On polynomial digraphs

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Abstract

Let $\Phi(x,y)$ be a bivariate polynomial with complex coefficients. The zeroes of $\Phi(x,y)$ are given a combinatorial structure by considering them as arcs of a directed graph $G(\Phi)$. This paper –the first of a series of two– studies the relationship between the polynomial $\Phi(x,y)$ and the structure of $G(\Phi)$.

1 Introduction

Let $\Phi(x,y) \in \mathbb{C}[x,y]$ be a bivariate polynomial with complex coefficients and let I be the ideal generated by $\Phi(x,y)$. The variety $\mathbb{V}(I)$ of I is the set of ordered pairs $(u,v) \in \mathbb{C}^2$ such that $\Phi(u,v) = 0$. We give a combinatorial structure to $\mathbb{V}(I)$ by taking its elements as arcs of a digraph $G(\Phi)$, and we explore the relationship between the polynomial $\Phi(x,y)$ and the digraph $G(\Phi)$. This paper is the first one of a series of two. In it we give a general overview of the topic and some concrete families of polynomials are studied. In the sequel [3] we adopt an algebraic approach to the problem of deciding which polynomials produce a given graph as a connected component of $G(\Phi)$.

The digraphs $G(\Phi)$ were introduced in [2] under the name of *Galois graphs*. Lengths of walks, distances and cycles were described in terms of $\Phi(x,y)$. Also, when the coefficients of $\Phi(x,y)$ belong to a field k and $\overline{k}$ is the algebraic closure of k, the action of the Galois group $G(\overline{k}/k)$ on $G(\Phi)$ was studied (and this was the motivation for the name of Galois graphs).

Let us fix some notation. In this paper digraphs are allowed to be infinite, and to have multiple arcs and loops. A *graph* is a digraph without loops nor multiple arcs such that for each arc (u,v) there exists the arc (v,u). The pair of arcs $(u,v),(v,u)$ are called an *edge* of the graph and it is denoted by uv. Let u be a vertex of a digraph D. The *strong (connected) component*, or *component*, of u is the subdigraph of D induced by u and the set of vertices v such that there exist a directed path from u to v. The *underlying graph* of a digraph D is the graph obtained by taking as edges the set of uv with (u,v) an arc of D and $u \neq v$. The *weakly (connected) component* of u in a digraph D is the component of u in the underlying graph of D. Note that, in a graph, components and weakly components

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coincide. For undefined concepts about graph theory, we refer to [5, 13]. The component of u in $G(\Phi)$ is denoted by $\vec{G}(\Phi, u)$ and the weakly component by $G(\Phi, u)$.

In a monomial cx^iy^j, $c \neq 0$, the non negative integer i is called the *partial degree* respect to x (analogously for j and y). The *total degree* or *degree* of the monomial is the integer $i+j$. The *partial degree* respect to x of a bivariate polynomial $\Phi(x, y)$ is the maximum of the partial degrees of its monomials; and the *total degree* or *degree* of $\Phi(x, y)$ is the maximum of the degrees of its monomials. If $\Phi(x, y) = F_1(x, y)^{n_1} \cdots F_k(x, y)^{n_k}$ is the expression of $\Phi(x, y)$ as a product of irreducible polynomials over $\mathbb{C}$, the *radical* of $\Phi(x, y)$ is the polynomial $\operatorname{rad}\Phi(x, y) = F_1(x, y) \cdots F_k(x, y)$. A polynomial is *radical* if $\operatorname{rad}\Phi(x, y) = \Phi(x, y)$. We refer to [6, 7] for any other undefined concept about polynomials.

The paper is organized as follows. In the rest of this section we give an informal description of its contents. In the next section we define the digraph $G(\Phi)$. We shall see that some natural conditions on $\Phi(x, y)$ (such as $\Phi(x, y)$ to be radical, $\Phi(u, y) \neq 0$ for all $u \in \mathbb{C}$, etc.) can be assumed. Polynomials satisfying such conditions are called *standard polynomials*. If $\Phi(x, y)$ is a standard polynomial, then $G(\Phi)$ has only a finite number of loops and multiple arcs. Moreover, all vertices have finite indegree and and all of them, but a finite number, have the same indegree; analogously for outdegrees. The components (resp. weakly components) of $G(\Phi)$ containing these vertices, multiple arcs and loops are called *singular components* (resp. *singular weakly components*).

In Section 3 we show that every finite d-regular digraph is isomorphic to a component of $G(\Phi)$ for an appropriate $\Phi(x, y)$. Nevertheless, the construction produces an infinite weakly component. The interesting cases arise when components and weakly components have the same set of vertices. A digraph is called *polynomial* if it is isomorphic to $G(\Phi)$ or to a non singular component $\vec{G}(\Phi, u)$ with the same set of vertices as $G(\Phi, u)$.

In Section 4 we will see that Cayley digraphs on the additive and multiplicative groups of $\mathbb{C}$ are polynomial and we give the corresponding polynomial $\Phi(x, y)$. This implies that directed and undirected cycles, finite complete graphs, finite bipartite complete graphs $K_{d,d}$ and, in general, circulant digraphs are polynomial.

In Section 5 we study polynomials of partial degree one in each indeterminate. It is shown that all non-singular components are isomorphic, and a characterization of polynomials of partial degree one which give n-cycles as non-singular components is given, as well as those giving infinite paths. By relating these polynomials to the group of linear fractional transformations, we prove that Cayley digraphs on dihedral groups and on the groups of symmetries of regular polyhedra are polynomial.

Finally, we consider symmetric polynomials of total degree 2. In this case the structure of $G(\Phi)$ is also completely determined.

