}^u \times \mathbb{T}_\omega$:

$$\pi_{x,y} \zeta^u(v, \theta) = \pi_{x,y} \hat{\zeta}^u(v, \theta) + \mathcal{O}(\delta^{p+4}). \tag{128}$$

By definition (125) of $r^{u,s}$ we are interested in computing $\zeta^u(v^u(u, \theta), \theta)$. For that we also need to study the difference between $\hat{\zeta}^u(u, \theta)$ and $\hat{\zeta}^u(v^u(u, \theta), \theta)$. We emphasize that by Lemma 4.10, taking δ sufficiently small we can ensure that $v_\lambda^u(u, \theta) := v^u(u, \theta) + \lambda(v^u(u, \theta) - u) \in D_{\kappa, \beta, T}^u$ for $u \in D_{\bar{\kappa}, \beta, \bar{T}}^u$ and $\lambda \in [0, 1]$. Then using Proposition 4.4, Lemma 4.10 and the mean value theorem:

$$\begin{aligned} |\pi_{x,y} \hat{\zeta}^u(v^u(u, \theta), \theta) - \pi_{x,y} \hat{\zeta}^u(u, \theta)| &\leq \sup_{\lambda \in [0, 1]} |\partial_v \pi_{x,y} \hat{\zeta}^u(v_\lambda^u(u, \theta))| |v^u(u, \theta) - u| \\ &\leq K \delta^{p+4} \sup_{\lambda \in [0, 1]} \frac{|\cosh(du)|^2}{|\cosh(d(v_\lambda^u(u, \theta)))|^2} \leq K \delta^{p+4}. \end{aligned}$$

Using this expression in (128) one obtains:

$$\pi_{x,y} \zeta^u(v^u(u, \theta), \theta) = \pi_{x,y} \hat{\zeta}^u(u, \theta) + \mathcal{O}(\delta^{p+4}).$$

Then, recalling that $\pi_{x,y} \zeta^u = (x^u, y^u)$ and using (127), we obtain:

$$\begin{aligned} r^u(u, \theta) &= \frac{1}{2} \left[(\pi_x \hat{\zeta}^u(u, \theta))^2 + (\pi_y \hat{\zeta}^u(u, \theta))^2 \right] + \mathcal{O} \left(\frac{\delta^{p+4}}{\cosh(du)} \right) \\ &= R_0(u) + R_1^u(u, \theta) + \mathcal{O} \left(\frac{\delta^{p+4}}{\cosh(du)} \right) \end{aligned}$$

and (126) is proven. We introduce $r_1^u = R_1^u + r_2^u$ and therefore, r^u is of the form $r^u = R_0 + r_1^u$. Note that, by construction, r_1^u satisfies the partial differential equation (26).

In addition, by the compact expression of \mathcal{G}^u in (115), the dominant part, r_{10}^u , of r_1^u , given in Theorem 2.4 is $r_{10}^u = \mathcal{G}^u \circ \mathcal{F}(0)$, where \mathcal{F} is the operator defined in (28). Then, by using the expression for R_1^u in Proposition 4.4, we obtain the decomposition:

$$r_1^u = r_{10}^u + r_{11}^u, \quad r_{10}^u = \mathcal{G}^u \circ \mathcal{F}(0), \quad r_{11}^u = \mathcal{G}^u \circ (\mathcal{F}^u(0) - \mathcal{F}(0)) + R_{11}^u + r_2^u, \tag{129}$$

where we have used that the operator \mathcal{G}^u is linear.

4.2.2 End of the Proof of Theorem 2.4

It remains to prove the bounds for r_{10}^u and r_{11}^u in Theorem 2.4 on $D_{\bar{\kappa}, \beta, \bar{T}}^{u,s} \times \mathbb{T}_\omega$. For this, it is convenient to define the auxiliary norms for functions $\phi : D_{\kappa, \beta, T}^{u,s} \times \mathbb{T}_\omega \rightarrow \mathbb{C}$:

$$\|\phi\|_{n, \omega}^{\kappa, \beta, T} = \sup_{\substack{\theta \in \mathbb{T}_\omega \\ v \in D_{\kappa, \beta, T}^u}} |\cosh^n(dv) \phi(v, \theta)|$$

which satisfies $\|\phi\|_{n,\omega}^{\kappa,\beta,T} \leq \|\phi\|_{n,0,\omega}^u$ if $\phi \in \mathcal{X}_{n,0,\omega}^u$. Moreover, if $m > 0$, then

$$\|\partial_u \phi\|_{n+1,\bar{\omega}}^{\bar{\kappa},\bar{T},\bar{\beta}} \leq M \|\phi\|_{n,\omega}^{\kappa,T,\beta} \tag{130}$$

with $\bar{\kappa} > \kappa + m$ satisfying condition (29), $0 < \bar{\beta} < \beta$, $0 < \bar{T} < T$ and $0 < \bar{\omega} < \omega$. This fact can be checked as Lemma 4.3 in Baldomá (2006).

Along this proof we will use that, if $u \in D_{\kappa,\beta}^u$ (see (30)), then $|\cosh(du)| \geq K\delta$ for some constant K . We also use decomposition (129) without mentioning it explicitly.

