A polynomial generalization of the power-compositions determinant*

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Abstract

Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$, and $\mathbf{x} = (x_1, \ldots, x_p)$ a vector of indeterminates. For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ two p-compositions of n, define $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1+\alpha_1)^{\beta_1} \cdots (x_p+\alpha_p)^{\beta_p}$. In this paper we prove an explicit formula for the determinant $\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)}((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}})$. In the case $x_1 = \cdots = x_p$ the formula gives a positive answer to a conjecture by C. Krattenthaler.

1 Introduction

Let us start with some notation. If $\mathbf{u} = (u_1, \ldots, u_\ell)$ and $\mathbf{v} = (v_1, \ldots, v_\ell)$ are two vectors of the same length, we define $\mathbf{u}^{\mathbf{v}} = u_1^{v_1} \cdots u_\ell^{v_\ell}$ (where, to be consistent $0^0 = 1$). In our case, the entries u_i and v_i of $\mathbf{u}$ and $\mathbf{v}$ will be nonnegative integers or polynomials. We use $\mathbf{x} = (x_1, \ldots, x_p)$ to denote a vector of indeterminates and $\mathbf{1} = (1, \ldots, 1)$. The lengths of $\mathbf{x}$ and $\mathbf{1}$ will be clear from the context. If $\mathbf{u} = (u_1, \ldots, u_\ell)$, then $s(\mathbf{u})$ denotes the sum of the entries of $\mathbf{u}$, i.e. $s(\mathbf{u}) = u_1 + \cdots + u_\ell$, and $\bar{\mathbf{u}}$ denotes the vector obtained from $\mathbf{u}$ by deleting the last coordinate, $\bar{\mathbf{u}} = (u_1, \ldots, u_{\ell-1})$.

Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$. If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$ are two p-compositions of n, using the above notation, we have $\boldsymbol{\alpha}^{\boldsymbol{\beta}} = \alpha_1^{\beta_1} \cdots \alpha_p^{\beta_p}$. In [1] the following explicit formula for the determinant $\Delta(n,p) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left(\boldsymbol{\alpha}^{\boldsymbol{\beta}}\right)$ was proved:

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}}. \tag{1.1}$$

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In a complement [4] to his impressive *Advanced Determinant Calculus* [3], C. Krattenthaler mentions this determinant, and after giving the alternative formula

$$\Delta(n,p) = n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}} \tag{1.2}$$

he states as a conjecture a generalization to univariate polynomials. Namely, let x be an indeterminate and

$$\Delta(n,p,x) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left((x\cdot\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

Note that $(x\cdot\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x+\alpha_1)^{\beta_1}\cdots(x+\alpha_p)^{\beta_p}$.

Conjecture [C. Krattenthaler]:

$$\Delta(n,p,x) = (px+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}. \tag{1.3}$$

The main goal of this paper is to prove the generalization of the formula (1.3) for p indeterminates. For this, let $\mathbf{x} = (x_1,\ldots,x_p)$ be a vector of indeterminates, and let

$$\Delta(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

(Recall that $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1+\alpha_1)^{\beta_1}\cdots(x_p+\beta_p)^{\beta_p}$). Then, we prove the following formula (Theorem 4.1):

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}. \tag{1.4}$$

As $s(\mathbf{x}) = x_1+\cdots+x_p$, if $x_1=\cdots=x_p=x$, then $s(\mathbf{x}) = px$ and the conjectured identity (1.3) follows.

We also prove a variant of this result for proper compositions. A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_p)$ of n such that $\alpha_i \geq 1$ for all $i=1,\ldots,n$. Denote by $C^*(n,p)$ the set of proper p-compositions of n and define

$$\Delta^*(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

We also prove the following identity (Theorem 5.1):

$$\Delta^*(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n-1}{p}} \left(\prod_{j=1}^{p}\prod_{i=1}^{n-p+1}(x_j+i)^{\binom{n-i-1}{p-2}}\right) \prod_{i=1}^{n-p+1} i^{(n-i-p+1)\binom{n-i-1}{p-2}}. \tag{1.5}$$

The paper is organized as follows. At the end of this section we collect some combinatorial identities for further reference. In next section we prove the equivalence between the formula (1.2) given by Krattenthaler and (1.1). In Section 3 we prove two lemmas. The first one is a generalization of the determinant for $p = 2$. The second lemma uses the first and corresponds to a property of a sequence of rational functions which appear in the triangulation process of the determinant. Section 4 contains the proof of the main theorem (4.1). Finally, Section 5 is devoted to proving (1.5).