2 The digraph of a polynomial

Let

$$\Phi(x, y) = \sum_{i=0}^{d} a_i(x) y^i = \sum_{j=0}^{e} b_j(y) x^j$$

be a polynomial with complex coefficients. The digraph $G(\Phi)$ has $\mathbb{C}$ as set of vertices and an ordered pair (u, v) is an arc of multiplicity m if v is a root of multiplicity m of the polynomial $\Phi(u, y)$. If $\Phi(u, y)$ is the zero polynomial for some $u \in \mathbb{C}$ the multiplicity of all arcs (u, v), $v \in \mathbb{C}$, is taken to be 1, and the vertex u is called a *source universal vertex*. Note that the source universal vertices are the roots of the polynomial $A(x) = \gcd(a_1(x), \ldots, a_d(x))$. Analogously, a vertex v is called a *sink universal vertex* if $\Phi(x, v)$ is the zero polynomial; the sink universal vertices are the roots of the $B(y) = \gcd(b_1(y), \ldots, b_e(y))$. In the following we assume that $G(\Phi)$ has no universal vertices, or, equivalently that the polynomials $A(x)$ and $B(y)$ are constant. If $\Phi(x, y)$ is constant then $G(\Phi)$ is the complete or the null digraph on $\mathbb{C}$ depending on whether the constant is zero or not. Therefore, we can also assume from now on that $e, d \geq 1$ and that $a_d(x)$ and $b_e(y)$ are non zero polynomials.

The structure of $G(\Phi)$ is given by the structure of its components. To study the components of $G(\Phi)$, it is useful to put aside some special cases.

Consider the discriminant

$$D(x) = \text{Resultant}(\Phi(x, y),\ \Phi'_y(x, y),\ y)\,,$$

where $\Phi'_y(x, y)$ denotes the partial derivative of $\Phi(x, y)$ with respect to y. We have that $D(x) = 0$ if and only if $\Phi(x, y)$ has a multiple factor of positive degree in y, say $\Phi(x, y) = \Phi_1(x, y))^k \Phi_2(x, y)$ with $k \geq 2$. In this case, the digraphs $G(\Phi)$ and $G(\operatorname{rad} \Phi)$ differ only in the multiplicity of the arcs corresponding to the factor $\Phi_1(x, y)$. Therefore, we can assume that $D(x)$ is not the zero polynomial.

Note that all vertices have outdegree at most d and a vertex u has outdegree $< d$ if and only if $a_d(u) = 0$. Analogously, all vertices have indegree at most e and a vertex u has indegree $< e$ if and only if $b_e(u) = 0$. The roots of $a_d(x)$ are called *out-defective vertices* and the roots of $b_e(y)$ are called *in-defective vertices*.

Let $D(x) \neq 0$ and $A(x) = 1$. The leading coefficients of $\Phi(x, y)$ and $\Phi'_y(x, y)$ as polynomials in the indeterminate y are $a_d(x)$ and $da_d(x)$ respectively, so $a_d(x)$ is a factor of $D(x)$. Thus, the out-defective vertices are roots of $D(x)$. If u is the origin of a multiple arc, then $\Phi(u, y)$ has a multiple root. Therefore, $D(u) = 0$. Conversely, if $D(u) = 0$, then either $a_d(u) = 0$ or $\Phi(u, y)$ has a multiple root, i.e. u is an out-defective vertex or it is the origin of a multiple arc. We conclude that the roots of $D(x)$ are the out-defective vertices and the origins of multiple arcs. Analogously, the roots of

$$E(x) = \text{Resultant}(\Phi(x, y),\ \Phi'_x(x, y),\ x)\,,$$

are the in-defective vertices and the ends of multiple arcs.

The vertices with a loop are the roots of $L(x) = \Phi(x, x)$. There is a loop at each vertex if and only if $L(x)$ is the zero polynomial, which means that $\Phi(x, y)$ admits a factorization $\Phi(x, y) = (y - x)^k \Phi_1(x, y)$ with $k \geq 1$ and $\Phi_1(x, y)$ not divisible by $y - x$. The structure of $G(\Phi)$ is then completely determined by the structure of $G(\Phi_1)$, and thus it is not a restriction to assume that $L(x)$ is not the zero polynomial.

A polynomial $\Phi(x, y)$ is *standard* if it is non constant ($G(\Phi)$ is neither the complete or the null graph); $A(x)B(x)$ is constant (there are not universal vertices), $D(x) \neq 0$ (the polynomial $\Phi(x,y)$ is a radical polynomial), and $L(x)$ is not the zero polynomial ($G(\Phi)$ has no loops at every vertex). For a standard polynomial $\Phi(x,y)$, the roots of $S(x) = L(x)D(x)E(x)$ are called *singular vertices*. They are the vertices with a loop, vertices which are origin or end of multiple arcs, and defective vertices. A *singular component* (resp. *singular weakly component*) is a component (resp. weakly component) which contains some singular vertex. Note that only a finite number of singular components (weakly components) exist. We denote by $G(\Phi)^*$ the digraph obtained from $G(\Phi)$ by removing all its weakly singular components.

Note that different polynomials $\Phi(x,y)$ can give isomorphic digraphs $G(\Phi)$, as stated in the following lemma of straightforward proof.

Lemma 2.1. Let $\Phi(x,y) \in \mathbb{C}[x,y]$ and $a,b,c \in \mathbb{C}$ with $a,c \neq 0$. If $\Psi(x,y) = c\Phi(ax+b, ay+b)$, then the mapping $u \mapsto au+b$ is an isomorphism from $G(\Psi)$ to $G(\Phi)$.