It can be straightforwardly proven (see Castejón 2015) that there exists $M > 0$, such that

$$\|\mathcal{F}(0)\|_{4,0,\omega}^{u,s} \leq M\delta^{p+3}, \quad \|\mathcal{F}(0) - \mathcal{F}^{u,s}(0)\|_{2,0,\omega}^{u,s} \leq M\delta^{p+4}.$$

Therefore, using the properties of \mathcal{G}^u in Lemma 4.6 we obtain

$$\|\mathcal{G}^u \circ \mathcal{F}(0)\|_{3,0,\omega}^u \leq K\delta^{p+3}, \quad \|\mathcal{G}(\mathcal{F}(0) - \mathcal{F}^{u,s}(0))\|_{1,0,\omega}^u \leq K\delta^{p+4},$$

so that the bounds for r_{10}^u are done using that $\|\cdot\|_{n,\omega}^{\kappa,\beta,T} \leq \|\cdot\|_{n,0,\omega}^u$. In addition, with respect to R_{11}^u , using Proposition 4.4, we have $\|R_{11}^u\|_{4,2,\omega}^u \leq K\delta^{2p+6}$ and with respect to r_2^u defined in (126), $\|r_2^u\|_{1,\omega}^{\bar{\kappa},\beta,\bar{T}} \leq K\delta^{p+4}$. Therefore, using also property (130), we have that

$$\|R_{11}\|_{4,\omega}^{\kappa,T,\beta}, \|\partial_u R_{11}\|_{5,\omega}^{\kappa,T,\beta} \leq K\delta^{2p+6} \quad \|r_2\|_{1,\omega}^{\bar{\kappa},\beta,\bar{T}}, \|\partial_u r_2\|_{1,\omega}^{\bar{\kappa},\beta,\bar{T}} \leq K\delta^{p+4}.$$

We point out that we have abused notation, using the same $\bar{\kappa}$ and \bar{T} although they are different from the previous ones. However, they still satisfy $\bar{\kappa} - \kappa > m$ and $0 < \bar{T} < T$. Now we are almost done. Notice that by definition of r_{11}^u

$$\begin{aligned} |r_{11}^u(u, \theta)| &\leq |\cosh(du)| (\|\mathcal{G}(\mathcal{F}(0) - \mathcal{F}^{u,s}(0))\|_{1,0,\omega}^u + \|r_2\|_{1,\omega}^{\bar{\kappa},\beta,\bar{T}}) \\ &\quad + |\cosh(du)|^4 \|R_{11}^u\|_{4,2,\omega}^u \end{aligned}$$

and then, using the above bounds, the statement for r_{11}^u in Theorem 2.4 is checked. The bound $\|\partial_u r_{11}^u\|_{4,\bar{\omega}}^{\bar{\kappa},\beta,\bar{T}} \leq K\delta^{p+3}$ is straightforward from the above bounds for $\partial_u r_{10}^u = \partial_u \mathcal{G}^u \circ \mathcal{F}(0)$ and $\partial_u r_{11}^u$ and from definition $r_1^u = r_{10}^u + r_{11}^u$. Finally, using that r_1^u satisfies Eq. (26), we easily obtain $\|\partial_\theta r_1\|_{4,\bar{\omega}}^{\bar{\kappa},\beta,\bar{T}} \leq K\delta^{p+4}$.

5 The Difference $\Delta(u, \theta)$: Proof of Theorem 2.7

In this section we will prove Theorem 2.7. It is clear that the difference $\Delta(u, \theta) = r_1^u(u, \theta) - r_1^s(u, \theta)$ is defined on $(u, \theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$, where $D_{\kappa,\beta} = D_{\kappa,\beta,T}^u \cap D_{\kappa,\beta,T}^s$ (see Fig. 6) so this will be our domain from now on.

5.1 Preliminary Considerations

As we explained in Sect. 2.4, the difference $\Delta = r_1^s - r_1^u$ satisfies the linear PDE (41) and can be expressed in the form (50): $\Delta(u, \theta) = P(u, \theta)\tilde{k}(\xi(u, \theta))$, with

$$\xi(u, \theta) = \theta + \delta^{-1}\alpha u + cd^{-1} \log \cosh(du) + C(u, \theta), \tag{131}$$

a solution of (47) and

$$P(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)) \tag{132}$$

a solution of (41). A straightforward computation shows that if C is a solution of

$$\begin{aligned} (-\delta^{-1}\alpha - cZ_0(u))\partial_\theta C + \partial_u C &= l_2(u, \theta)(\delta^{-1}\alpha + cZ_0(u) + \partial_u C) \\ &+ l_3(u, \theta)(1 + \partial_\theta C), \end{aligned} \tag{133}$$

then, ξ is a solution of (47). Conversely, if P_1 is a solution of

$$\begin{aligned} (-\delta^{-1}\alpha - cZ_0(u)) \partial_\theta P_1 + \partial_u P_1 &= (2\sigma + l_1(u, \theta) + 2Z_0(u)l_2(u, \theta))(1 + P_1) \\ &+ l_2(u, \theta)\partial_u P_1 + l_3(u, \theta)\partial_\theta P_1, \end{aligned} \tag{134}$$

then P as in (132) is a solution of (41).