Lemma 1.1. *Let a, b, c, d, m and n be nonnegative integers. Then, the following equalities hold.*

(i) $\sum_{k\in\mathbb{Z}} \binom{a}{c+k}\binom{b}{d-k} = \binom{a+b}{c+d}$;

(ii) $\sum_{k\le n} \binom{a+k}{a} = \sum_{k\le n} \binom{a+k}{k} = \binom{n+a+1}{a+1}$;

(iii) $\sum_{r=1}^{n} r\binom{n+a-r}{a} = \binom{n+a+1}{a+2}$;

(iv) $\sum_{t=1}^{m-1} t\binom{a+t}{a} = (m-1)\binom{a+m}{a+1} - \binom{a+m}{a+2}$.

Proof. (i) is the well known Vandermonde's convolution, see [2, p. 169]. The formulas in (ii) are versions of the parallel summation [2, p. 159]. We prove (iii) and (iv). Part (iii) follows from

$$\begin{aligned}\sum_{r=1}^{n} r\binom{n+a-r}{a} &= \sum_{r=1}^{n} r\binom{n+a-r}{n-r} = \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{a+i}{a}\\ &= \sum_{k=0}^{n-1}\binom{a+k+1}{a+1} = \binom{a+n+1}{a+2}.\end{aligned}$$

For (iv), let S be the sum to be calculated. Consider the sum

$$\begin{aligned}T \;=\; &\binom{a+0}{a} + \overset{(m-1)}{\cdots} + \binom{a+0}{a}\\ &+\binom{a+1}{a} + \overset{(m-1)}{\cdots} + \binom{a+1}{a}\\ &\cdots\\ &+\binom{a+m-1}{a} + \overset{(m-1)}{\cdots} + \binom{a+m-1}{a}.\end{aligned}$$

Summing by columns gives $T = (m-1)\binom{a+m}{a+1}$. Now S is exactly the sum of the terms below the diagonal. Thus,

$$\begin{aligned}(m-1)\binom{a+m}{a+1} = T &= S + \sum_{i=0}^{m-1}\sum_{t=0}^{i}\binom{a+t}{a}\\ &= S + \sum_{i=0}^{m-1}\binom{a+i+1}{a+1}\\ &= S + \binom{a+m}{a+2},\end{aligned}$$

and the formula for S follows. $\square$

2 Equivalence between the two formulas for $\mathbf{x} = 0$

Here we prove the equivalence beetween the formulas (1.1) and (1.2) for $\Delta(n,p)$. Obviously, the result of substituting $x = 0$ in formula (1.3) of the Conjecture gives formula (1.2) for $\Delta(n,p)$.

Proposition 2.1. *Formulas* (1.1) *and* (1.2) *are equivalent.*

Proof. We derive formula (1.2) from (1.1), which was already proved in [1]. First, note that if $p < k \le n$, the binomial coefficient $\binom{p}{k}$ is zero. Thus, we can replace $\min\{p,n\}$ by n in formula (1.1). Analogously, if $n-k+1 < i \le n$, the binomial coefficient $\binom{n-i-1}{k-2}$ is zero, and we can replace the upper value $n-k+1$ by n in the inner product. Second, the case $a = n-1$, $b = d = p$ and $c = 0$ of Lemma 1.1 (i) yields

$$\sum_{k=1}^{n} \binom{n-1}{k}\binom{p}{k} = -1 + \sum_{k=0}^{n} \binom{n-1}{k}\binom{p}{p-k} = \binom{n+p-1}{p} - 1,$$

and, if $i \ge 1$, by taking $a = n-i-1$, $b = d = p$ and $c = -2$ in Lemma 1.1 (i), we obtain

$$\sum_{k=1}^{n-1} \binom{n-i-1}{k-2}\binom{p}{k} = \sum_{k} \binom{n-i-1}{k-2}\binom{p}{p-k} = \binom{n+p-i-1}{p-2}.$$