3 Finite components

Our immediate goal is to show that every finite strongly connected d-regular digraph can be seen as a component of $G(\Phi)$ for some appropriate $\Phi(x,y)$. We need the following Lemma:

Lemma 3.1. Every d-regular digraph admits a 1-factorization.

Lemma 3.1 can be proved by using that every regular graph of even degree admits a 2-factorization [12]. See also [8] for a detailed proof.

Theorem 3.2. Let $D = (V,E)$ be a finite strongly connected d-regular digraph of order $n \geq 2$ with $V \subset \mathbb{C}$. Then there exists a polynomial $\Phi(x,y)$ such that D is a component of $G(\Phi)$.

Proof. Lemma 3.1 ensures that D admits a 1-factorization. Let $F_1,\ldots,F_d$ be the set of arcs of the 1-factors. For each $i \in [d] = \{1,\ldots,d\}$ let $L_i(x)$ be the interpolation polynomial such that $L_i(u) = v$ for each $(u,v) \in F_i$. In this way, the vertices adjacent from $u \in V$ are $L_1(u),\ldots,L_d(u)$. Define $\Phi(x,y) = (y-L_1(x))\cdots(y-L_d(x))$. In $G(\Phi)$, the vertex u is also adjacent to $L_1(u),\ldots L_d(u)$. Therefore $D = \vec{G}(\Phi,u)$. □

In the proof of Theorem 3.2, the polynomials $L_i(x)$ have degree $n-1$, so $\Phi(x,y)$ has degree d in y and degree $d(n-1)$ in x. If the given digraph D has order $n \geq 3$, then $d(n-1) > d$ and the weakly component $G(\Phi,u)$ is infinite, while the component $\vec{G}(\Phi,u) = D$ is finite. For instance, take $D = K_3$, the complete digraph of order 3, and choose $1,2$ and 3 as the vertices of D. The digraph D is 2-regular and admits the factorization F_1,F_2 where $F_1 = \{(1,2),(2,3),(3,1)\}$ and $F_2 = \{(1,3),(3,2),(2,1)\}$. Figure 1 shows the factorization of K_3; the arcs of F_1 are the thick ones.

The polynomial of degree 2 such that $L_1(1) = 2$, $L_1(2) = 3$ and $L_1(3) = 1$ is $L_1(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2$, and the polynomial of degree 2 such that $L_2(1) = 3$, $L_2(3) = 2, L_2(2) = 1$ is $L_2(x) = \frac{3}{2}x^2 - \frac{13}{2}x + 8$. If $\Phi(x,y) = (y-L_1(x))(y-L_2(x))$, then $\vec{G}(\Phi,u)$ is $D = K_3$. Nevertheless, the weakly component $G(\Phi,1)$ is infinite. In the Figure 2 vertices adjacent to 1, 2 and 3 which are not in $\vec{G}(\Phi,1)$ are shown.

Figure 1: A factorization of the digraph K_3

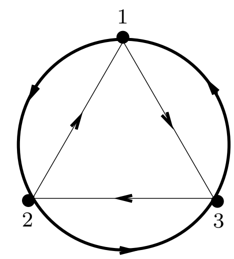
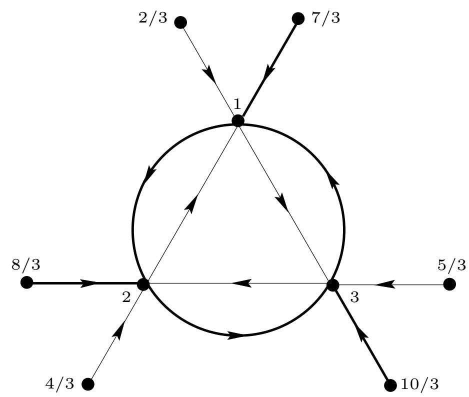


Figure 2: Part of $G(\Phi, 1)$



By Theorem 3.2 the condition on a d-regular digraph of being isomorphic to a component of $G(\Phi)$ for some polynomial $\Phi(x, y)$ is not restrictive at all. A d-regular digraph D is said to be *polynomial* if, for some standard polynomial $\Phi(x, y)$, the digraph D is isomorphic to $G(\Phi)^*$ or to a component of $G(\Phi)$, say $\vec{G}(\Phi, u)$, such that $\vec{G}(\Phi, u)$ and $G(\Phi, u)$ have the same set of vertices. Note that the vertices in $G(\Phi, u)$ are the vertices in $\vec{G}(\Psi, u)$, where $\Psi(x, y) = \mathrm{rad}(\Phi(x, y)\Phi(y, x))$. If $\Phi(x, y)$ is a standard symmetric polynomial, then $\Psi(x, y) = \Phi(x, y)$. Thus, a graph D is polynomial if it is isomorphic to $G(\Phi)$ or to a component of $G(\Phi)^*$ for some $\Phi(x, y)$. Next section show examples of polynomial graphs and digraphs.

4 Cayley digraphs

Cayley digraphs are relevant structures in different contexts such as modelling interconnection networks [10, 11], tessellations of the sphere and of the Euclidean Plane [1] and in combinatorial group theory [14]. Recall that, given a group Γ and a set $S \subseteq \Gamma$ with

$1 \notin S$, the *Cayley digraph* $\mathrm{Cay}(\Gamma, S)$ is defined by taking the elements in Γ as vertices and an ordered pair (u, v) is an arc if $v = su$ for some $s \in S$. A Cayley digraph $\mathrm{Cay}(\Gamma, S)$ is strongly connected if and only if S is a generating set of Γ (this is the reason why the condition of S being a generating system is often included in the definition). As we are interested in locally finite digraphs locally finite (all vertices have finite indegree and finite outdegree), in this paper the sets S defining Cayley digraphs are assumed to be finite. If $s^{-1} \in S$ for all $s \in S$, then $\mathrm{Cay}(\Gamma, S)$ is a graph. Cayley digraphs are known to be vertex transitive. This implies that all components of a Cayley digraph are isomorphic.