Therefore, we focus to prove the existence and the properties stated in Theorem 2.7 of the functions C and P_1 . To this aim, first we point out that the linear operator on the left-hand side of Eqs. (133) and (134) is in both cases:

$$\hat{\mathcal{L}}(\phi) = (-\delta^{-1}\alpha - cZ_0(u))\partial_\theta \phi + \partial_u \phi. \tag{135}$$

In order to prove the existence of C and P_1 , we will use a right inverse of the operator $\hat{\mathcal{L}}$, which we will call $\hat{\mathcal{G}}$.

As we did with $\mathcal{G}^{u,s}$, we look for an expression of $\hat{\mathcal{G}}$ by solving the ordinary differential equations that its Fourier coefficients satisfy. Proceeding in this way and taking into account that our functions are defined in $D_{\kappa,\beta} \times \mathbb{T}_\omega$, we define $\hat{\mathcal{G}}$ as the operator acting on functions ϕ defined in $D_{\kappa,\beta} \times \mathbb{T}_\omega$ as:

$$\hat{\mathcal{G}}(\phi)(u, \theta) = \sum_{l \in \mathbb{Z}} \hat{\mathcal{G}}^{[l]}(\phi)(u)e^{il\theta}, \tag{136}$$

where

$$\begin{aligned}
 \hat{\mathcal{G}}^{[l]}(\phi)(u) &= \int_{u_+}^u e^{-il\delta^{-1}\alpha(w-u)-ilcd^{-1}\log\left(\frac{\cosh(dw)}{\cosh(dw)}\right)} \phi^{[l]}(w)dw, & \text{if } l < 0, \\
 \hat{\mathcal{G}}^{[0]}(\phi)(u) &= \int_{u_R}^u \phi^{[0]}(w)dw, & (137) \\
 \hat{\mathcal{G}}^{[l]}(\phi)(u) &= \int_{u_-}^u e^{-il\delta^{-1}\alpha(w-u)-ilcd^{-1}\log\left(\frac{\cosh(dw)}{\cosh(dw)}\right)} \phi^{[l]}(w)dw, & \text{if } l > 0,
 \end{aligned}$$

and $u_{\pm} = \pm i(\pi/(2d) - \delta\kappa)$ and $u_R \in \mathbb{R}$ is the point of $D_{\kappa,\beta}$ with largest real part (see Fig. 6 in Sect. 2.4).

In the next section, we introduce the Banach spaces we will work with, give some bounds of the functions l_i and finally check that the operator $\hat{\mathcal{G}}$ is well defined and has appropriate properties.

5.2 Banach Spaces and Properties of $\hat{\mathcal{G}}$

We will consider functions $\phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \rightarrow \mathbb{C}$. Again, they can be written in their Fourier series $\phi(v, \theta) = \sum_{l \in \mathbb{Z}} \phi^{[l]}(v)e^{il\theta}$. In a similar way as we did in Sect. 4.1.2 we define the norms:

$$\begin{aligned}
 \|\phi^{[l]}\|_n &= \sup_{v \in D_{\kappa,\beta}} \left| \cosh^n(dv)\phi^{[l]}(v) \right|, & \|\phi\|_{n,\omega} &= \sum_{l \in \mathbb{Z}} \|\phi^{[l]}\|_n e^{|l|\omega}, \\
 \|\phi\|_{n,\omega} &= \|\phi\|_{n,\omega} + \|\partial_v \phi\|_{n+1,\omega} + \delta^{-1} \|\partial_\theta \phi\|_{n+1,\omega}
 \end{aligned}$$

and we consider the Banach spaces endowed with these norms:

$$\begin{aligned}
 \mathcal{X}_{n,\omega} &:= \left\{ \phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \rightarrow \mathbb{C} : \phi \text{ is analytic, such that } \|\phi\|_{n,\omega} < +\infty \right\}, \\
 \tilde{\mathcal{X}}_{n,\omega} &:= \left\{ \phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \rightarrow \mathbb{C} : \phi \text{ is analytic, such that } \|\phi\|_{n,\omega} < +\infty \right\}.
 \end{aligned}$$

Remark 5.1 From Theorem 2.4, $r_1^{u,s} \in \tilde{\mathcal{X}}_{3,\omega}$ and that there exists a constant M such that $\|r_1^{u,s}\|_{3,\omega} \leq M\delta^{p+3}$.

Next lemma provides bounds for l_1, l_2 and l_3 . Its proof is given in Castejón (2015).

Lemma 5.2 *Let $l_i(u, \theta)$, $i = 1, 2, 3$, be the functions defined in (42)–(44). There exists a constant M such that:*

$$\|l_1\|_{2,\omega} \leq M\delta^{p+3}, \quad \|l_2\|_{1,\omega} \leq M\delta^{p+3}, \quad \|l_3\|_{2,\omega} \leq M\delta^{p+3}.$$

To finish this section we enunciate the properties of the linear operator $\hat{\mathcal{G}}$ which turns out to be very similar to the ones of \mathcal{G} in Lemma 4.6. Its proof involves several technicalities and can be found in Castejón (2015).