Therefore,

$$\begin{aligned}
\Delta(n,p) &= \prod_{k=1}^{\min\{n,p\}} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}} \\
&= \prod_{k=1}^{n} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}} \\
&= \left(\prod_{k=1}^{n} n^{\binom{n-1}{k}\binom{p}{k}} \right) \left(\prod_{k=1}^{n}\prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}\binom{p}{k}} \right) \\
&= n^{\binom{n+p-1}{p}-1} \left(\prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}} \right) n^{\sum_{k=1}^{n} \binom{-1}{k-2}\binom{p}{k}} \\
&= n^{\binom{n+p-1}{p}+p-1} \prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}} \\
&= n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}. \qquad \square
\end{aligned}$$

3 A recurrence

The next lemma evaluates the determinant

$$D_r(n,y,z) = \det_{0 \le i,j \le r} \left((y-i)^{n-j}(z+i)^j \right),$$

which corresponds to a variation of the determinant $\Delta(n,2,\mathbf{x})$.

Lemma 3.1.

$$D_r(n,y,z) = (y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r}(y-i)^{n-r}\right)\left(\prod_{i=1}^{r} i^{r-i+1}\right).$$

Proof.

$$\begin{aligned}
D_r(n,y,z) &= \begin{vmatrix} (y-0)^n(z+0)^0 & (y-0)^{n-1}(z+0)^1 & \cdots & (y-0)^{n-r}(z+0)^r \\ (y-1)^n(z+1)^0 & (y-1)^{n-1}(z+1)^1 & \cdots & (y-1)^{n-r}(z+1)^r \\ \vdots & \vdots & & \vdots \\ (y-r)^n(z+r)^0 & (y-r)^{n-1}(z+r)^1 & \cdots & (y-r)^{n-r}(z+r)^r \end{vmatrix} \\
&= \left(\prod_{i=0}^{r}(y-i)^n\right) \begin{vmatrix} 1 & (z+0)/(y-0) & \cdots & (z+0)^r/(y-0)^r \\ 1 & (z+1)/(y-1) & \cdots & (z+1)^r/(y-1)^r \\ \vdots & \vdots & & \vdots \\ 1 & (z+r)/(y-r) & \cdots & (z+r)^r/(y-r)^r \end{vmatrix} \\
&= \left(\prod_{i=0}^{r}(y-i)^n\right) \prod_{0\le i<j\le r} \left(\frac{z+j}{y-j} - \frac{z+i}{y-i}\right) \\
&= \left(\prod_{i=0}^{r}(y-i)^n\right) \prod_{0\le i<j\le r} \frac{(y+z)(j-i)}{(y-j)(y-i)} \\
&= \left(\prod_{i=0}^{r}(y-i)^n\right) (y+z)^{\binom{r+1}{2}} \frac{\prod_{i=1}^{r} i^{r-i+1}}{\prod_{i=0}^{r}(y-i)^r} \\
&= (y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r}(y-i)^{n-r}\right)\left(\prod_{i=1}^{r} i^{r-i+1}\right). \quad \square
\end{aligned}$$

Lemma 3.2. *Define* $f_r\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Q}(y,z)$ *recursively by*

$$\begin{aligned}
f_0(i,j) &= (z+i)^j; \\
f_{r+1}(i,j) &= f_r(i,j) \quad \text{if} \quad j \le r; \\
f_{r+1}(i,j) &= f_r(i,j) - \left(\frac{y-i}{y-r}\right)^{j-r} \frac{f_r(i,r)f_r(r,j)}{f_r(r,r)} \quad \text{if} \quad j > r.
\end{aligned}$$

Then

(i) $f_{r+1}(r,j) = 0$ *for* $j \ge r+1$;

(ii) $f_r(r,r) = (y+z)^r \dfrac{r!}{\prod_{i=0}^{r-1}(y-i)}$.