Theorem 4.1. 1. Let $\Phi(x, y)$ be a standard polynomial. If $\Phi(x, y) = f(y - x)$ for some univariate polynomial $f(s) \in \mathbb{C}[s]$, then $G(\Phi)$ is a Cayley digraph on $(\mathbb{C}, +)$.

2. Let $D = \mathrm{Cay}(\mathbb{C}, S)$ be a Cayley digraph on $(\mathbb{C}, +)$. Then there exists a univariate polynomial $f(s) \in \mathbb{C}[s]$ such that D is isomorphic to $G(\Phi)$ where $\Phi(x, y) = f(y - x)$ is a standard polynomial.

Proof. 1. Let $s_1, \ldots, s_d$ be the roots of $f(s)$. Then $\Phi(x, y) = f(y-x) = c(y-x-s_1)\cdots(y-x-s_d)$ with $c \neq 0$. In $G(\Phi)$ a vertex u is adjacent to the vertices $u+s_1, \ldots, u+s_d$. Because $\Phi(x, y)$ is standard, $s_i \neq 0$ for all i and $s_i \neq s_j$ for $i \neq j$. If $S = \{s_1, \ldots, s_d\}$, then we get $G(\Phi) = \mathrm{Cay}(\mathbb{C}, S)$.

2. Given $\mathrm{Cay}(\mathbb{C}, S)$ with $S = \{s_1, \ldots, s_d)$, consider the polynomial $f(s) = (s-s_1)\cdots(s-s_d)$ and take $\Phi(x, y) = f(y - x)$. The polynomial $\Phi(x, y)$ is standard and $\mathrm{Cay}(\mathbb{C}, S) = G(\Phi)$. □

Note that for a polynomial $\Phi(x, y) = f(y - x)$ as in Theorem 4.1 the components are always infinite. For instance, if $\Phi(x, y) = (y-x)^4-1 = (y-x-1)(y-x+1)(y-x-i)(y-x+i)$ then $G(\Phi)$ has no singular components and it is isomorphic to $\mathrm{Cay}(\mathbb{C}, \{1, -1, i, -i\})$. In this example, components and weakly components coincide and all of them are isomorphic to $G(\Phi, 0)$, the grid of integer coordinates.

Next theorem is the corresponding to Theorem 4.1 for Cayley digraphs on the multiplicative group of $\mathbb{C}$. As usual, $\mathbb{C}^*$ denotes $\mathbb{C} \setminus \{0\}$.

Theorem 4.2. 1. Let $\Phi(x, y)$ be an homogeneous standard polynomial. Then $G(\Phi)^*$ is a Cayley digraph on $(\mathbb{C}^*, \cdot)$.

2. Let $\mathrm{Cay}(\mathbb{C}^*, S)$ be a Cayley digraph on $(\mathbb{C}^*, \cdot)$. Then, there exists an homogeneous standard polynomial $\Phi(x, y)$ such that $\mathrm{Cay}(\mathbb{C}^*, S) = G(\Phi)$.

Proof. 1. Let $\Phi(x, y)$ be an homogeneous standard polynomial of total degree d. Note that $\Phi(x, y)$ being standard, it must be also of partial degree d in both indeterminates. We have $\Phi(x, sx) = x^d f(s)$ where $f(s)$ is a univariate polynomial in s of degree d. Let $s_1, \ldots, s_d$ be the roots of $f(s)$. Then, $\Phi(x, s_i x) = 0$ for $1 \leq i \leq d$ and $\Phi(x, y) = c(y-s_1x)\cdots(y-s_dx)$ for some $c \neq 0$. As $\Phi(x, y)$ is standard, $s_i \neq 1$ and $s_i \neq 0$ for all i, and $s_i \neq s_j$ for $i \neq j$. Each vertex $u \in \mathbb{C}^*$ is adjacent to the d vertices $s_1u, \ldots, s_du$. Therefore, $G(\Phi) = \mathrm{Cay}(\mathbb{C}^*, S)$.

2. Given a Cayley digraph $\mathrm{Cay}(\mathbb{C}^*, S)$ on the multiplicative group on $(\mathbb{C}^*, \cdot)$ and $S = \{s_1, \ldots, s_d)$, then $\Phi(x, y) = (y - s_1 x) \cdots (y - s_d x)$ is a standard polynomial and $G(\Phi)^* = \mathrm{Cay}(\mathbb{C}^*, S)$. □

As a Corollary of Theorems 4.1 and 4.2 we have:

Corollary 4.3. Cayley digraphs on the additive and multiplicative groups of $\mathbb{C}$ are polynomial.

A *circulant* digraph is a strongly connected Cayley digraph on a finite cyclic group. It is not a restriction to take the group U_n of the n-th roots of the unity as the cyclic group of order n. Then, a circulant digraph is a Cayley digraph of the form $\mathrm{Cay}(U_n, S)$, where S is a generating set of U_n. Now, $\mathrm{Cay}(U_n, S)$ is the component of 1 in $\mathrm{Cay}(\mathbb{C}^*, S)$, so we conclude

Corollary 4.4. Circulant digraphs are polynomial.