Lemma 5.3 *Let $l \in \mathbb{Z}$, $n \geq 1$ and $\phi \in \mathcal{X}_{n,\omega}$. There exists $M > 0$ such that:*

1. *If $n > 1$, then $\|\hat{\mathcal{G}}^{[l]}(\phi)\|_{n-1} \leq M \|\phi^{[l]}\|_n$.*
2. *If $l \neq 0$, then $\|\hat{\mathcal{G}}^{[l]}(\phi)\|_n \leq \frac{\delta M \|\phi^{[l]}\|_n}{|l|}$.*
3. *As a consequence, if $n > 1$, $\|\hat{\mathcal{G}}(\phi)\|_{n-1,\omega} \leq M \|\phi\|_{n,\omega}$. Moreover, if $\phi^{[0]}(v) = 0$, then for all $n \geq 1$, $\|\hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq \delta M \|\phi\|_{n,\omega}$.*
4. $\|\partial_\theta \hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq \delta M \|\phi\|_{n,\omega}$.
5. $\|\partial_u \hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq M \|\phi\|_{n,\omega}$.
6. *In conclusion, if $n > 1$ and $\phi \in \mathcal{X}_{n,\omega}$, then $\hat{\mathcal{G}}(\phi) \in \tilde{\mathcal{X}}_{n-1,\omega}$ and there exists $M > 0$ such that:*

$$\|\hat{\mathcal{G}}(\phi)\|_{n-1,\omega} \leq M \|\phi\|_{n,\omega}.$$

In the following two sections we will prove the results related to the functions C and P_1 .

5.3 Existence and Properties of C

We enunciate the results about the function C that we will prove. They give a more precise information than the ones in Theorem 2.7.

Proposition 5.4 *There exists a particular solution C of (133) of the form:*

$$C(u, \theta) = \delta^{-1} \alpha \int_0^u l_2^{[0]}(w) dw + C_1(u, \theta), \tag{138}$$

with $l_2^{[0]}(u)$ the average of the function l_2 defined in (43).

$$\|C_1\|_{1,\omega} \leq M \delta^{p+3}, \quad \|\partial_u C\|_{1,\omega} \leq M \delta^{p+2}, \quad \|\partial_\theta C\|_{1,\omega} \leq M \delta^{p+3}. \tag{139}$$

Finally, $(\xi(u, \theta), \theta)$, with ξ given by (131), is injective in $D_{\kappa,\beta} \times \mathbb{T}_\omega$.

Remark 5.5 Assuming that C actually exists and recalling that Δ has the form (48), the function

$$k(u, \theta) := \tilde{k}(\theta + \delta^{-1} \alpha u + cd^{-1} \log \cosh(du) + C(u, \theta)),$$

has to be 2π -periodic in θ , which implies that $\tilde{k}(\tau)$ is 2π -periodic in τ .

Now we make some further considerations on the integral $\int_0^u l_2^{[0]}(w) dw$. First of all, we point out that using Lemma 5.2 and the fact that for $w \in D_{\kappa,\beta}$ one has $|\cosh(dw)| \geq K |w^2 - \pi^2/(2d)^2|$, one obtains:

$$\left| \delta^{-1} \alpha \int_0^u l_2^{[0]}(w) dw \right| \leq K \delta^{p+2} \int_0^u |\cosh(dw)|^{-1} dw \leq K \delta^{p+2} |\log(\delta \kappa)|.$$

Hence, in the regular case $p > -2$ this integral is small, even for complex values of $u \in D_{\kappa,\beta}$, and one can avoid to take into account its contribution to the function $C(u, \theta)$ defined in (138). Notice that when $u \in \mathbb{R}$, this integral is $\mathcal{O}(\delta^{p+2})$. However, in the singular case $p = -2$, one needs to have some more precise knowledge of its behavior.

The following result deals with this integral. Its proof is given with detail in Castejón (2015).

Lemma 5.6 Define L_0 as the following limit that is well defined:

$$L_0 = \lim_{u \rightarrow i\frac{\pi}{2d}} \lim_{\delta \rightarrow 0} \delta^{-p-3} l_2^{[0]}(u) \tanh^{-1}(du).$$

Then, there exist functions $L(u)$ and $\Lambda(u)$ such that for all $u \in D_{\kappa,\beta}$:

$$\int_0^u l_2^{[0]}(w)dw = \delta^{p+3} d^{-1} L_0 \log \cosh(du) + \delta L(u) + \delta \Lambda(u).$$

Moreover, $L_0 \in \mathbb{R}$, $L(0) = 0$ and $L(u)$ is defined on the limit $u \rightarrow i\pi/(2d)$ and $\|L\|_0 \leq M\delta^{p+2}$, $\|L'\|_0 \leq M\delta^{p+2}$ and $\|\Lambda\|_1 \leq M\delta^{p+3}$, for some $M > 0$.