Proof. Part (i) is trivial using induction. To obtain $f_r = f_r(r,r)$, we take $n \ge r$ and calculate $D(n,y,z) = D_n(n,y,z)$ by a triangulation method.

The entry (i,j) of $D(n,y,z)$ is $(y-i)^{n-j}(z+i)^j = (y-i)^{n-j} f_0(i,j)$. If $j \geq 1$, add to the column j the column 0 multiplied by

$$-\frac{1}{(y-0)^{j-0}} \frac{f_0(0,j)}{f_0(0,0)}.$$

Then, the entry (i,j) with $j \geq 1$ is modified to

$$\begin{aligned} & (y-i)^{n-j} f_0(i,j) - (y-i)^{n-0} f_0(i,0) \frac{1}{(y-0)^{j-0}} \frac{f_0(0,j)}{f_0(0,0)} \\ = \ & (y-i)^{n-j} \left\{ f_0(i,j) - \left(\frac{y-i}{y-0} \right)^{j-0} \frac{f_0(i,k) f_0(k,j)}{f_0(0,0)} \right\} \\ = \ & (y-i)^{n-j} f_1(i,j). \end{aligned}$$

Therefore, $D(n,y,z) = \det_{0 \leq i,j \leq r} \big((y-i)^{n-j} f_1(i,j) \big)$ and $f_1(0,j) = 0$ for $j \geq 1$.

Now, assume that $D(n,y,z) = \det_{0 \leq i,j \leq n} \big((y-i)^{n-j} f_k(i,j) \big)$ for $k \geq 1$ with $f_k(i,j) = 0$ for $k, j > i$. Add to the column $j \geq k+1$ the column k multiplied by

$$-\frac{1}{(y-k)^{j-k}} \frac{f_k(k,j)}{f_k(k,k)}.$$

The entry (i,j) is modified to

$$\begin{aligned} & (y-i)^{n-j} f_k(i,j) - (y-i)^{n-k} f_k(i,k) \cdot \frac{1}{(y-k)^{j-k}} \cdot \frac{f_k(k,j)}{f_k(k,k)} \\ = \ & (y-i)^{n-j} \left\{ f_k(i,j) - \left(\frac{y-i}{y-k} \right)^{j-k} \frac{f_k(i,k) f_k(k,j)}{f_k(k,k)} \right\} \\ = \ & (y-i)^{n-j} f_{k+1}(i,j). \end{aligned}$$

Clearly $f_{k+1}(k,j) = 0$ for $j > k$. After n iterations, we get the determinant of a triangular matrix. Hence

$$D(n,y,z) = \det_{0 \leq k \leq n} \Big((y-k)^{n-k} f_k(k,k) \Big) = \prod_{r=0}^{n} (y-k)^{n-k} f_k.$$

The principal minor of order $r+1$ is $D_r(n,y,z) = \prod_{k=0}^{r} (y-k)^{n-k} f_k$. Therefore,

$$\frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} = (y-r)^{n-r} f_r. \tag{3.6}$$

On the other hand, by Lemma 3.1 we obtain

$$\begin{aligned} \frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} \ &= \ \frac{(y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r} (y-i)^{n-r} \right) \left(\prod_{i=1}^{r} i^{r-i+1} \right)}{(y+z)^{\binom{r}{2}} \left(\prod_{i=0}^{r-1} (y-i)^{n-r-1} \right) \left(\prod_{i=1}^{r-1} i^{r-i} \right)} \\ &= \ (y+z)^r \cdot r! \cdot \frac{(y-r)^{n-r}}{\prod_{i=0}^{r-1} (y-i)}. \end{aligned}$$

Comparing with (3.6), we have arrived at

$$f_r = (y+z)^r \frac{r!}{\prod_{i=0}^{r-1}(y-i)}. \quad \square$$

4 Proof of the main theorem

We sort $C(n,p)$ in lexicographical order. For instance, for $n=5$, and $p=3$, we obtain

$$\begin{aligned} C(5,3) = \ \{ & (5,0,0),(4,1,0),(3,2,0),(2,3,0),(1,4,0),(0,5,0), \\ & (4,0,1),(3,1,1),(2,2,1),(1,3,1),(0,4,1), \\ & (3,0,2),(2,1,2),(1,2,2),(0,3,2), \\ & (2,0,3),(1,1,3),(0,2,3), \\ & (1,0,4),(0,1,4), \\ & (0,0,5)\}. \end{aligned}$$

Let $M(n,p,\mathbf{x})$ be the matrix with rows and columns labeled by the p-compositions of n in lexicographical order and with the entry $(\boldsymbol{\alpha},\boldsymbol{\beta})$ equal to $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$. We have $\Delta(n,p,\mathbf{x}) = \det M(n,p,\mathbf{x})$.