For instance, if ω is a primitive n-root of unity and we define $\Phi(x, y) = \prod_{i=1}^{n-1}(y - \omega^i x)$, the components of $G(\Phi)^*$ are complete graphs K_n. If ω is a primitive $2d$-root of unity and $\Phi(x, y) = \prod_{i=1}^{d}(y - \omega^{2i-1} x)$, then the components of $G(\Phi)^*$ are complete bipartite graphs $K_{d,d}$.

The n-prisms are a family of Cayley digraphs over non cyclic groups that are also polynomial. The *n-prism* is the Cayley graph

$$\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(1, 0), (n - 1, 0), (0, 1)\}.$$

For instance, the 3-dimensional cube is the 4-prism. The n-prism can be obtained by the polynomial $\Phi(x, y) = (y - \omega x)(y - \omega^{n-1} x)(xy - 2)$, where ω is a n-th primitive root of the unity.

5 Polynomials of partial degree one

In this section we give a complete description of the digraphs $G(\Phi)$ when $\Phi(x, y)$ is a polynomial of partial degree one. Note that a strongly connected digraph of indegree and outdegree equal to 1 is isomorphic either to a directed n-cycle $\vec{C}_n = \mathrm{Cay}(\mathbb{Z}_n, \{1\})$ for some n or to an infinite path $\vec{P} = \mathrm{Cay}(\mathbb{Z}, \{1\})$. We shall see that in any case, all components of $G(\Phi)^*$ are isomorphic. This result is applied to exhibit examples of Cayley digraphs on non commutative groups that are polynomial digraphs. First, we characterize the standard polynomials of partial degree 1.

Lemma 5.1. A polynomial $\Phi(x, y) = (cx + d)y - (ax + b)$ is standard if and only if $ad - bc \neq 0$ and it is not divisible by $y - x$.

Proof. Assume that $\Phi(x, y) = (cx + d)y - (ax + b) = (cy - a)x + dy - b$ is standard. Then the polynomials $D(x) = cx + d$ and $E(x) = cy - a$ are not the zero polynomials. First, consider the case $c = 0$. Then $d \neq 0$ and $a \neq 0$, so $ad - bc = ad \neq 0$. Second, assume $a = 0$. Then $c \neq 0$. If $b = 0$, then $B(y) = \gcd(cy - a, dy - b) = \gcd(cy, dy) \neq 1$, a contradiction.

Thus, $b \neq 0$ and $ad - bc = -bc \neq 0$. Finally, let $ac \neq 0$. By dividing $cx + d$ by $ax + b$, the remainder is $-bc/a + d = (ad - bc)/a$. As $A(x) = \gcd(cx + d, ax + b) = 1$, this remainder must be $\neq 0$. Therefore $ad - bc \neq 0$. The condition of not being divisible by $y - x$ ensures that $L(x)$ is not the zero polynomial.

Conversely, assume that $ad - bc \neq 0$. Then c and d can not be simultaneously zero, so $D(x) = cx + d$ is not the zero polynomial. Analogously, $E(x) = cy - a$ is not the zero polynomial. If $A(x) = \gcd(cx + d, ax + b)$ is non constant, then $c = a\lambda$ and $d = b\lambda$, which implies $ad - bc = 0$, a contradiction. Analogously, $B(x) = \gcd(cy - a, dy - b)$ non constant implies $ad - bc = 0$. Finally, if $L(x) = cx^2 + (d - a)x - b$ is the zero polynomial, then $b = c = 0$ and $d = a$. Therefore $\Phi(x, y) = d(y - x)$ is divisible by $y - x$. □

For standard polynomials $\Phi(x, y) = (cx + d)y - (ax + b)$ of partial degree one, the structure of $G(\Phi)^*$ is closely related to the properties of the maps $f(z) = (az + b)/(cz + d)$ with $ad - bc \neq 0$ (or, equivalently, with $ad - bc = 1$). These maps, called *linear fractional transformations* [9] or *Moebius transformations* [4], apply the complex plane minus $-d/c$ to the complex plane minus a/c. They are examples of conformal maps and form a group $M(\mathbb{C})$ under composition. It is useful to represent the group $M(\mathbb{C})$ as a quotient of the group $S_2(\mathbb{C})$ of square matrices of order 2 with determinant 1 as follows. The mapping

$$\begin{array}{ccc} S_2(\mathbb{C}) & \longrightarrow & M(\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & f(z) = \dfrac{az + b}{cz + d} \end{array}$$

is an exhaustive group homomorphism and its kernel is $\{1, -1\}$. Therefore $M(\mathbb{C}) \simeq S_2(\mathbb{C})/\{+1, -1\}$. Given $f \in M(\mathbb{C})$ defined by $f(z) = (az + b)/(cz + d)$ we denote $A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $[A_f]$ the class of A_f in $S_2(\mathbb{C})/\{+1, -1\}$.

Theorem 5.2. If $\Phi(x, y) = (cx+d)y-(ax+b)$ is a standard polynomial, then all components of $G(\Phi)^*$ are isomorphic.

