Software for computing the Gröbner cover of a parametric ideal

Antonio Montes*
Universitat Politècnica de Catalunya, Spain.
antonio.montes@upc.edu

Extended abstract

The objective of this software presentation is to show the behavior and applications of the Singular library grobcov.lib that we have recently developed [Mo-web] to compute the Gröbner cover [MoWi10] of a parametric ideal. The content of the talk is the following:

1. Brief history of the study of parametric ideals using Gröbner bases.
2. Objective and definition of the Gröbner cover.
3. Some insight about the algorithm.
4. Some examples to show its interest and comparison with Comprehensive Gröbner systems (CGS).

1. Since the introduction by Weispfenning [We92] of Comprehensive Gröbner bases and systems (CGB and CGS), different algorithms have been developed for obtaining a CGS. Let K be a computable field and $\overline{K}$ an algebraically closed extension of K. Given a generating set $\{p_1, \cdots, p_s\}$ of the parametric ideal $I \subset K[\overline{a}][\overline{x}]$, where $\overline{a} = a_1, \ldots, a_m$ are the parameters and $\overline{x} = x_1, \ldots, x_n$ the variables, and a monomial order $\succ_{\overline{x}}$ in the variables, a CGS is a set of pairs (S_i, B_i) with $S_i \subset \overline{K}^m$ (called *segments*) and $B_i \subset K[\overline{a}][\overline{x}]$ (called *bases*) that specialize to a Gröbner basis in $\overline{K}[\overline{x}]$ for all points $a \in S_i$.

Probably the fastest family of algorithms are based [Ka97] on the stability of Gröbner bases in $K[\overline{x}, \overline{a}]$ with a product order with $\overline{x} \succ \overline{a}$ by specialization. Sato and Suzuki [SuSa06], Nabeshima [Na07] and Kapur-Sun-Wang [KaSuWa10] have successively improved the most popular of these algorithms. The objective of them is to obtain a CGS without any more requirements, and they can benefit from the existing fast implementations of ordinary Gröbner bases computations. Moreover, the last version in [KaSuWa10] already obtains a disjoint CGS.

2. Another family of algorithms [Mo02, We03, MaMo09], have as objective obtaining a canonical CGS with good properties for applications. These algorithms work in $K[\overline{a}][\overline{x}]$ and use only the $\overline{x}$ as variables, but on the counterpart they need specific algorithms for Gröbner bases computations. Based on Wibmer's Theorem [Wi07] we have introduced the canonical Gröbner cover [MoWi10] and implemented it in the Singular grobcov.lib library whose beta version and an executable tutorial can be downloaded at [Mo-web].

The canonical Gröbner cover consist of a set of triplets $\{(\text{lpp}_1, B_1, S_1), \ldots, (\text{lpp}_r, B_r, S_r)\}$ with the following properties:

- the $S_i \subset \overline{K}^m$ are *locally closed, disjoint segments* forming a partition of the parameter space ($\bigcup_i (S_i) = \overline{K}^m$), and are given in canonical prime-representation (P-representation, see [MoWi10]),

*Work partially supported by the Spanish Ministerio de Ciencia y Tecnología under project MTM2009-07242, and by the Generalitat de Catalunya under project 2009SGR1040.

- the B_i are a set of *monic I-regular functions* $f : S_i \to \mathcal{O}(S_i)[\overline{x}]$ (not simply polynomials) having *constant* lpp_i (leading power products) on S_i, such that, for every point $a \in S_i$, *B_i is the reduced Gröbner basis* of I_a. They are provided in full representation (or optionally in generic representation). See [MoWi10] for the detailed definitions. The full representation of an element $f : S_i \to \mathcal{O}(S_i)[\overline{x}]$ of the basis consist of one or more polynomials in $K[\overline{a}][\overline{x}]$ that for every point $(a_1, \ldots, a_m) \in S_i$ if one of them does not specialize to 0, then it specializes (after normalizing) to the corresponding polynomial of the reduced Gröbner basis, and at least one of these polynomials specializes to non-zero.

- Moreover *if* the ideal is *homogeneous*, then *different segments have different* lpp*'s.*

- For *non-homogeneous* ideals, homogenizing, computing the Gröbner cover and dehomogenizing leads also to a Gröbner cover. It is canonical if we homogenize the ideal and not only a basis, but even being canonical, it can contain more than one segment with the same lpp.

3. The algorithm [MoWi10] begins homogenizing the ideal (optionally only the basis) and then computes an ordinary CGS, that has to be disjoint with constant lpp on the segments that should be provided in P-representation. For this purpose it uses `cgsdr` (the implementation of the `buildtree` routine described in [MoWi10]). (Alternatively KLW-algorithm can substitute it, but then the output should be transformed into P-representation). Next, the segments with common lpp are added together using `LCUnion`, that by Wibmer's Theorem [Wi07] results in a locally closed segment. Then the generic representation of the basis (it is a subset of $K[\overline{a}][\overline{x}]$ that specializes to the Gröbner basis almost everywhere and to zero in the remaining points) is evaluated using the algorithm `Combine`. Then the basis is dehomogenized and reduced, and the full representation (optionally) is computed using `Extend` algorithm.

4. We will give some examples to show the usefulness of the Gröbner cover. The most emblematic ones are the Casas-Alberó conjecture [Dr06] and the Generalized Steiner-Lehmus Theorem. [Wa04, LoReVa10, StLe-web]. For other potential examples see [MoRe07]. Let us summarize here the results on the Casas Alberó conjecture.

