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A New Algorithm for Discussing Gröbner Bases with Parameters

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Let F be a set of polynomials in the variables $\overline{x} = x_1, \ldots, x_n$ with coefficients in $R[\overline{a}]$, where R is a UFD and $\overline{a} = a_1, \ldots, a_m$ a set of parameters. In this paper we present a new algorithm for discussing Gröbner bases with parameters. The algorithm obtains all the cases over the parameters leading to different reduced Gröbner basis, when the parameters in F are substituted in an extension field K of R. This new algorithm improves Weispfenning's comprehensive Gröbner basis CGB algorithm, obtaining a reduced complete set of compatible and disjoint cases. A final improvement determines the minimal singular variety outside of which the Gröbner basis of the generic case specializes properly. These constructive methods provide a very satisfactory discussion and rich geometrical interpretation in the applications.

1. Introduction

In many practical applications it is necessary to determine a Gröbner basis of an ideal of polynomials whose coefficients depend on some parameters. The main problem in this context is to obtain the distinct reduced Gröbner basis for all possible values of the parameters.

Examples of this situation can be found, for instance, in constructive algebraic geometry when determining conditions for a family of curves to have singular points (see Section 11.2); in robotics, in order to determine conditions on the magnitudes of a robot configuration with certain degrees of freedom and to solve the inverse kinematics problem (see Section 11.3); in the load-flow problem for a given electrical network (Montes, 1995, 1998), (see Section 11.4); in automatic theorem proving, and so on.

A direct approach to this problem can be derived applying the comprehensive Gröbner basis (CGB) algorithm of Weispfenning (1992) (see also Becker and Weispfenning, 1993). The goal of Weispfenning's algorithm is to obtain a CGB that specializes for all possible values of the parameters. For this purpose it constructs a Gröbner system, i.e. a complete set of constructible sets over the parameters, in order to add new polynomials to the initial basis and achieve the CGB. Nevertheless, the Gröbner system derived from CGB is, in general, not simple enough for applications, as it generates more cases than necessary and leads to a complex discussion. In this paper, we present an improved algorithm, called DISPGB, that abandons the objective of obtaining a CGB and focuses its interest on the essentially different reduced Gröbner basis (Gröbner system), simplifying the general discussion. A dichotomic tree discussion is carried out using quasi-canonical representations

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of specialization. A further improvement of the output (GENCASE algorithm) provides a minimal singular variety, outside of which the Gröbner basis of the generic case is specialization-invariant, and a reduced complete set of special cases inside the given minimal singular variety. This is an important point that is not obtained by Weispfenning's CGB algorithm. Although the question about the algorithm independence of the singular variety remains open, its practical determination by the algorithm provides a satisfactory discussion leading to a rich geometrical interpretation in applications (see Section 11). All the algorithms described in this paper are implemented in *Maple V*, release 6.[†]

In the case of linear systems, Sit (1992) gives an algorithm for the discussion using determinants. For this case, determinants provide the theoretical elements for defining the minimal singular variety. Our algorithm does not use determinants at all, but provides a polynomial in the parameters directly related to the value of the system determinant. Moreover, this "discriminant" polynomial is also obtained for general nonlinear systems, where no analogue of the system determinant is known.

In order to state precisely the problem, let $F = \{f_1, \ldots, f_s\} \subseteq R[\overline{x}, \overline{a}]$ be a set of polynomials in the variables $\overline{x} = x_1, \ldots, x_n$ and the parameters $\overline{a} = a_1, \ldots, a_m$, where R is a unique factorization domain (UFD). We can consider F as a polynomial set in the variables $\overline{x}$ with coefficients in $R[\overline{a}]$.

The goal of DISPGB is to obtain the distinct reduced Gröbner basis for all possible values of the parameters. Under this perspective, a case is a set of polynomial equalities and inequalities over the parameters $\overline{a}$ accepting the same expression for the reduced Gröbner basis. It would be desirable to distinguish cases only when their corresponding reduced Gröbner bases have different leading power product sets in the variables $\overline{x}$. This is a very demanding requirement. Nevertheless, using GENCASE, this objective is always reached for the generic case, and usually we get very close to it for the special cases as it will become clear in the examples.

The problem of specialization of Gröbner bases has been actually studied by many authors. The general basic problem can be formulated in the following way. Let $\mathcal{R}, \mathcal{R}'$ be Noetherian commutative rings with identity, and $\sigma : \mathcal{R} \to \mathcal{R}'$ be a ring homomorphism. When does a Gröbner basis G of an ideal $I \subseteq \mathcal{R}[\overline{x}]$ map to a Gröbner basis of $\langle \sigma(I) \rangle \subset \mathcal{R}'[\overline{x}]$ under the natural extension $\sigma : \mathcal{R}[\overline{x}] \to \mathcal{R}'[\overline{x}]$?

In Kalkbrener (1997) conditions obtained by different authors for this question are reviewed (see Gianni, 1987; Kalkbrenner, 1987; Adams and Boyle, 1992; Pauer, 1992; Gräbe, 1993; Assi, 1994; Becker, 1994). Kalkbrener (1997) also proves a necessary and sufficient condition when R' is a field.