Proof. Let ℓ_1, ℓ_2 be the roots of $L(x) = \Phi(x, x) = cx^2 + (d - a)x - b$. Note that if $c = 0$ all vertices have outdegree 1; otherwise, $-d/c$ is the unique out-defective vertex. Let $V = \mathbb{C}$ if $c = 0$ and $V = \mathbb{C} \setminus \{-d/c\}$ if $c \neq 0$. A vertex u in V is adjacent to the vertex $f(u) = (au+b)/(cu+d)$. Consider the function $f(x) = (ax+b)/(cx+d)$ defined on V. Assume that there exist components of $G(\Phi)^*$ which are directed cycles and let $n \geq 2$ be the minimum of the lengths of these cycles. Then n is the minimum positive integer such that there exists a vertex u in a non-singular component such that $f^n(u) = u$, or equivalently,

$$u = f^n(u) = \frac{a_n u + b_n}{c_n u + d_n} \text{ where } \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = A_f^n.$$

Thus, the solutions of $f^n(x) = x$ are the roots of $F(x) = c_n x^2 + (d_n - a_n)x - b_n$. But ℓ_1, ℓ_2 and u are three roots of $F(x)$ (if $\ell_1 = \ell_2$, then ℓ_1 has multiplicity 2), so $F(x)$ is the zero polynomial and $f^n(v) = v$ for all v. This implies that all components of $G(\Phi)^*$ are isomorphic to $\vec{C}_n$. □

Table 1:

n	$\vec{C}_n(a,b,c,d)$
2	$a+d$
3	$a^2+b\,c+a\,d+d^2$
4	$a^2+2\,b\,c+d^2$
5	$a^4+3\,a^2\,b\,c+b^2\,c^2+a^3\,d+4\,a\,b\,b\,d+a^2\,d^2$ $+3\,b\,c\,d^2+a\,d^3+d^4$
6	$3\,b\,c+a^2-a\,d+d^2$
7	$8\,c\,a^3\,b\,d+9\,c\,a^2\,b\,d^2+6\,c^2\,a^2\,b^2+9\,c^2\,a\,b^2\,d+a^6$ $+8\,c\,a\,b\,d^3+5\,a^4\,b\,c+c^3\,b^3+6\,c^2\,b^2\,d^2$ $+5\,d^4\,c\,b+d\,a^5+d^2\,a^4+d^3\,a^3+d^4\,a^2$ $+d^5\,a+d^6$
8	$2\,c^2\,b^2+a^4+d^4+4\,a^2\,b\,c+4\,d\,a\,b\,c+4\,c\,b\,d^2$
9	$c^3\,b^3+9\,c^2\,b^2\,d^2+15\,c^2\,a\,b^2\,d+9\,c^2\,a^2\,b^2$ $+6\,c\,a^3\,b\,d+6\,d^4\,c\,b+3\,c\,a^2\,b\,d^2$ $+6\,a^4\,b\,c+6\,c\,a\,b\,d^3+d^6+a^6+d^3\,a^3$
10	$5\,c^2\,b^2-d\,a^3+d^2\,a^2+5\,c\,b\,d^2+5\,a^2\,b\,c-d^3\,a$ $+d^4+a^4$

From a computational point of view, conditions on a, b, c and d for the components of $G(\Phi)^*$ being directed n-cycles are easily obtained. The polynomial $F(x)$ in the above proof is of the form $F(x) = c_n x^2 + (d_n - a_n)x - b_n = F_n(a,b,c,d)L(x) = F_n(a,b,c,d)(cx^2 + (d-a)x - b)$. Therefore, $F_n(a,b,c,d) = c_n/c = b_n/b = (d_n - a_n)/(d-a)$. Now, for each divisor k of n, the polynomial $F_k(a,b,c,d)$ must be a factor of $F_n(a,b,c,d)$. By dividing $F_n(a,b,c,d)$ by all the factors corresponding to digraphs $G(\Phi)^*$ with directed k-cycles as components (k divisor of n), we obtain the condition $\vec{C}_n(a,b,c,d)$ for the components of $G(\Phi)^*$ to be directed cycles of length n. In the table 5 the conditions $\vec{C}_n(a,b,c,d)$ for n from 2 to 10 are given.

The above proposition can be interpreted in the sense that if a linear fractional transformation $f(z) = (az+b)/(cz+d)$ generates a cyclic group of order n, then all its orbits but the fixed points (the loops in the graph) have n points (and therefore the components of $G(\Phi)^*$, where $\Phi(x,y) = (cx+d)y - (ax+b)$, are directed n-cycles).

The following theorem is a characterization of polynomials of partial degree one which give directed n-cycles. By a *primitive* n-root of a complex number z we mean a complex r such that n is the minimum positive integer with $r^n = z$.

Theorem 5.3. Let $\Phi(x,y) = (cx+d)y - (ax+b)$ be a standard polynomial and let r be a square root of $(a+d)^2 - 4$. Then the components of $G(\Phi)^*$ are directed n-cycles if and only if $a + d \notin \{+2, -2\}$ and $(a+d \pm r)/2$ are both n-th roots of 1 or both n-roots of -1, and at least one of them primitive.

Proof. Let $f \in M(\mathbb{C})$ defined by $f(z) = (az+b)/(cz+d)$ with $ad - bc = 1$. We have the following equivalences:

The components of $G(\Phi)^*$ are isomorphic to $\vec{C}_n$
$\Leftrightarrow f^n$ is the identity mapping
$\Leftrightarrow$ the order of $[A_f]$ is n
$\Leftrightarrow A_f$ has two distinct eigenvalues which are n-th roots of 1 or -1 and at least one of them primitive. The characteristic polynomial of A_f is $\lambda^2 - (a+d)\lambda + 1$ and the discriminant $(a+d)^2 - 4$. Therefore A_f has two distinct eigenvalues if and only if $a + d \neq \pm 2$. In this case the eigenvalues are $(a+d \pm r)/2$ and $[A_f]$ is of order n if they are n-th roots of 1 or -1 and at least one primitive. $\square$

For instance, consider the polynomial $\Phi(x, y) = (x - 1 + \sqrt{3})y - (x - 2 + \sqrt{3})$. We have the values $a = c = 1$, $b = -2 + \sqrt{3}$ and $d = -1 + \sqrt{3}$. They satisfy the 6 cycle-condition $3bc + a^2 - ad + d^2 = 0$ of table 5. On the other hand, we can use the Theorem 5.3. The matrix associated to the corresponding f is $A_f = \begin{pmatrix} 1 & -2+\sqrt{3} \\ 1 & -1+\sqrt{3} \end{pmatrix}$. The eigenvalues of A_f are $\lambda = (+\sqrt{3} \pm i)/2$ which are primitive 6-roots of -1. Therefore $[A_f]$ has order 6 and all components of $G(\Phi)^*$ are directed 6-cycles.