An entry $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$ in $M(n,p,\mathbf{x})$ can be written in the form $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}(x_p+\alpha_p)^{\beta_p}$. For $0\le i,j\le n$, let S_{ij} be the matrix with entries $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $\alpha_p=i$ and $\beta_p=j$. Thus, the submatrix of $M(n,p,\mathbf{x})$ formed by the entries labeled $(\boldsymbol{\alpha},\boldsymbol{\beta})$ with $\alpha_p=i$ and $\beta_p=j$ can be written $\big(S_{ij}(x_p+i)^j\big)$. Note that

$$S_{kk} = M(n-k,p-1,\bar{\mathbf{x}}).$$

Define $f_0(i,j)=(x_p+i)^j$. Therefore,

$$M(n,p,\mathbf{x}) = (S_{ij}f_0(i,j)).$$

The idea is to triangulize the matrix $M(n,p,\mathbf{x})$ by blocks (i,j) in such a way that at each step only the last factor of each block is modified.

Theorem 4.1.

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}.$$

Proof. The proof is by induction on p. For $p=1$, $\Delta(n,p,x)$ is the determinant of the 1×1 matrix $((x+n)^n)$. Hence $\Delta(n,p,x)=(x+n)^n$. This value coincides with the right hand side of the formula for $p=1$.

Consider now the case $p=2$. Any 2-composition of n is of the form $(n-i,i)$ for some i, $0\le i\le n$. The determinant to be calculated is $\Delta(n,2,\mathbf{x}) =$

$\det_{0\le i,j\le n}\left((x_1+n-i)^{n-j}(x_2+i)^j\right)$. By taking $r=n$, $y=x_1+n$ and $z=x_2$ in Lemma 3.1, we get

$$\Delta(n,2,\mathbf{x}) = D_n(n,x_1+n,x_2) = (x_1+x_2+n)^{\binom{n+1}{2}}\prod_{i=1}^{n} i^{n-i+1}.$$

Therefore, the formula holds for $p=2$.

Now, let $p>2$ and assume that the formula holds for $p-1$. Begin with the matrix $M(n,p,\mathbf{x}) = (S_{ij}f_0(i,j))$.

Assume $\Delta(n,p,\mathbf{x}) = \det(S_{ij}f_r(i,j))$ where $S_{ij} = ((\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}})$, with $\alpha_p=i$, $\beta_p=j$, and $f_r(i,j)=0$ for $j>r,i$.

Fix a column $\boldsymbol{\beta}$ with $\beta_p=j>r$. For each $\boldsymbol{\gamma}\in C(n,p)$ with $\gamma_p=r$ and $\gamma_k\ge\beta_k$ for $k\in[p-1]$, add to the column $\boldsymbol{\beta}$ the column $\boldsymbol{\gamma}$ multiplied by

$$-\frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\binom{j-r}{\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}}\frac{f_r(r,j)}{f_r(r,r)}.$$

The differences $\bar{\boldsymbol{\delta}}=\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}$ are exactly the $(p-1)$-compositions of $j-r$. Also note that by the multinomial theorem,

$$(s(\bar{\mathbf{x}})+n-i)^{j-r} = ((x_1+\alpha_1)+\cdots+(x_{p-1}+\alpha_{p-1}))^{j-r} = \sum_{\bar{\boldsymbol{\delta}}}\binom{j-r}{\bar{\boldsymbol{\delta}}}(s(\bar{\mathbf{x}})+\bar{\boldsymbol{\alpha}})^{\bar{\delta}}.$$