Remark 5.7 One can obtain explicit expressions for L_0 , $L(u)$ and $\Lambda(u)$ depending only on $\mathcal{F}(0)$ or equivalently, on $F(0)$, $G(0)$ and $H(0)$ defined in (24). See (Castejón 2015) for details.

We write the formulas for L_0 and $L(u)$. We define the constants

$$\begin{aligned} \rho_0 &= \frac{(d+1)}{2bd(3d+2)} \left[\frac{(d+1)}{4b} (f_{3120} + g_{3210} + 3f_{3300} + 3g_{3030}) - (f_{3102} + g_{3012}) \right. \\ &\quad \left. - \frac{d+1}{b} (h_{3201} + h_{3021}) + 2h_{3003} \right] \\ H_0 &= -h_{3003} + \frac{d+1}{2b} (h_{3021} + h_{3201}), \end{aligned}$$

where the coefficients f_{qkmn} , g_{qkmn} and h_{qkmn} were defined as in (62).

In the conservative case, $L_0 = -h_{3003}$ while in the general one:

$$L_0 = -\frac{2b}{d} \rho_0 - \frac{1}{d} H_0.$$

With respect to $L(u)$, we define

$$\begin{aligned} H_1(u) &= \frac{\cosh^3(du) \lim_{\delta \rightarrow 0} \delta^{-3} (H(0))^{[0]} - H_0 \sinh(du)}{\cosh(du)}, \\ \rho_1^{u,s}(u) &= \delta^{-(p+3)} \sigma \int_{\mp\infty}^u \frac{R_0(w)}{\cosh^{4+\frac{2}{d}}(dw)} dw \end{aligned}$$

$$\begin{aligned}
 & + \cosh^{\frac{2}{d}}(du) \int_{\mp\infty}^u \frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{d}}(dw)} dw - \rho_0 \frac{\tanh(du)}{\cosh^2(du)}, \\
 \mathcal{F}_1(u) = \lim_{\delta \rightarrow 0} & \delta^{-p-3} \cosh^4(u) [\mathcal{F}^{[0]}(u) - 2\sigma R_0(u)],
 \end{aligned}$$

where we take $-$ for the unstable case and $+$ for the stable one. Then

$$L(u) = -\delta^{p+2} \int_0^u \frac{b}{d} \cosh^2(dw) (\rho_1^u(u) + \rho_1^s(w)) + \frac{1}{d} H_1(w) dw.$$

5.3.1 Proof of Proposition 5.4

Let us define:

$$\hat{l}_2(u, \theta) = l_2(u, \theta) - l_2^{[0]}(u).$$

It is easy to see that in order that C defined in (138) satisfies (133) it is enough that C_1 satisfies the following equation:

$$\begin{aligned}
 (-\delta^{-1}\alpha - cZ_0(u))\partial_\theta C_1 + \partial_u C_1 = & \delta^{-1}\alpha \hat{l}_2(u, \theta) + l_3(u, \theta)(1 + \partial_\theta C_1) \quad (140) \\
 & + l_2(u, \theta)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u) + \partial_u C_1).
 \end{aligned}$$

We define the operator \mathcal{A}_1 as:

$$\mathcal{A}_1(\phi) = \frac{\alpha}{\delta} \hat{l}_2(u, \theta) + l_2(u, \theta)(cZ_0(u) + \frac{\alpha}{\delta} l_2^{[0]}(u) + \partial_u \phi) + l_3(u, \theta)(1 + \partial_\theta \phi). \quad (141)$$

Then Eq. (140) can be rewritten as $\hat{\mathcal{L}}(C_1) = \mathcal{A}_1(C_1)$, where $\hat{\mathcal{L}}$ was defined in (135). It is enough then to solve the fixed point equation:

$$C_1 = \tilde{\mathcal{A}}_1(C_1),$$

where $\tilde{\mathcal{A}}_1 = \hat{\mathcal{G}} \circ \mathcal{A}_1$, and $\hat{\mathcal{G}}$ is the operator defined in (136).

Lemma 5.8 *For κ big enough and $p \geq -2$, the operator $\tilde{\mathcal{A}}_1 : \tilde{\mathcal{X}}_{0,\omega} \rightarrow \tilde{\mathcal{X}}_{0,\omega}$. Moreover, there exists a constant M such that $\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega} \leq M\delta^{p+2}$, and $\tilde{\mathcal{A}}_1$ has a unique fixed point in the ball $B(2\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega}) \subset \tilde{\mathcal{X}}_{0,\omega}$.*

Proof First of all we shall prove that:

$$\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega} \leq K\delta^{p+2}. \quad (142)$$

We have:

$$\mathcal{A}_1(0) = \delta^{-1}\alpha \hat{l}_2(u, \theta) + l_2(u, \theta)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u) + l_3(u, \theta)).$$

To prove (142) we shall bound the Fourier coefficients of $\mathcal{A}_1(0)$ and then use Lemma 5.3. On the one hand, since by definition \hat{l}_2 has zero average, one has:

$$\mathcal{A}_1^{[0]}(0) = l_2^{[0]}(u)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u)) + l_3^{[0]}(u).$$