Taking into account that $M(\mathbb{C})$ is a non commutative group, we can find polynomial Cayley digraphs on non abelian groups. The following theorem opens the way.

Theorem 5.4. Let Γ be a subgroup of $M(\mathbb{C})$ generated by a set $S = \{f_1, \ldots, f_d\}$ of d linear fractional transformations. Then $\mathrm{Cay}(\Gamma, S)$ is polynomial.

Proof. Let $f \in \Gamma$ be a transformation defined by $f(z) = (az + b)/(az + d)$. We associate to f the polynomial $\Phi_f(x, y) = (cx + d)y - (ax + b)$. The group Γ is generated by a finite set, so it is a countable set. Therefore, the set of $u \in \mathbb{C}$ such that there exists $f \in \Gamma$ with u belonging to a singular component of $G(\Phi_f)$ is also a countable set. Thus, we can choose $u \in \mathbb{C}$ such that $\vec{G}(u, \Phi_f)$ is not a singular component of $G(\Phi_f)$ for all $f \in \Gamma$.

We shall see that the mapping $\mathrm{Cay}(\Gamma, S) \to \vec{G}(u, \Phi)$ defined by $f \mapsto f(u)$ is a digraph isomorphism. Indeed: it is injective, because $f(u) = u$ implies that f is the identity (otherwise u would be a loop vertex in $G(\Phi_f)$. It is exhaustive, because if v is a vertex in $\vec{G}(u, \Phi)$, then there exists a path $u = u_0, \ldots, u_\ell = v$. Now, each arc is of the form $(u_j, f_{i_j}(u_j))$ for some $f_{i_j} \in S$. If $f = f_{i_\ell} \cdots f_{i_1}$, we have $f(u) = v$. Finally, it preserves adjacencies: (g, h) is an arc in $\mathrm{Cay}(\Gamma, S) \Leftrightarrow g = f_i h$ for some $f_i \in S \Leftrightarrow g(u) = f_i(h(u))$ for some i, $\Leftrightarrow \Phi_{f_i}(h(u), g(u)) = 0$ for some $i \Leftrightarrow \Phi(x, y) = 0$. $\square$

The finite subgroups of $M(\mathbb{C})$ are determined ([9], Chapter VI). In particular, dihedral groups and the groups of symmetries of regular polyhedra are finite subgroups of $M(\mathbb{C})$. Therefore, we have:

Corollary 5.5. Cayley digraphs on dihedral groups and on the groups of symmetries of regular polyhedra are polynomial.

For instance, the dihedral group $D_{2n} = \langle f, t \mid f^n = t^2 = 1,\ tftf = 1\rangle$ is the subgroup of $M(\mathbb{C})$ generated by $f(z) = \omega z$ and $t(z) = 2/z$, where ω is a primitive n-root of unity. Then, the Cayley digraph $\mathrm{Cay}(D_{2n}, \{f, t\})$ is obtained by the polynomial $\Phi(x, y) = (y - \omega x)(xy - 2)$.

6 Symmetric polynomials of degree two

In this section we analyze the components of $G(\Phi)$ for a symmetric polynomial $\Phi(x, y)$ of total degree two. In particular, it is shown that all components of $G(\Phi)^*$ are isomorphic. As $G(\Phi)^*$ is a 2-regular graph, a component of $G(\Phi)^*$ is isomorphic to a (undirected) cycle $C_n = \mathrm{Cay}(\mathbb{Z}_n, \{1, -1\})$ or to the double ray graph $R = \mathrm{Cay}(\mathbb{Z}, \{1, -1\})$. From Theorem 5.2 we can assume that the polynomial is not of partial degree one, so it is of the form $\Phi(x, y) = x^2 + y^2 + axy + b(x + y) + c$. As a consequence of the discussion, the following Theorem is obtained.

Theorem 6.1. Let $\Phi(x, y) = x^2 + y^2 + axy + b(x + y) + c$ be a standard polynomial. If the equation $\lambda^2 + a\lambda + 1 = 0$ has two different roots ω_1 and ω_2 which are primitive n-th roots of the unity, then all components of $G(\Phi)^*$ are isomorphic to C_n, the cycle of length n. Otherwise, they are isomorphic to the double ray graph R.

Let v_0, v_1 be two adjacent vertices (non necessarily distinct) in $G(\Phi)$. We define v_n recurrently as follows: the polynomial $\Phi(v_{n-1}, y)$ has two roots; one of them is v_{n-2}; the other is, by definition, v_n. This means that the quotient of dividing $\Phi(v_{n-1}, y)$ by $y - v_{n-2}$ is $y - v_n$ (and the remainder is 0). The division algorithm gives $y - v_n = y + av_{n-1} + v_{n-2} + b$, hence

$$v_n + av_{n-1} + v_{n-2} = -b. \tag{1}$$

The solutions of the above recurrence with initial values v_0 and v_1 are the vertices of $G(\Phi, v_0)$, and the edges are of the form $v_{n-1}v_n$. Recurrence (1) is a linear recurrence of second order with characteristic equation $\lambda^2 + a\lambda + 1 = 0$. The discriminant is $\Delta(a) = a^2 - 4$. To solve the recurrence three cases have to be analyzed: $a = -2$, $a = 2$, and $a^2 - 4 \neq 0$. The discussion is a routine, but long. So we give only the results.

