Gröbner Cover

Canonical discussion of polynomial systems with parameters

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References

Based on

- Antonio Montes, Michael Wibmer. “Gröbner Bases for Polynomial Systems with Parameters".
  Journal of Symbolic Computation **45** (2010) 1391 - 1425.
- Antonio Montes, Tomás Recio. “Generalization of Steiner-Lehmus Theorem using the Gröbner Cover". Work in progress.
- Software download (beta version):
  http://www-ma2.upc.edu/∼montes/
- Standard software version will be distributed with the next Singular release.

Outline

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Parametric polynomial discussion

Goal

Data: Parametric polynomial system of equations

$$\begin{cases} p_1(a_1,\dots,a_m,x_1,\dots,x_n) = 0 \\ \dots \\ p_r(a_1,\dots,a_m,x_1,\dots,x_n) = 0 \end{cases}$$

Goal: describe the different kind of solutions $(x_1,\dots,x_n)$ *in dependence of the parameters* $a_1,\cdots,a_m$.

Some notations

Let:
K be a computable field (in practice $\mathbb{Q}$).
$\overline{K}$ be an algebraically closed extension of K (in practice $\mathbb{C}$).

$K[\overline{a}]$ the polynomial ring in the parameters $\overline{a} = a_1, \ldots, a_m$ over K.
$K[\overline{a}][\overline{x}]$ the polynomial ring in the variables $\overline{x} = x_1, \ldots, x_n$ over $K[\overline{a}]$.
$\overline{K}^m$ is the parameter space.

Fix: $\succ_{\overline{x}}$ monomial ordering wrt $\overline{x}$ and the ideal
$I = \langle p_1(\overline{a}, \overline{x}), \cdots, p_r(\overline{a}, \overline{x}) \rangle \subset K[\overline{a}][\overline{x}]$
$\mathrm{lpp}(G)$ = set of leading power products wrt $\succ_{\overline{x}}$ of the polynomials in G.

Specialization:
$a = (a_1^0, \cdots, a_m^0) \in \overline{K}^m$
$I_a = \langle p_1(a, \overline{x}), \cdots, p_r(a, \overline{x}) \rangle \subset \overline{K}[\overline{x}]$

Goal

Gröbner bases are the computational method par excellence for studying polynomial systems.

The set of lpp of the reduced Gröbner basis determines the type of solutions of the system.

In the case of parametric polynomial systems the goal is to describe the reduced Gröbner basis of $I_a \subset \overline{K}[\overline{x}]$ (with respect to $\succ_{\overline{x}}$) in dependence of $a \in \overline{K}^m$.

Precedents

Weispfenning (1992)

Given $I = \langle p_1, \dots, p_r \rangle \subset K[\overline{a}][\overline{x}] = K[\overline{a}, \overline{x}]$ *and* $\succ_{\overline{x}}$

A Comprehensive Gröbner System (CGS) for I *and* $\succ_{\overline{x}}$ *is a finite set of pairs* $\{(S_1, B_1), \dots, (S_s, B_s)\}$ *(Segments:* S_i*, Bases:* B_i*) such that*

1. *The* S_i*'s are constructible subsets of* $\overline{K}^m$ *such that* $\overline{K}^m = \cup S_i$.
2. *The* B_i*'s are finite subsets of* $K(\overline{a})[\overline{x}]$ *and* $B_i(a) = \{p(a, \overline{x}) : p \in B_i\}$ *is a Gröbner basis of* I_a *with respect to* $\succ_{\overline{x}}$ *for every* $a \in S_i$.

Faithful: $B_i \subset I$. *Leads to a Comprehensive Gröbner Basis*
Non-faithful: B_i *reduced.*

Historical development

Two directions:

- Speed up. Duval (1995), Dellière (1999), Kapur (1995), Kalkbrenner (1997), Sato (2003), Suzuki & Sato (2006), Nabeshima (2006), Deepak Kapur & Yao Sun & Dingkang Wang (2010).
- Improve output. Montes (2002), Weispfenning (2003), Wibmer (2007), Manubens & Montes (2009), Montes & Wibmer (2010).

Our goal:

- best output for applications,
- disjoint segments,
- segments with constant lpp,
- minimal number of segments,
- canonical output,
- if possible, locally closed segments.

A simple but critical example

Consider the ideal $F = \langle ax + by, cx + dy \rangle$.
It is elementary to obtain the following discussion ($\mathrm{lex}(x, y)$):

Num.	segment	basis	lpp
1	$\mathbb{C}^4 \setminus \mathbb{V}(ad - bc)$	$[y, x]$	$[y, x]$
2	$\mathbb{V}(ad - bc) \setminus \mathbb{V}(a, c)$	$[x + \left\{\frac{b}{a}, \frac{d}{c}\right\} y]$	$[x]$
3	$\mathbb{V}(a, c) \setminus \mathbb{V}(a, b, c, d)$	$[y]$	$[y]$
4	$\mathbb{V}(a, b, c, d)$	$[\ ]$	$[\ ]$

To summarize into a unique segment each set of solotions with the same lpp, ordinary polynomials are not sufficient. We need more general functions:

regular functions $\quad f : \quad S = \mathbb{V}(ad - bc) \setminus \mathbb{V}(a, b) \longrightarrow \mathbb{C}$

$$(a, b, c, d) \mapsto \left\{\frac{b}{a}, \frac{d}{c}\right\}$$

I-regular functions $\quad g : \quad S = \mathbb{V}(ad - bc) \setminus \mathbb{V}(a, b) \longrightarrow \mathbb{C}[x, y]$

$$(a, b, c, d) \mapsto x + fy$$

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Locally closed sets

Definition

A subset $S \subset \overline{K}^m$ is locally closed, if it is difference of two varieties: $S = \mathbb{V}(M) \setminus \mathbb{V}(N)$.