Using Lemma 5.2 and the properties of the norm:

$$\|\mathcal{A}_1^{[0]}(0)\|_2 \leq \|l_2^{[0]}\|_1(c\|Z_0\|_1 + \delta^{-1}\alpha\|l_2^{[0]}\|_1) + \|l_3^{[0]}\|_2 \leq K\delta^{p+3}. \tag{143}$$

Then, by item 1 of Lemma 5.3 one has:

$$\|\hat{\mathcal{G}}^{[0]}(\mathcal{A}_1(0))\|_1 \leq K\|\mathcal{A}_1^{[0]}(0)\|_2 \leq K\delta^{p+3}. \tag{144}$$

On the other hand, for the remaining Fourier coefficients one has:

$$\mathcal{A}_1^{[l]}(0) = l_2^{[l]}(u)(\delta^{-1}\alpha + cZ_0(u) + \delta^{-1}\alpha l_2^{[l]}(u)) + l_3^{[l]}(u) \quad l \neq 0.$$

Again, using Lemma 5.2 and the properties of the norm, we obtain:

$$\|\mathcal{A}_1^{[l]}(0)\|_1 \leq \|l_2^{[l]}\|_1(\delta^{-1}\alpha + c\|Z_0\|_0 + \delta^{-1}\alpha\|l_2^{[l]}\|_0) + \|l_3^{[l]}\|_1 \leq K\delta^{p+2}. \tag{145}$$

Then by item 2 of Lemma 5.3 and taking into account that $l \neq 0$, we have:

$$\|\hat{\mathcal{G}}^{[l]}(\mathcal{A}_1(0))\|_1 \leq \frac{K\delta\|\mathcal{A}_1^{[l]}(0)\|_1}{|l|} \leq \frac{K\delta^{p+3}}{|l|}. \tag{146}$$

From (144) and (146) we obtain:

$$\|\tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\delta^{p+3}, \tag{147}$$

and as a consequence:

$$\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega} \leq K\frac{\delta^{p+2}}{\kappa} \leq K\delta^{p+2}. \tag{148}$$

We note that from bounds (143) and (145) we have $\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+2}$, and then from items 4 and 5 of Lemma 5.3 we obtain directly:

$$\|\partial_u \tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+2}, \tag{149}$$

and:

$$\|\partial_\theta \tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\delta\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+3}. \tag{150}$$

From bounds (148), (149) and (150), one obtains bound (142).

It is not difficult to check that given two functions $\phi_1, \phi_2 \in \tilde{\mathcal{X}}_{0,\omega}$:

$$\|\tilde{\mathcal{A}}_1(\phi_1) - \tilde{\mathcal{A}}_1(\phi_2)\|_{0,\omega} \leq \frac{K}{\kappa} \delta^{p+2} \|\phi_1 - \phi_2\|_{0,\omega}. \tag{151}$$

To finish the proof, we take κ sufficiently large such that the Lipschitz constant in (151) is smaller than 1. Then $\tilde{\mathcal{A}}_1$ sends $B\left(2\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega}\right)$ to itself and since it is contractive, it has a unique fixed point in this ball. \square

End of the proof of Proposition 5.4 Let us define C_1 as the unique fixed point of the operator $\tilde{\mathcal{A}}_1$ in the ball $B\left(2\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega}\right)$, whose existence is proven by Lemma 5.8. Let C be defined as in (138) and ξ the function defined as (131). It remains to check that bounds (139) hold and that $(\xi(\theta, u), \theta)$ is injective.

First we shall see that C_1 satisfies the bound in (139). We point out that this is not given directly by Lemma 5.8, but it can be obtained *a posteriori*. Indeed, by definition C_1 satisfies: $C_1 = \hat{\mathcal{G}}(\mathcal{A}_1(C_1))$. By definition (141) of the operator \mathcal{A}_1 , and since $\hat{\mathcal{G}}$ is linear, we can write:

$$C_1 = \hat{\mathcal{G}}(\mathcal{A}_1(0)) + \hat{\mathcal{G}}(l_2(u, \theta)\partial_u C_1 + l_3(u, \theta)\partial_\theta C_1). \tag{152}$$

On the one hand, we recall bound (147) which stated:

$$\|\hat{\mathcal{G}}(\mathcal{A}_1(0))\|_{1,\omega} \leq K\delta^{p+3}. \tag{153}$$

On the other hand, since $C_1 \in B\left(2\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega}\right)$, by the definition of the norm $\|\cdot\|_{0,\omega}$ and the bound of $\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega}$ provided by Lemma 5.8, one has:

$$\|\partial_u C_1\|_{1,\omega} \leq K\delta^{p+2}, \quad \|\partial_\theta C_1\|_{1,\omega} \leq K\delta^{p+3}. \tag{154}$$

Then, using Lemma 5.2 and bounds (154) it is easy to see that:

$$\|l_2(u, \theta)\partial_u C_1 + l_3(u, \theta)\partial_\theta C_1\|_{2,\omega} \leq K\delta^{2p+5},$$

so that by item 3 of Lemma 5.3 we obtain:

$$\|\hat{\mathcal{G}}(l_2(u, \theta)\partial_u C_1 + l_3(u, \theta)\partial_\theta C_1)\|_{1,\omega} \leq K\delta^{2p+5}. \tag{155}$$

Using that $p \geq -2$ and bounds (153) and (155) in Eq. (152), we obtain $\|C_1\|_{1,\omega} \leq K\delta^{p+3}$, and then bound (139) is obtained.