6.1 $a = -2$

For $a = -2$ the polynomial is $\Phi(x, y) = (x - y)^2 + b(x + y) + c$. Because $\Phi(x, y)$ is standard, b and c can not be simultaneously zero. In this case all components of $G(\Phi)$ are infinite and all components of $G(\Phi)^*$ are isomorphic to R. The structure of the singular components depends on b and c.

If $b = 0$ and $c \neq 0$, there are no singular components.

If $b \neq 0$, there are two singular components, one with the loop $\ell = -c/(2b)$ and the other with a double arc with origin at $m = (b^2 - 4c)/(8b)$.

For $a \neq -2$, it follows from Lemma 2.1 that if $\Psi(x, y) = (a + 2)\Phi(x - b/(a + 2), y - b/(a+2)) = x^2 + y^2 + axy + (c - b^2/(a+2))$, then $G(\Phi)$ is isomorphic to $G(\Psi)$. Therefore we can assume without loss of generality that the polynomial $\Phi(x, y)$ is of the form $\Phi(x, y) = x^2 + y^2 + axy + c$.

6.2 $a = 2$

The polynomial to consider is $\Phi(x, y) = (x + y)^2 + c$. As $\Phi(x, y)$ is standard, we have $c \neq 0$. In this case all components of $G(\Phi)$ are infinite and all components of $G(\Phi)^*$ are isomorphic to R.

There exist two singular vertices $\ell_1 = \sqrt{-c}/4$ and $\ell_2 = -\ell_1$, where $\sqrt{-c}$ is one of the square roots of $-c$. The components of $G(\Phi, \ell_1)$ and $G(\Phi, \ell_2)$ are the unique singular components of $G(\Phi)$.

6.3 $a^2 - 4 \neq 0$

Let ω_1 and $\omega_2 = 1/\omega_1$ the two different roots of $\lambda^2 + a\lambda + 1$.

6.3.1 ω_1 is not a n-th root of unity

In this case all components of $G(\Phi)^*$ are isomorphic to R. The singular components depend on c.

If $c = 0$ there exists a unique singular vertex $\ell_0 = 0$, with a loop of multiplicity two. The component $G(\Phi, 0)$ contains only the vertex 0.

If $c \neq 0$ there exist two loops ℓ_1 and ℓ_2 at the two square roots of $-c/(a+2)$, and two origins of multiple arcs at the vertices $m_1 = 2\sqrt{c/(a^2-4)}$ and $m_2 = -m_1$. There exist four singular components, all of them infinite, which are $G(\Phi, \ell_1)$, $G(\Phi, \ell_2)$, $G(\Phi, m_1)$ and $G(\Phi, m_2)$.

6.3.2 ω_1 is a n-th primitive root of unity

In this case all components of $G(\Phi)$ are finite and all components of $G(\Phi)^*$ are isomorphic to the cycle C_n.

The structure of the singular components depends on c and on the parity of n.

If $c = 0$ there exists a unique singular vertex $\ell_0 = 0$, which is a loop of multiplicity two. The component $G(\Phi, 0)$ contains only the vertex 0.

If $c \neq 0$, then two cases have to be considered.

- n even. Then there exist two singular components, which are $G(\Phi, \ell_1) = G(\Phi, \ell_2)$ and $G(\Phi, m_1) = G(\Phi, m_2)$.
- n odd. Then there exist two singular components, which are $G(\Phi, \ell_1) = G(\Phi, m_1)$ and $G(\Phi, \ell_2) = G(\Phi, m_2)$.

7 A conjecture

If $\Phi(x, y)$ is an homogeneous standard polynomial, then $G(\Phi)^*$ is a Cayley digraph (Theorem 4.2). Therefore if a component of $G(\Phi)^*$ is finite, all of them are finite and isomorphic. In Section 5 we have seen that if $\Phi(x, y)$ is a polynomial of partial degree 1, then if a component of $G(\Phi)^*$ is a directed n-cycle, then all of them are directed n-cycles. Also, for a symmetric polynomials $\Phi(x, y)$ of partial and total degree 2, if a component of $G(\Phi)^*$ is a n-cycle, then all of them are n-cycles. In all these examples components and weakly components have the same set of vertices.

Conjecture 7.1. Assume that H is a finite digraph isomorphic to a component $\vec{G}(\Phi, u)$ of $G(\Phi)^*$ such that $\vec{G}(\Phi, u)$ and $G(\Phi, u)$ have the same set of vertices. Then all components of $G(\Phi)^*$ are isomorphic to H.

For symmetrical polynomials the conjecture above is as follows.

Conjecture 7.2. If $\Phi(x, y)$ is a symmetric standard polynomial of partial degree d and H is a finite graph isomorphic to a component of $G(\Phi)^*$, then all components of $G(\Phi)^*$ are isomorphic to H.

In [3] more evidence of Conjecture 7.2 is given. For instance, it is shown that if a symmetric polynomial $\Phi(x, y)$ of partial degree 2 $G(\Phi)^*$ has a component which is a n-cycle (for small values of n) then all of them are n-cycles. Also, if $\Phi(x, y)$ is a polynomial such that $G(\Phi)^*$ has a component isomorphic to K_n ($2 \leq n \leq 6$) then all of them are isomorphic to K_n.

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