Definition (Open subset)

A subset $U \subset S$ is said to be open on S if $\overline{S \setminus U} \subsetneq S$.

Proposition (Canonical representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist uniquely determined radical ideals $\mathfrak{a} \subset \mathfrak{b}$ of $K[\overline{a}]$, with $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$, such that

- $\overline{S} = \mathbb{V}(\mathfrak{a})$,
- $\overline{S} \setminus S = \mathbb{V}(\mathfrak{b})$.

The pair $(\mathfrak{a}, \mathfrak{b})$ -top,hole- is called the canonical representation of S.

Locally closed sets

Proposition (Canonical prime representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist a unique canonical prime representation of S given the prime components of $\mathfrak{a}$, say $\mathfrak{p}_i$, and associated to each, a set of prime ideals $\mathfrak{p}_{ij}$ (holes) in the form $((\mathfrak{p}_1, (\mathfrak{p}_{11}, \dots, \mathfrak{p}_{1j_1})), \dots, (\mathfrak{p}_k, (\mathfrak{p}_{k1}, \dots, \mathfrak{p}_{kj_k})))$ so that

$$S = \bigcup_{i=1}^{k} \left(\mathbb{V}(\mathfrak{p}_i) \setminus (\bigcup_{j=1}^{j_i} \mathbb{V}(\mathfrak{p}_{ij})) \right).$$

and $\mathfrak{p}_i \subset \mathfrak{p}_{ij}$ for all i, j, such that

- $\overline{S} = \mathbb{V}(\mathfrak{p}_1) \cup \ldots \cup \mathbb{V}(\mathfrak{p}_r)$ *and*
- $(\overline{S} \setminus S) \cap \mathbb{V}(\mathfrak{p}_i) = \mathbb{V}(\mathfrak{p}_{i1}) \cup \cdots \cup \mathbb{V}(\mathfrak{p}_{ir_i})$

are the minimal decompositions into irreducible closed sets.

I-regular functions

Definition (I-Regular function)

Let S be a locally closed subset of $\overline{K}^m$. We call a function $f : S \longrightarrow \overline{K}[\overline{x}]$ I-regular, if $\forall a \in S$ it exists an open $U \subset S$ with $a \in U$ and

$$f(b) = \frac{P(b, \overline{x})}{Q(b)} \quad \text{for all } b \in U,$$

where $P \in I$ and $Q \in K[\overline{a}]$ and $Q(b) \neq 0$ for all $b \in U$.

Remark

Let P and Q be a polynomials as above, (they are not unique), $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$ and $p(b, \overline{x}) = P(b, \overline{x}) \mod \mathfrak{a}$. If f is monic and $\mathrm{lpp}(f)$ is constant on S, then, for all $b \in U$ is

- $\mathrm{lpp}_{\overline{x}}(p(b, \overline{x})) = \mathrm{lpp}_{\overline{x}}(f)$, *and*
- $\mathrm{lc}_{\overline{x}}(p(b, \overline{x})) = Q(b) \mod \mathfrak{a}$.

Parametric subsets

Definition (Parametric subset of $\overline{K}^m$)

A locally closed subset $S \in \overline{K}^m$ is called parametric (wrt to I and $\succ_{\overline{x}}$) if there exist monic I-regular functions $\{g_1, \ldots, g_s\}$ over S so that $\{g_1(a, \overline{x}), \ldots, g_s(a, \overline{x})\}$ is the reduced Gröbner basis of I_a for all $a \in S$.

Note

Note that the definition immediately implies that if a, b lie in a parametric set S, then $\text{lpp}_{\overline{x}}(I_a) = \text{lpp}_{\overline{x}}(I_b)$.
The amazing thing is that the converse also holds if we additionally assume that $I \subset K[\overline{a}][\overline{x}]$ is homogeneous (wrt to the variables).

Wibmer's Theorem

Theorem (M. Wibmer)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal and $a \in \overline{K}^m$. Then the set

$$S_a = \{b \in \overline{K}^m \ : \ \mathrm{lpp}_{\overline{x}}(I_b) = \mathrm{lpp}_{\overline{x}}(I_a)\}$$

is parametric.
In particular, S_a is locally closed.

Gröbner cover

Definition (Gröbner cover)

By a Gröbner cover of $\overline{K}^m$ wrt to I and $\succ_{\overline{x}}$ we mean a finite set of pairs $\{(S_1, B_1), \ldots, (S_r, B_r)\}$ such that

1. *the S_i's are parametric and so, $B_i \subset \mathcal{O}(S_i)[\overline{x}]$ is the reduced Gröbner basis of I over S_i for $i = 1, \ldots, r$, and*
2. *the union of all S_i's equals $\overline{K}^m$.*

Theorem (Canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.