The bounds in (139) for C are consequence of (154) and Lemma 5.2. It only remains to prove that $(\xi(\theta, u), \theta)$ is injective. Let us assume $\xi(u_1, \theta) = \xi(u_2, \theta)$. This means:

$$\begin{aligned} u_1 - u_2 &= \delta d^{-1} \alpha^{-1} c(\log \cosh(du_1) - \log \cosh(du_2)) \\ &\quad + \delta \alpha^{-1} (C(u_1, \theta) - C(u_2, \theta)). \end{aligned}$$

On the one hand, for $u_1, u_2 \in D_{\kappa, \beta}$ we have:

$$\delta d^{-1} \alpha^{-1} c |\log \cosh(du_1) - \log \cosh(du_2)| \leq \frac{K}{\kappa} |u_1 - u_2|.$$

On the other hand, using the mean value theorem and bound (139):

$$\delta |C(u_1, \theta) - C(u_2, \theta)| \leq \frac{K \delta^{p+2}}{\kappa} |u_1 - u_2|,$$

Thus, since $p \geq -2$, we know that there exists a constant K such that: $|u_1 - u_2| \leq \kappa^{-1} K |u_1 - u_2|$. Taking κ sufficiently large yields $u_1 = u_2$. \square

5.4 Existence and Properties of P_1

Our goal is to find a particular solution of Eq. (134) satisfying the properties stated in Theorem 2.7. We introduce the operator

$$\mathcal{B}(\phi) = (2\sigma + l_1(u, \theta) + 2Z_0(u)l_2(u, \theta))(1 + \phi) + l_2(u, \theta)\partial_u \phi + l_3(u, \theta)\partial_\theta \phi, \quad (156)$$

in such a way that Eq. (134) can be written as:

$$\hat{\mathcal{L}}(P_1) = \mathcal{B}(P_1). \quad (157)$$

We distinguish between the general and the conservative case, since in the first case P_1 will be found using a fixed point equation, while in the latter it will be defined in terms of the function C of Theorem 2.7.

5.4.1 The General Case

In this subsection we will follow the same steps as in Sect. 5.3 to prove the existence of C_1 . Since P_1 has to be a solution of (157), we shall do it by solving the fixed point equation:

$$P_1 = \tilde{\mathcal{B}}(P_1), \quad (158)$$

where $\tilde{\mathcal{B}} = \hat{\mathcal{G}} \circ \mathcal{B}$, and $\hat{\mathcal{G}}$ is the operator defined by (136) and (137), and \mathcal{B} is defined in (156).

Lemma 5.9 *For κ big enough and $p \geq -2$, the operator $\tilde{\mathcal{B}} : \tilde{\mathcal{X}}_{1,\omega} \rightarrow \tilde{\mathcal{X}}_{1,\omega}$, and it has a unique fixed point in the ball $B(2\|\tilde{\mathcal{B}}(0)\|_{1,\omega}) \subset \tilde{\mathcal{X}}_{1,\omega}$. Moreover, there exists a constant M such that $\|\tilde{\mathcal{B}}(0)\|_{1,\omega} \leq K \delta^{p+3}$.*

Proof First we deal with $\tilde{\mathcal{B}}(0)$. Indeed, using Lemma 5.2 and that $|\sigma| \leq \sigma^* \delta^{p+3}$, it is straightforward to prove that there exists a constant K such that: $\|\mathcal{B}(0)\|_{2,\omega} \leq K \delta^{p+3}$. Then item 6 of Lemma 5.3 yields $\|\tilde{\mathcal{B}}(0)\|_{1,\omega} \leq K \delta^{p+3}$.

Next step is to find the Lipschitz constant of the operator $\tilde{\mathcal{B}}$. To do so, let us fix $\phi_1, \phi_2 \in B\left(2\|\tilde{\mathcal{B}}(0)\|_{1,\omega}\right)$. We have:

$$\begin{aligned} \|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2)\|_{2,\omega} &\leq 2|\sigma|\|\phi_1 - \phi_2\|_{2,\omega} + \|I_1\|_{1,\omega}\|\phi_1 - \phi_2\|_{1,\omega} \\ &\quad + 2\|Z_0\|_{1,\omega}\|I_2\|_{0,\omega}\|\phi_1 - \phi_2\|_{1,\omega} + \|I_2\|_{0,\omega}\|\partial_u(\phi_1 - \phi_2)\|_{2,\omega} \\ &\quad + \|I_3\|_{1,\omega}\|\partial_\theta(\phi_1 - \phi_2)\|_{1,\omega}. \end{aligned} \tag{159}$$

First we note that, since $\sigma = \mathcal{O}(\delta^{p+3})$:

$$|\sigma|\|\phi_1 - \phi_2\|_{2,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \|\phi_1 - \phi_2\|_{1,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \|\phi_1 - \phi_2\|_{1,\omega}. \tag{160}$$