Then, a term of column $\boldsymbol{\beta}$ is modified to

$$\begin{aligned}
&(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\beta}}f_r(i,j) - \sum_{\bar{\gamma}}\frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\binom{j-r}{\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}}\frac{f_r(r,j)}{f_r(r,r)}(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\gamma}}f_r(i,r)\\
= &(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\beta}}\left\{f_r(i,j)-\frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\left(\sum_{\bar{\delta}}\binom{j-r}{\bar{\boldsymbol{\delta}}}(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\delta}}\right)\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}\right\}\\
= &(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\beta}}\left\{f_r(i,j)-\frac{(s(\bar{\mathbf{x}})+n-i)^{j-r}}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}\right\}.
\end{aligned}$$

Now, define $f_{r+1}(i,j)=f_r(i,j)$ for $j\le r$ and

$$f_{r+1}(i,j) = f_r(i,j) - \frac{(s(\bar{\mathbf{x}})+n-i)^{j-r}}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}$$

for $j>r$. Note that $f_{r+1}(r,j)=0$ for $j>r$. After n iterations, we arrive at the matrix $(S_{ij}f_n(i,j))$ where $f(i,j)=0$ for $j>i$. Thus, the determinant $\Delta(n,p,\mathbf{x})$ is the product of the determinants of the diagonal blocks:

$$\Delta(n,p,\mathbf{x}) = \prod_{r=0}^{n}\det(S_{rr}f_r(r,r)).$$

Now, $S_{rr}=M(n-r,p-1,\bar{\mathbf{x}})$, a square matrix of order $\binom{n-r+p-2}{p-2}$. Therefore

$$\Delta(n,p,\mathbf{x}) = \prod_{r=0}^{n}\left(\Delta(n-r,p-1,x)f_r(r,r)^{\binom{n-r+p-2}{p-2}}\right).$$

Now, observe that the rational funcions f_r satisfy the hypothesis of Lema 3.2 with $y = s(\bar{\mathbf{x}}) + n = x_1 + \cdots + x_{p-1} + n$ and $z = x_p$. Thus,

$$f_r = f_r(r,r) = (s(\mathbf{x}) + n)^r \cdot \frac{r!}{\prod_{i=0}^{r-1}(s(\bar{\mathbf{x}}) + n - i)}.$$

By the induction hypothesis,

$$\begin{aligned}\Delta(n,p,\mathbf{x}) \quad &= \quad \prod_{r=0}^{n}\left((s(\bar{\mathbf{x}}) + n - r)^{\binom{n-r+p-2}{p-1}} \prod_{i=1}^{n-r} i^{(n-r-i+1)\binom{n-r+p-i-2}{p-3}}\right) \\ &\quad \cdot \prod_{r=0}^{n}\left((s(\mathbf{x}) + n)^r \cdot r! \cdot \frac{1}{\prod_{i=0}^{r-1}(s(\bar{\mathbf{x}}) + n - i)}\right)^{\binom{n-r+p-2}{p-2}}\end{aligned}$$

It remains to count how many factors of each type there are in the above product.

The number of factors $(s(\mathbf{x}) + n)$ is $\sum_{r=1}^{n} r\binom{n+p-r-2}{p-2}$. ¿From Lemma 1.1 (iii) for $a = p - 2$ this coefficient is $\binom{n+p-1}{p}$.

The number of factors $s(\bar{\mathbf{x}}) + n - i$, for $0 \le i \le n-1$, is (by using Lemma 1.1 (ii) with $a = p - 2$)

$$\binom{n-i+p-2}{p-1} - \sum_{r=i+1}^{n}\binom{n-r+p-2}{p-2} = \binom{n-i+p-2}{p-1} - \binom{n-i+p-2}{p-1} = 0.$$

Finally, for $1 \le i \le n$, the number of factors equal to i is

$$\sum_{r=0}^{n-i}(n-r-i+1)\binom{n+p-i-r-2}{p-3} + \sum_{r=i}^{n}\binom{n+p-r-2}{p-2}.$$