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Non-homogeneous ideals

Note (Homogenization and dehomogenization)

For homogenization introduce a new variable x_0 and extend $\succ_{\overline{x}}$ to the monomials in $\overline{x}, x_0$ by setting

$$\overline{x}^{\alpha} x_0^i \succ_{\overline{x},x_0} \overline{x}^{\beta} x_0^j \text{ iff } (\overline{x}^{\alpha} \succ_{\overline{x}} \overline{x}^{\beta}) \text{ or } (\overline{x}^{\alpha} = \overline{x}^{\beta} \text{ and } i > j)$$

Denote τ the dehomogenization consisting of substituting $x_0 = 1$.

Proposition (Preserving Gröbner bases)

Let $I \subset K[\overline{x}]$ be an ideal and $J \subset K[\overline{x}, x_0]$ a homogeneous ideal such that $\tau(J) = I$. Then, if $\{g_1, \ldots, g_r\}$ is a Gröbner basis of J wrt $\succ_{\overline{x},x_0}$ and the g_i's are homogeneous, then $\{\tau(g_1), \ldots, \tau(g_r)\}$ is a Gröbner basis of I wrt $\succ_{\overline{x}}$.

Non-homogeneous ideals

Proposition (Preserving the parametric character)

Let $J \subset K[\overline{a}][\overline{x}, x_0]$ be a homogeneous ideal such that $\tau(J) = I$ and $S \subset \overline{K}^m$ parametric wrt J and $\succ_{\overline{x},x_0}$. Then S is parametric wrt I and $\succ_{\overline{x}}$.

Definition (Affine canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a non-homogeneous ideal and let $J \subset K[\overline{a}][\overline{x}, x_0]$ denote its homogenization. The disjoint Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$ obtained by dehomogenization and reduction will be called the canonical Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$.

Remark

The affine canonical Gröbner cover does not necessarily summarize in a unique segment all the points corresponding to the same lpp. Nevertheless it is canonical, and when two segments occur with the same lpp they correspond to different kind of solutions at infinity.

Representation of I-regular functions

Definition (Generic representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \to \overline{K}[\overline{x}]$ a monic I-regular function. We say that $p \in K[\overline{a}][\overline{x}]$ generically represents f if

- $\mathrm{lpp}(f) = \mathrm{lpp}(p)$,
- $\mathrm{lc}(p)(a) \neq 0$ *on an open and dense set of points in* S,
- *if* $\mathrm{lc}(p)(a) \neq 0$ *then* $f(a,\overline{x}) = p(a,\overline{x})/\,\mathrm{lc}(p)(a)$, *otherwise is* $p(a,\overline{x}) = 0$.

Proposition

Every monic I-regular function $f : S \to \overline{K}[\overline{x}]$ admits a generic representation.

Representation of I-regular functions

Definition (Full representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \to \overline{K}[\overline{x}]$ a monic I-regular function. We say that a the set of polynomials $\{p_1, \cdots, p_r\} \subset K[\overline{a}][\overline{x}]$ fully represents f if

- $\text{lpp}(f) = \text{lpp}(p_i)$, *for* $1 \leq i \leq r$,
- *for* $a \in S$ *and* $1 \leq i \leq r$ *either* $\text{lc}(p_i)(a) \neq 0$ *or* $p_i(a, \overline{x}) = 0$,
- *for all* $a \in S$ *it exist al least one* i *and an open* $U \subset S$ *such that for every* $b \in U$ *is* $\text{lc}(p_i)(a) \neq 0$ *and* $f(a, \overline{x}) = p(a, \overline{x}) / \text{lc}(p)(a)$.

Proposition

Given a generic representation of a monic I-regular function $f : S \to \overline{K}[\overline{x}]$, the algorithm Extend *computes a full representation.*

Representation of I-regular functions

Example

Let $I = \langle ax + by, cx + dy \rangle$ and F be the monic I-regular function

$$F: \quad S = \mathbb{V}(ad - bc) \setminus \mathbb{V}(a, c) \subset \mathbb{C}^4 \quad \rightarrow \quad \mathbb{C}[x, y]$$

$$(a, b, c, d) \quad \mapsto \quad \begin{cases} x + \frac{b}{a} y & \text{if } a \neq 0 \\ x + \frac{d}{c} y & \text{if } c \neq 0 \end{cases}$$

Then

Generic representation of F: $\quad p = ax + by$

Full representation of F: $\quad \{p_1 = ax + by, p_2 = cx + dy\}$

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The Gröbner Cover algorithm

Input: A generating set $\{p_1, \cdots, p_s\} \subset K[\overline{a}][\overline{x}]$ of the ideal I and a monomial order $\succ_{\overline{x}}$.

Output: A set of pairs $\{(S_1, B_1), \ldots, (S_r, B_r)\}$ determining the canonical Gröbner cover of $\overline{K}^m$ wrt I, where

- the S_i are locally closed segments given in canonical prime-representation (P-representation),
- the B_i are a set of monic I-regular functions given in full representation.