Similarly, by Lemma 5.2:

$$\|I_1\|_{1,\omega} \leq K \frac{\delta^{p+2}}{\kappa}, \quad \|I_2\|_{0,\omega} \leq K \frac{\delta^{p+2}}{\kappa}, \quad \|I_3\|_{1,\omega} \leq K \frac{\delta^{p+2}}{\kappa}. \tag{161}$$

Finally, we just need to note that:

$$\|\partial_\theta(\phi_1 - \phi_2)\|_{1,\omega} \leq \frac{K}{\delta\kappa} \|\partial_\theta(\phi_1 - \phi_2)\|_{2,\omega} \leq K \|\phi_1 - \phi_1\|_{1,\omega}. \tag{162}$$

Using the definition of the norm $\|\cdot\|_{1,\omega}$, the fact that $\|Z_0\|_{1,\omega} \leq K$ and the previous bounds (160), (161) and (162) in Eq. (159) we immediately obtain

$$\|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2)\|_{2,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \|\phi_1 - \phi_2\|_{1,\omega}.$$

Again, by item 6 of Lemma 5.3

$$\|\tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2)\|_{1,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \|\phi_1 - \phi_2\|_{1,\omega}. \tag{163}$$

To finish the proof, we take κ large enough such that the Lipschitz constant in (163) is smaller than 1. Then the fixed point theorem yields the result. \square

The fact P_1 satisfies Eq. (157) (and consequently (134)) is clear since it is a solution of Eq. (158). Clearly, using Lemma 5.9, one has, $\|P_1\|_{1,\omega} \leq 2\|\tilde{\mathcal{B}}(0)\|_{1,\omega} \leq K\delta^{p+3}$. Since:

$$\sup_{(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega} |P_1(u, \theta)| \leq \|P_1\|_{1,\omega} \sup_{(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega} |\cosh^{-1}(du)| \leq K \frac{\delta^{p+2}}{\kappa},$$

taking κ sufficiently large we obtain $|1 + P_1(u, \theta)| \geq 1 - K\kappa^{-1}\delta^{p+2} \neq 0$. Finally, since $\cosh^{2/d}(du) \neq 0$ for $u \in D_{\kappa,\beta}$ we can ensure that, for $(u, \theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$, $P(u, \theta) = \cosh^{2/d}(du)(1 + P_1(u, \theta)) \neq 0$.

5.4.2 The Conservative Case

We recall that in the conservative case we have $d = 1$ and $\sigma = 0$. Let:

$$P_1(u, \theta) = \frac{\partial_u C(u, \theta) - l_3(u, \theta)}{\delta^{-1}\alpha + cZ_0(u) + l_3(u, \theta)}, \tag{164}$$

where $C(u, \theta)$ is the function given by Proposition 5.4 and $l_3(u, \theta)$ is defined in (44). First let us check that it satisfies bound (54), that is:

$$|P_1(u, \theta)| \leq \frac{K}{\kappa} \delta^{p+2}, \tag{165}$$

for all $(u, \theta) \in D_{\kappa, \beta} \times \mathbb{T}_\omega$. On the one hand, note that by Proposition 5.4 and Lemma 5.2 we have:

$$|\partial_u C(u, \theta)| \leq \frac{K}{\kappa} \delta^{p+1}, \quad |l_3(u, \theta)| \leq \frac{K}{\kappa^2} \delta^{p+1}. \tag{166}$$

On the other hand, taking κ sufficiently large, we also have:

$$\left| \frac{1}{\delta^{-1}\alpha + cZ_0(u) + l_3(u, \theta)} \right| \leq K\delta. \tag{167}$$

Then (165) follows directly from using (166) and (167) in (164).

It only remains to prove that P_1 defined in (164) satisfies Eq. (134):

$$\begin{aligned} (-\delta^{-1}\alpha - cZ_0(u)) \partial_\theta P_1 + \partial_u P_1 &= (l_1(u, \theta) + 2Z_0(u)l_2(u, \theta))(1 + P_1) \\ &\quad + l_2(u, \theta)\partial_u P_1 + l_3(u, \theta)\partial_\theta P_1. \end{aligned} \tag{168}$$

Tedious but standard computations yield that P_1 is a solution of

$$\begin{aligned} (-\delta^{-1}\alpha - cZ_0(u) - l_3(u, \theta))\partial_\theta P_1 + \partial_u P_1 \\ = (\partial_u l_2(u, \theta) + \partial_\theta l_3(u, \theta))(1 + P_1) + l_2(u, \theta)\partial_u P_1. \end{aligned} \tag{169}$$

Therefore, Eqs. (169) and (168) are the same, if and only if:

$$l_1(u, \theta) + 2Z_0(u)l_2(u, \theta) = \partial_u l_2(u, \theta) + \partial_\theta l_3(u, \theta).$$

This equality can be checked using the definitions of l_2 and l_3 as well as the fact that the vector field is divergence free, that is:

$$\partial_z H(r) + \partial_\theta G(r) = -\partial_r F(r).$$

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