The conjecture is the following: *If a polynomial of degree n in x has a common root which each of its $n-1$ derivatives (not assumed to be the same), then it is of the form $P(x) = k(x+a)^n$, i.e. the common roots must be all the same.*

Recently Hans-Christian Graf von Bothmer, Oliver Labs, Joseph Schicho, and Christiaan van de Woestijne proved that it is true whenever n is a prime power or 2 times a prime power. Except for the fact that this shows that it is true for infinitely many n, the conjecture remains widely open.

For fixed n we can verify it solving a system of equations. For $\deg(f) = 5$ the system of equations is:

$$\begin{aligned}
\{&x_1^5 + (5a_4)x_1^4 + (10a_3)x_1^3 + (10a_2)x_1^2 + (5a_1)x_1 + (a_0),\ x_1^4 + (4a_4)x_1^3 + (6a_3)x_1^2 + (4a_2)x_1 + (a_1),\\
&x_2^5 + (5a_4)x_2^4 + (10a_3)x_2^3 + (10a_2)x_2^2 + (5a_1)x_2 + (a_0),\ x_2^3 + (3a_4)x_2^2 + (3a_3)x_2 + (a_2),\\
&x_3^5 + (5a_4)x_3^4 + (10a_3)x_3^3 + (10a_2)x_3^2 + (5a_1)x_3 + (a_0),\ x_3^2 + (2a_4)x_3 + (a_3),\\
&x_4^5 + (5a_4)x_4^4 + (10a_3)x_4^3 + (10a_2)x_4^2 + (5a_1)x_4 + (a_0),\ x_4 + (a_4).\}
\end{aligned}$$

The call to `multigrobcov(F)` leads to the following Gröbner cover with only the two expected segments:

Segment	Basis
$\mathbb{C}^5 \setminus \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{1\}$
$\mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{x_4 + a_4, (x_3 + a_4)^2, (x_2 + a_4)^3, (x_1 + a_4)^4\}$

proving the conjecture for $\deg(f) = 5$, and showing that the polynomial is $F_5(x, 0) = (x + a_4)^5$. The Casas-Alberó conjecture is equivalent to the following conjecture of the Gröbner cover for $\deg(f) = n$:

Segment	Basis
$\mathbb{C}^n \setminus \mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{1\}$
$\mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{x_{n-1} + a_{n-1}, \ldots, (x_1 + a_{n-1})^{n-1}\}$

corresponding to the polynomial: $F_n(x, 0) = (x + a_{n-1})^n$, suggested by the Casas Alberó conjecture.

The Singular implementation (compreg.lib) of Suzuki-Sato's algorithm only solves the conjecture for $\deg(f) = 3$ giving there 8 segments being difficult to be interpreted. For higher degrees it does not answer.

References

[Dr06] J. Draisma. On the Casas-Alberó conjecture. (2006) www.linkpdf.com/.../on-the-casas-alvero-conjecture-1-the-problem-eduardo-casas-alvero–.pdf

[Ka97] M. Kalkbrenner, (1997). On the stability of Gröbner bases under specializations. *Jour. Symb. Comp.*, **24**:1, (1997), 51–58.

[KaSuWa10] D. Kapur, Y. Sun, and D.K. Wang. A new algorithm for computing comprehensive Gröbner systems. Proceedings of ISSAC'2010, ACM Press, (2010) 29–36.

[LoReVa10] R. Losada, T. Recio, J.L. Valcarce, On the automatic discovery of Steiner-Lehmus generalizations. Proceedings of ADG 2010, (J. Richter-Gebert, P. Schreck, editors), München, (2010) 171–174.

[MaMo09] M. Manubens, A. Montes, Minimal Canonical Comprehensive Gröbner Systems, *Jour. Symb. Comp.*, **44**:5, (2009), 463–478.

[Mo02] A. Montes. New Algorithm for Discussing Gröbner Bases with Parameters. *Jour. Symb. Comp.* **33**:1-2 (2002), 183–208.

[MoRe07] A. Montes, T. Recio, (2007). Automatic discovery of geometry theorems using minimal canonical comprehensive Groebner systems. Proceedings of ADG 2006, L.N.A.I., Springer, **4869**, (2007), 113–138.

[Mo-web] http://www-ma2.upc.edu/∼montes/ download a beta version of the software grobcov.lib. (2011).

[MoWi10] A. Montes, M. Wibmer. Gröbner Bases for Polynomial Systems with Parameters. *Jour. Symb. Comp.*, **45**, (2010), 1391–1425.

[Na07] K. Nabeshima. A speed-up of the algorithm for computing comprehensive Gröbner systems. Proceedings of ISSAC'2007, ACM Press, (2007), 299–306.

[StLe-web] http://www.mathematik.uni-bielefeld.de/∼sillke/PUZZLES/steiner-lehmus

[SuSa06] A. Suzuki and Y. Sato. A simple algorithm to compute comprehensive Gröbner bases using Gröbner bases. Proceedings of ISSAC'2006, ACM Press, (2006), 326–331.

[Wa04] D. Wang, Elimination practice: software tools and applications, Imperial College Press, London, (2004), p. 144–159.

[We92] V. Weispfenning. Comprehensive Gröbner bases. *Jour. Symb. Comp.* **14**, (1992), 1–29.

[We03] V. Weispfenning. Canonical comprehensive Gröbner bases. *Jour. Symb. Comp.* **36**, (2003), 669–683.

[Wi07] M. Wibmer, Gröbner bases for families of affine or projective schemes. *Jour. Symb. Comp.*, **42**:8 (2007), 803–834.