Gröbner Cover algorithm

Algorithm (Homogeneous GröbnerCover)

GCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
  T :=**BuildTree**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$. *(Initial disjoint and reduced CGS)*
  $G := \emptyset$
  Group the segments of T *by* lpp*'s:* $T = \{T_i : 1 \leq i \leq s\}$.
    where $T_i = \{(S_{i_j}, B_{i_j}) : 1 \leq j \leq s_i\}$ *with* $\mathrm{lpp}(B_{ij}) = \mathrm{lpp}(B_{ik})$
  For *each* lpp*-segment* T_i
    S_i :=**LCUnion**$(S_{i_j} : 1 \leq j \leq s_i)$. *(Summarizing* lpp*-segments)*
    B_i :=**Basis**(S_i, T_i). *(Determining the generic basis for* S_i *using* T_i*.)*
    $G := G \cup (S_i, B_i)$
  end for
Return G

Gröbner Cover algorithm

Algorithm (Affine GröbnerCover)

GröbnerCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
  If F *is homogeneous* **then** $G :=$ **GCover**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
  else
    $F' :=$ **Homogenize**(F, x_0), $\overline{y} := \overline{x}, x_0$, $\succ_{\overline{y}} = \succ_{\overline{x}, x_0}$
    $G :=$ **GCover**$(F', \succ_{\overline{y}}, \succ_{\overline{a}})$
    $\overline{y} := \overline{x}, 1$, *(***Dehomogenize** *the bases in* G*)*
    Reduce *the bases in* G
  end if
  Extend *the bases in* G *(to obtain a full representation)*
Return G

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Consider a geometrical construction depending on a set of points $A_1, \dots, A_s$, whose coordinates are taken as parameters $\bar{a}$.

The construction produces some new points $P_1, \dots, P_r$, whose coordinates are taken as variables $\bar{x}$.

The problem is determining the configuration of the points A in order that the points P verify some property (example, they are aligned).

For this, write the equations reflecting the geometrical construction, and consider the corresponding parametric ideal I.

Let $\{(S_i, B_i) : 1 \leq i \leq s\}$ be the Gröbner cover of the parameter space wrt to I.

Automatic Discovery of Geometric Theorems

- As the locus does have dimension less than the whole parameter space, the generic segment must correspond to $\mathrm{lpp} = \{1\}$. The generic segment will be of the form

$$S_1 = \overline{K}^m \setminus \bigcup_i \mathbb{V}(\mathfrak{p}_i)$$

- The remaining segments will be all inside $\bigcup_i V(\mathfrak{p}_i)$
- If the construction is acceptable, the points P_i are, in general, uniquely determined by the points A_j. In that case we expect for the locus a segment S_2 corresponding to a solution in $\overline{x}$ whose reduced Gröbner basis has the set of coordinates as lpp.
- They can exist segments with more than one solution that we have then to analyze.
- They can also exist segments corresponding to degenerate constructions in which we are in general not interested.

S92. Casas Alberó conjecture

Conjecture

If a polynomial of degree n in x has a common root which each of its $n-1$ derivatives (not assumed to be the same), then it is of the form $P(x) = k(x+a)^n$, i.e. the common roots must be all the same.

Let

$$f(x) = x^n + \sum_{i=0}^{n-1} \binom{n}{i} a_i x^i.$$

We have

$$F_n(x,j) = \frac{j!}{n!} f^{(j)}(x) = x^{n-j} + \sum_{i=0}^{n-j-1} \binom{n-j}{i} a_{i+j} x^i$$

The system of the hypothesis becomes

$$\{F_n(x_1,0), F_n(x_1,1), \ldots, F_n(x_n,0), F_n(x_n,n-1)\}$$

S92. Casas Alberó Conjecture

```
> ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
> proc Fn(poly x,int n,int j)
  {
    int i; poly f=x^n;
    for(i=0;i<=n-1;i++)
    {
      f=f+binomial(n,i)*par(i+1+j)*x^i;
    }
    return(f);
  }
> int n=5;  ideal F;
> for (i=1;i<=n-1;i++)
  {
    F[size(F)+1]=Fn(var(i),n,0);
    F[size(F)+1]=Fn(var(i),n-i,i);
  }
```

```
> F;
F[1]=x1^5+(5*a4)*x1^4+(10*a3)*x1^3+(10*a2)*x1^2+(5*a1)*x1+(a0)
F[2]=x1^4+(4*a4)*x1^3+(6*a3)*x1^2+(4*a2)*x1+(a1)
F[3]=x2^5+(5*a4)*x2^4+(10*a3)*x2^3+(10*a2)*x2^2+(5*a1)*x2+(a0)
F[4]=x2^3+(3*a4)*x2^2+(3*a3)*x2+(a2)
F[5]=x3^5+(5*a4)*x3^4+(10*a3)*x3^3+(10*a2)*x3^2+(5*a1)*x3+(a0)
F[6]=x3^2+(2*a4)*x3+(a3)
F[7]=x4^5+(5*a4)*x4^4+(10*a3)*x4^3+(10*a2)*x4^2+(5*a1)*x4+(a0)
F[8]=x4+(a4)
```

```
> multigrobcov(F);
[1]:
   [1]:
      [1]:
         _[1]=1
      [2]:
         _[1]=1
      [3]:
         [1]:
            [1]:
               _[1]=0
            [2]:
               [1]:
                  _[1]=(a3-a4^2)
                  _[2]=(a2-a4^3)
                  _[3]=(a1-a4^4)
                  _[4]=(a0-a4^5)
```

```
[2]:
   [1]:
      [1]:
         _[1]=x4
         _[2]=x3^2
         _[3]=x2^3
         _[4]=x1^4
      [2]:
         _[1]=x4+(a4)
         _[2]=x3^2+(2*a4)*x3+(a4^2)
         _[3]=x2^3+(3*a4)*x2^2+(3*a4^2)*x2+(a4^3)
         _[4]=x1^4+(4*a4)*x1^3+(6*a4^2)*x1^2+(4*a4^3)*x1+(a4^4)
      [3]:
         [1]:
            [1]:
               _[1]=(a3-a4^2)
               _[2]=(a2-a4^3)
               _[3]=(a1-a4^4)
               _[4]=(a0-a4^5)
            [2]:
               [1]:
                  _[1]=1
```