By taking $t = n - r - i + 1$, $a = p - 3$ and $m = n - i + 2$ in Lemma 1.1 (iv), the first sum is

$$\sum_{r=0}^{n-i}(n-r-i+1)\binom{n+p-i-r-2}{p-3} = (n-i+1)\binom{n+p-i-1}{p-2} - \binom{n+p-i-1}{p-1}.$$

According to Lemma 1.1 (ii), the second sum is

$$\sum_{r=i}^{n}\binom{n+p-r-2}{p-2} = \binom{n+p-i-1}{p-1}.$$

Hence, the number of factors i is $(n-i+1)\binom{n+p-i-1}{p-2}$. $\square$

5 Proper compositions

A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ of n such that $\alpha_i \ge 1$ for all $i = 1, \ldots, n$. We denote by $C^*(n,p)$ the set of proper p-compositions of n. In [1] the following formula was given:

$$\Delta^*(n,p) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C^*(n,p)}(\boldsymbol{\alpha}^{\boldsymbol{\beta}}) = n^{\binom{n-1}{p}} \prod_{i=1}^{n-p+1} i^{(n-i+1)\binom{n-i-1}{p-2}}.$$

Here, we study the corresponding generalization

$$\Delta^*(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

Theorem 5.1. *If $p \le n$, then*

$$\Delta^*(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n-1}{p}} \left(\prod_{i=1}^{n-p+1}\prod_{j=1}^{p}(x_j+i)^{\binom{n-i-1}{p-2}}\right)\prod_{i=1}^{n-p+1} i^{(n-i-p+1)\binom{n-i-1}{p-2}}.$$

Proof. The mapping $C^*(n,p)\to C(n-p,p)$ defined by $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_p)\mapsto \boldsymbol{\alpha}-\mathbf{1} = (\alpha_1-1,\ldots,\alpha_p-1)$ is bijective. Thus, we have

$$\begin{aligned}
\Delta^*(n,p,\mathbf{x}) &= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)} \left((\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha}-\mathbf{1})^{\boldsymbol{\beta}-\mathbf{1}+\mathbf{1}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n-p,p)} \left((\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}\right)\\
&= \Delta(n-p,p,\mathbf{x}+\mathbf{1})\prod_{\boldsymbol{\alpha}\in C(n-p,p)}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}.
\end{aligned}$$

The number of times that an integer i, $0\le i\le n-p$ appears as the first entry of p-compositions of $n-p$ is the number of solutions $(\alpha_2,\ldots,\alpha_{n-p})$ of $i+\alpha_2+\cdots+\alpha_p = n-p$, which is $\binom{n-p-i+p-2}{p-2}=\binom{n-i-2}{p-2}$. The count is the same for every coordinate. Then, in the product $\prod_{\boldsymbol{\alpha}\in C(n-p,p)}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}$, the number of factors equal to x_j+1+i is $\binom{n-i-2}{p-2}$; equivalently, for $1\le i\le n-p+1$, the number of factors equal to x_j+i is $\binom{n-i-1}{p-2}$. Therefore,

$$\begin{aligned}
\Delta^*(n,p,\mathbf{x}) &= \Delta(n-p,p,\mathbf{x}+\mathbf{1})\prod_{\boldsymbol{\alpha}\in C(n-p,p)}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}\\
&= (s(\mathbf{x})+n)^{\binom{n-1}{p}} \left(\prod_{i=1}^{n-p+1}\prod_{j=1}^{p}(x_j+i)^{\binom{n-i-1}{p-2}}\right)\prod_{i=1}^{n-p+1} i^{(n-i-p+1)\binom{n-i-1}{p-2}}. \quad \square
\end{aligned}$$

References

[1] J. M. Brunat and A. Montes, *The power-compositions determinant and its application to global optimization*, SIAM J. Matrix Anal. Appl. **23** (2001) 459-471.

[2] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics, A Foundation for Computer Science*, Addison-Wesley, 1989.

[3] C. Krattenthaler, *Advanced Determinant Calculus,* Séminaire Lotharingien Combin. **42** (1999), Article B42q.

[4] C. Krattenthaler, *Advanced Determinant Calculus: A complement,* `http://igd.univ-lyon1.fr/~kratt`.