S92. Casas Alberó conjecture

If we can solve the system for every n we are done.
But for concrete values of n we can compute the Gröbner cover.
For $n = 5$ we obtain two segments:

Segment	Basis
$\mathbb{C}^5 \setminus \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{1\}$
$\mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{x_4 + a_4, (x_3 + a_4)^2, (x_2 + a_4)^3, (x_1 + a_4)^3\}$

Thus the polynomial is $F_5(x, 0) = (x + a_4)^5$.

And the conjecture for the Gröbner cover for n becomes:

Segment	Basis
$\mathbb{C}^n \setminus \mathbb{V}\left(a_{n-2} - a_{n-1}^2, \dots, a_0 - a_{n-1}^n\right)$	$\{1\}$
$\mathbb{V}\left(a_{n-2} - a_{n-1}^2, \dots, a_0 - a_{n-1}^n\right)$	$\{x_{n-1} + a_{n-1}, \dots, (x_1 + a_{n-1})^{n-1}\}$

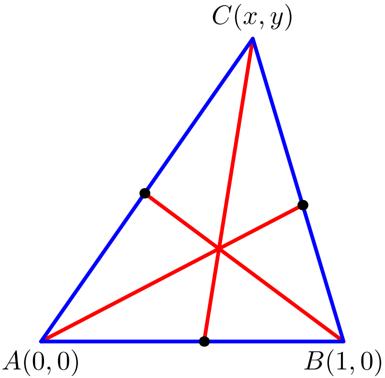
Thus the polynomial is $F_n(x, 0) = (x + a_{n-1})^n$.

Classical Steiner-Lehmus Theorem

Theorem (Classical Steiner-Lehmus)

The inner bisectors of angles A and B of a triangle ABC are of equal length if and only if the triangle is isosceles with AC=BC.

Proved: 1848



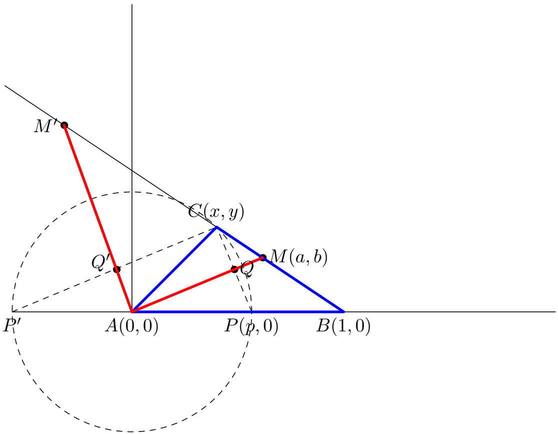
Recent studies

Generalization of the Steiner-Lehmus Theorem using automatic deduction of geometrical theorems.

- [Wa04] D. Wang, Elimination practice: software tools and applications, Imperial College Press, London, (2004), p. 144-159.
- [LoReVa09] R. Losada, T. Recio, J.L. Valcarce, Sobre el descubrimiento automático de diversas generalizaciones del Teorema de Steiner-Lehmus, Boletín de la Sociedad Puig Adam, no. 82, pp. 53-76, (2009).
- http://www.mathematik.uni-bielefeld.de/∼sillke/PUZZLES/steiner-lehmus

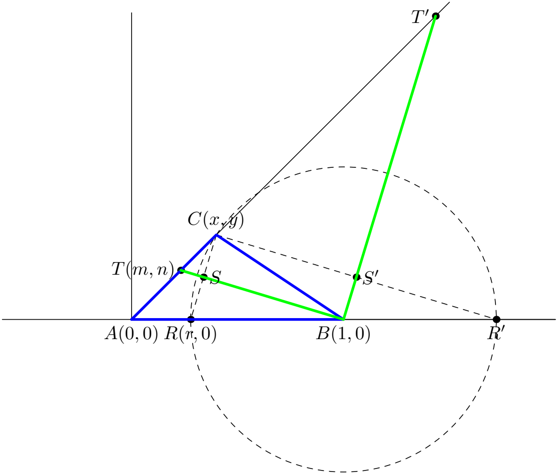
The novelty of our approach comes from the use of the Gröbner cover, and the rich information that this provide.

Trying to prove it automatically



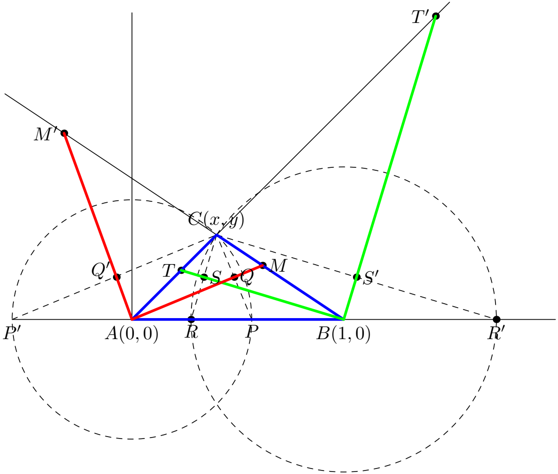
$$x^2 + y^2 = p^2, \begin{vmatrix} 0 & 0 & 1 \\ (x+p)/2 & y/2 & 1 \\ a & b & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0,$$

Trying to prove it automatically



$$(1-x)^2+y^2=(1-r)^2, \begin{vmatrix} 1 & 0 & 1 \\ (x+r)/2 & y/2 & 1 \\ m & n & 1 \end{vmatrix}=0, \begin{vmatrix} 0 & 0 & 1 \\ m & n & 1 \\ x & y & 1 \end{vmatrix}=0,$$

Trying to prove it automatically



$$a^2 + b^2 = (1 - m)^2 + n^2$$

Trying to prove it automatically

One bisector of A equal to one bisector of B.

System of equations:

$$\begin{cases} x^2 + y^2 - p^2, \\ (a-1)y + b(1-x), \\ -ay + b(x+p), \\ (1-x)^2 + y^2 - (1-r)^2, \\ my - xn, \\ (1-m)y + (x+r-2)n, \\ a^2 + b^2 = (1-m)^2 + n^2. \end{cases}$$

Parameters: x, y Variables: a, b, m, n, p, r

Solutions:

	$+$	$-$
p	i_A	e_A
$1-r$	i_B	e_B

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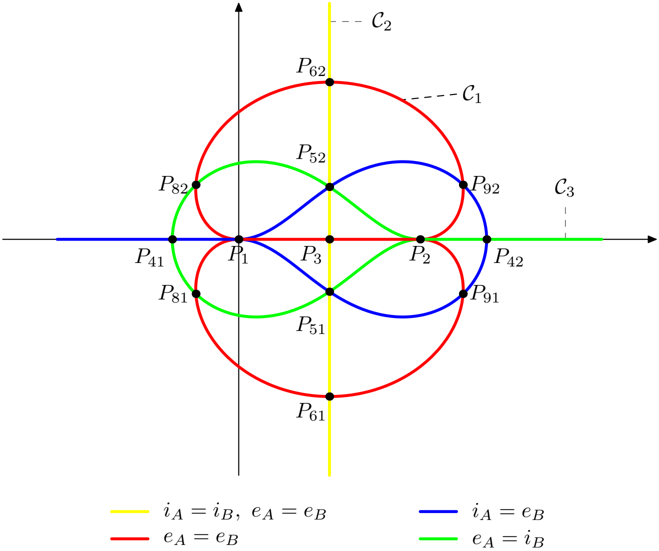
The Gröbner cover of the Steiner-Lehmus system



The Gröbner cover of the Steiner-Lehmus system

The algorithm is used taking $\mathrm{grevlex}(a, b, m, n, p, r)$ order for the variables. The parameters are (x, y).
In the result they appear the following curves:

$$
\begin{aligned}
\mathcal{C}_1 &= \mathbb{V}(8x^{10} + 41x^8y^2 + 84x^6y^4 + 86x^4y^6 + 44x^2y^8 + 9y^{10} - 40x^9 \\
&\quad -164x^7y^2 - 252x^5y^4 - 172x^3y^6 - 44xy^8 + 76x^8 + 246x^6y^2 \\
&\quad +278x^4y^4 + 122x^2y^6 + 14y^8 - 64x^7 - 164x^5y^2 - 136x^3y^4 \\
&\quad -36xy^6 + 16x^6 + 31x^4y^2 + 14x^2y^4 - y^6 + 8x^5 + 20x^3y^2 + 12xy^4 \\
&\quad -4x^4 - 10x^2y^2 - 6y^4 + y^2), \\
\mathcal{C}_2 &= \mathbb{V}(2x - 1). \\
\mathcal{C}_3 &= \mathbb{V}(y),
\end{aligned}
$$

The Gröbner cover of the Steiner-Lehmus system

and the following varieties representing points (only the real points are represented in the table):

Varieties	Real points
$V_1 = \mathbb{V}(y, x)$	$P_1 = (0, 0)$
$V_2 = \mathbb{V}(y, x - 1)$	$P_2 = (1, 0)$
$V_3 = \mathbb{V}(y, 2x - 1)$	$P_3 = (\frac{1}{2}, 0)$
$V_4 = \mathbb{V}(y, 2x^2 - 2x - 1)$	$P_{4,12} = \left(\frac{1 \pm \sqrt{3}}{2}, 0\right)$
$V_5 = \mathbb{V}(12y^2 - 1, 2x - 1)$	$P_{5,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right)$
$V_6 = \mathbb{V}(4y^2 - 3, 2x - 1)$	$P_{6,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
$V_7 = \mathbb{V}(4y^4 + 5y^2 + 2, 2x - 1)$	
$V_8 = \mathbb{V}(y^4 + 11y^2 - 1, 5x + 2y^2 + 1)$	$P_{8,12} = \left(2 - \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$
$V_9 = \mathbb{V}(y^4 + 11y^2 - 1, 5x - 2y^2 - 6)$	$P_{9,12} = \left(-1 + \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$

The Gröbner cover of the Steiner-Lehmus system

1. Segment with $\mathrm{lpp} = \{1\}$ Generic segment

Segment: $\mathbb{C}^2 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$

Basis: $\{1\}$

3. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$

Segment: $\mathcal{C}_2 \setminus (V_3 \cup V_5 \cup V_6))$

Basis:

$$\{p + r - 1, (4y^2 - 3)n + (4y)r, (4y^2 - 3)m + 2r, (4y^2 - 3)b + (4y)r, (4y^2 - 3)a - 2r + (-4y^2 + 3), 4r^2 - 8r + (-4y^2 + 3)\}.$$

The Gröbner cover of the Steiner-Lehmus system

2. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: $\mathcal{C}_1 \setminus (V_1 \cup V_2 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8 \cup V_9)$
Basis:

$$
\begin{aligned}
B_2 = \{&(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 + 3y^4 + 5y^2 - 1)r \\
&+(x^5 - 10x^4 + 2x^3y^2 + 17x^3 - 18x^2y^2 - 10x^2 + xy^4 + 17xy^2 - x - 8y^4 - 10y^2 + 2), \\
&(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 - 4x + 3y^4 + 5y^2 + 1)p \\
&+(x^5 + 2x^4 + 2x^3y^2 - 7x^3 + 6x^2y^2 + 4x^2 + xy^4 - 7xy^2 - x + 4y^4 + 4y^2), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)n \\
&+(-3x^4y + 6x^3y - 6x^2y^3 - 5x^2y + 6xy^3 - 3y^5 - 5y^3 + y), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)m \\
&+(-3x^5 + 6x^4 - 6x^3y^2 - 5x^3 + 6x^2y^2 - 3xy^4 - 5xy^2 + x), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)b \\
&+(3x^4y - 6x^3y + 6x^2y^3 + 5x^2y - 6xy^3 - 4xy + 3y^5 + 5y^3 + y), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)a \\
&+(2x^5 - 8x^4 + 4x^3y^2 + 12x^3 - 12x^2y^2 - 8x^2 + 2xy^4 + 12xy^2 + 2x - 4y^4 - 4y^2)\}
\end{aligned}
$$

The Gröbner cover of the Steiner-Lehmus system

4. Segment with $\mathrm{lpp} = \{n, b, r^2, p^2, a^2\}$
Segment: $\mathcal{C}_3 \setminus (V_1 \cup V_2)$
Includes the points $V_3 \cup V_4$
Basis: $\{n, b, r^2 - 2r - x^2 + 2x, p^2 - x^2, a^2 - m^2 + 2m - 1\}$

5. Segment with $\mathrm{lpp} = \{n, m, b, a, r^2, p^2\}$
Segment: V_5
Basis:

$$\{2n - 3yr, 4m - 3r, 2b + 3yp - 3y, 4a - 3p - 1, 3r^2 - 6r + 2, 3p^2 - 1\}$$

6. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: V_6
Basis: $\{r, p - 1, 2n - y, 4m - 1, 2b - y, 4a - 3\}$

The Gröbner cover of the Steiner-Lehmus system

7. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$
Segment: $V_7 \cup V_8$
Basis:

$$
\begin{aligned}
B_7 = \Big\{ & (7284y^6 + 88197y^4 - 15633y^2 - 3849)p + (8820y^6 + 97285y^4 \\
& -5905y^2 - 265)r + (-11380y^6 - 103045y^4 + 1425y^2 - 1015), \\
& (116y^6 + 1493y^4 + 2403y^2 + 179)n + (660y)r, \\
& (116y^6 + 1493y^4 + 2403y^2 + 179)m + (-72y^6 - 866y^4 - 1006y^2 - 58)r, \\
& (87932y^6 + 779351y^4 + 109221y^2 - 31747)b + (-35280y^7 - 389140y^5 \\
& +23620y^3 + 1060y)r + (16384y^7 + 59392y^5 + 56832y^3 + 19456y), \\
& (87932y^6 + 779351y^4 + 109221y^2 - 31747)a + (17640y^6 + 194570y^4 \\
& -11810y^2 - 530)r + (-51068y^6 - 786519y^4 - 157349y^2 + 5123), \\
& 660r^2 - 1320r + (-116y^6 - 1493y^4 - 2403y^2 - 179)\Big\} .
\end{aligned}
$$

The Gröbner cover of the Steiner-Lehmus system

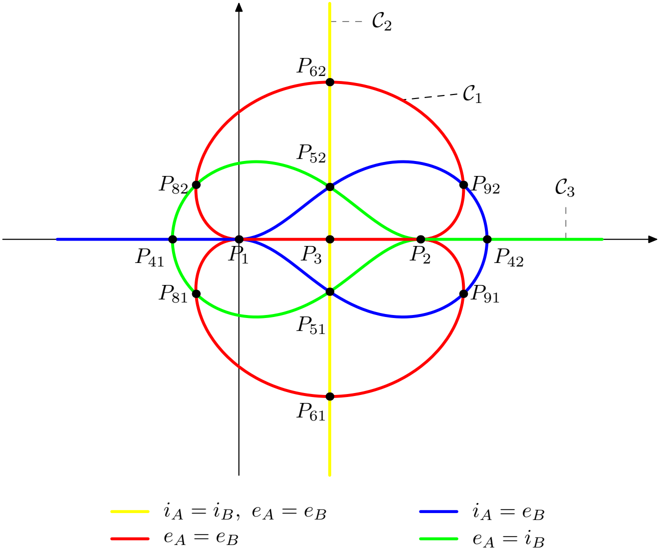
8. Segment with $\mathrm{lpp} = \{r, n, m, b, a, p^2\}$
Segment: V_9
Basis:

$$\{(23y^2 - 1)r + (-83y^2 + 6), (134y^2 - 13)n + (83y^3 - 6y),$$
$$(134y^2 - 13)m + (-268y^2 + 26),$$
$$(y^2 + 3)b + (-5y)p + (5y), (y^2 + 3)a + (-2y^2 - 1)p + (y^2 - 2),$$
$$5p^2 + (-y^2 - 8)\} .$$

9. Segment with $\mathrm{lpp} = \{b, r^2, nr, p^2, a^2\}$
Segment: V_1
Basis: $\{b, r^2 - 2r, nr - 2n, p^2, a^2 - m^2 - n^2 + 2m - 1\}$

10. Segment with $\mathrm{lpp} = \{n, r^2, p^2, bp, a^2\}$
Segment: V_2
Basis: $\{n, r^2 - 2r + 1, p^2 - 1, bp + b, a^2 + b^2 - m^2 + 2m - 1\}$

The Gröbner cover of the Steiner-Lehmus system



The classical Steiner-Lehmus theorem enhanced

- Segment 3 corresponds to the mediatrix of side AB except the points $P_{51}, P_{52}, P_{61}, P_{62}, P_3$.
- On segment 2 there are two solutions corresponding to

$$\left.\begin{array}{l} p+r-1 \\ 4r^2-8r+(-4y^2+3) \end{array}\right\} \Rightarrow p = 1-r = \pm\sqrt{1+4y^2}$$

- Thus there are two solutions corresponding to

$$i_A = i_B, \qquad e_A = e_B.$$

Solutions at the special points

$$s_A = p, \quad s_B = 1 - r$$

Point	(s_A, s_B)	Bisectors
P_{51}, P_{52}	$(0.5773502693, 0.5773502693)$	$i_A = i_B$
	$(0.5773502693, -0.577350269)$	$i_A = e_B$
	$(-0.5773502693, 0.5773502693)$	$e_A = i_B,$
	$(-0.5773502693, -0.5773502693)$	$e_A = e_B$
P_{61}, P_{62}	(1,1)	$i_A = i_B$
P_{81}, P_{82}	$(-0.3819659526, -1.272019650)$	$e_A = e_B$
	$(-0.3819659526, 1.272019650)$	$e_A = i_B$
P_{91}, P_{92}	$(-1.272019650, -0.381965976)$	$e_A = e_B$
	$(1.272019650, -0.381965976)$	$i_A = e_B$

Table: Coincidences of bisectors of A and B at the special points

Point	Branch	(s_A, s_B)	Bisectors
$(0, .7013671986)$	P_{62}-P_{82}	$(-.7013, -1.2214)$	$e_A = e_B$
$(0, .4190287818)$	P_{52}-P_{82}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.4190287818)$	P_{51}-P_{81}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.7013671986)$	P_{61}-P_{81}	$(-.7013, -1.2214)$	$e_A = e_B$
$(1, .7013671986)$	P_{62}-P_{92}	$(-1.2215, -0.7013)$	$e_A = e_B$
$(1, .4190287818)$	P_{52}-P_{92}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.4190287818)$	P_{51}-P_{91}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.7013671986)$	P_{61}-P_{91}	$(-1.2215, -0.7013)$	$e_A = e_B$

Table: Coincidences of bisectors of A and B at some points of curve $\mathcal{C}_1$.

Generalized Steiner-Lehmus Theorem

Theorem (Generalized Steiner-Lehmus)

Let ABC be a triangle and i_A, e_A, i_B, e_B the lengths of the inner and outer bisectors of the angles A and B. Then, considering the conditions for the equality of some bisector of A and some bisector of B the following excluding situations occur:

- *the triangle ABC is degenerate (i.e. C is aligned with A and B);*
- *ABC is equilateral and then $i_A = i_B$ whereas e_A and e_B become infinite, (P_{61}, P_{62});*
- *point C is in the center of an equilateral triangle, and then $i_A = i_B = e_A = e_B$, (P_{51}, P_{52});*
- *the triangle is isosceles but not of the special form of cases 2) or 3) and then $i_A = i_B \neq e_A = e_B$, (ordinary Theorem);*

continues in the next slice ..

Theorem (continues)

- $\frac{\overline{AC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, $\frac{\overline{BC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, *and then* $e_A = e_B = i_B$, (P_{81}, P_{82});
- $\frac{\overline{AC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, $\frac{\overline{BC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, *and then* $e_A = e_B = i_A$, (P_{91}, P_{92});
- *C lies in the curve of degree 10 relative to points A and B (case of curve $\mathcal{C}_1$) passing through all the special points above but is none of these points, and then only one of the following things arrive: either $e_A = e_B$ or $i_A = e_B$ or $e_A = i_B$ depending on the branch of the curve (see Figure, the color representing which of the situations occur);*
- *none of the above cases occur, and then no bisector of A is equal to no bisector of B.*