We restrict ourselves to the case where the original ring $\mathcal{R}$ is a polynomial ring $R[\overline{a}]$ over a UFD R and $\mathcal{R}'$ some extension field K of the quotient field $\mathrm{Quot}(R)$. The variables $\overline{a}$ are considered as parameters. In this case $R[\overline{a}]$ can be embedded in the quotient field $R(\overline{a})$ and we do not need the general algorithm for computing Gröbner basis over rings (Möller, 1988), but only the classical (Buchberger, 1965, 1985; Gebauer and Möller, 1987; Gianni and Mora, 1987) algorithm over fields.

The structure of this paper is as follows:

Sections 3 and 4 are devoted to the study of how specialization acts on the pseudo-division and Gröbner basis algorithms. It is proved that they specialize to the correspond-

[†]The program and help libraries implementing the algorithms and a tutorial containing all the applications given in this paper with its corresponding time evaluation are available at the author's url http://www-ma2.upc.es/∼montes. Identification JSC3091.

ing algorithms in $K[\overline{x}]$ except for an $m-1$ dimensional variety (hypersurface). Any hypersurface outside which Buchberger's algorithm specializes will be called a *singular variety*. It is not difficult to complete the algorithm in order to obtain a singular variety.

Nevertheless, the singular variety computed by Buchberger's algorithm is in general larger than is strictly necessary, and is algorithm dependent. In order to minimize it we introduce a generalized Gaussian elimination (GGE) algorithm. This algorithm produces a new basis specializing properly for any specialization and is a good input for Buchberger's algorithm. It is also directly useful in applications, because it is able to simplify problems. Section 5 is devoted to the study of the GGE algorithm (Montes, 1999).

In Section 6 we describe how specializations can be specified in a quasi-canonical form in order to obtain a dichotomic discussion, that is a central point of DISPGB.

We are now ready to deal with the general DISPGB algorithm. Section 7 provides a general description of DISPGB that tries to make it understandable. Section 8 is devoted to two essential sub-algorithms. Section 9 is devoted to the control flow sub-algorithms; the final theorem describing DISPGB is also given there.

In Section 10 the algorithm GENCASE, designed to improve the output of DISPGB is described, and the theorem allowing definition and determination of the unique *generic case* and the associated *minimal singular variety* is proved.

Finally some classical applications are given in the last section, showing both the elegance and power of DISPGB.

2. Some Notations

In what follows we assume that $\succ_{\overline{x}}$ is a given monomial order for the variables $\overline{x}$. We use the following notations. A power product in the variables $\overline{x}$ is denoted $\overline{x}^{\alpha}$. A monomial is a product of a coefficient in the coefficient ring or field and a power product of the variables. The leading monomial, coefficient and power product are denoted:

$$\mathrm{lm}(f, \succ_{\overline{x}}) = c\,\overline{x}^{\alpha}, \qquad \mathrm{lc}(f, \succ_{\overline{x}}) = c, \qquad \mathrm{lpp}(f, \succ_{\overline{x}}) = \overline{x}^{\alpha}.$$

We will also need a monomial order $\succ_{\overline{a}}$ involving the parameters $\overline{a}$ and the compatible elimination product order $\succ_{\overline{x}\,\overline{a}}$ (see Bayer and Stillman, 1987). It is defined by:

For all $\alpha, \gamma \in \mathbb{Z}^n_{\geq 0}$ and $\beta, \delta \in \mathbb{Z}^m_{\geq 0}$

$$\overline{x}^{\alpha}\overline{a}^{\beta} \succ_{\overline{x}\,\overline{a}} \overline{x}^{\gamma}\overline{a}^{\delta} \iff \left\{\overline{x}^{\alpha} \succ_{\overline{x}} \overline{x}^{\gamma} \quad \text{or} \quad \left(\overline{x}^{\alpha} = \overline{x}^{\gamma} \quad \text{and} \quad \overline{a}^{\beta} \succ_{\overline{a}} \overline{a}^{\delta}\right)\right\}. \tag{1}$$

In order to avoid denominators and unnecessary factors in S-polynomials we define them as follows:

$$S(f,g) = \frac{\Gamma\,\overline{x}^{\gamma}}{\mathrm{lm}(f)}\,f - \frac{\Gamma\,\overline{x}^{\gamma}}{\mathrm{lm}(g)}\,g \tag{2}$$

where $\Gamma = \mathrm{lcm}(\mathrm{lc}(f), \mathrm{lc}(g))$ and $\overline{x}^{\gamma} = \mathrm{lcm}(\mathrm{lpp}(f), \mathrm{lpp}(g))$, with a slightly different normalization than in Weispfenning (1992).

For specializations coming from replacement of parameters, the corresponding ring homomorphism is:

$$\sigma : R[\overline{a}] \longrightarrow K \tag{3}$$

where K is any field extension of R and $\sigma|R = Id$. The discussion is complete when we consider the algebraic closure $\overline{K}$ of K. Providing a specialization is equivalent to choosing $\overline{a}_0 \in K^m$, and setting $\sigma(\overline{a}) = \overline{a}_0$. We will also denote by σ the natural extension $\sigma' : R[\overline{a}][\overline{x}] \longrightarrow K[\overline{x}]$ which is the identity on $\overline{x}$.

We can identify $\sigma_{\overline{a}_0}$ with $\overline{a}_0$, and extend σ for certain elements of $R(\overline{a})$ (more precisely to the local ring $R[\overline{a}]_{\ker(\sigma)}$). When we say that $\sigma(f)$ makes sense for $f \in R(\overline{a})[\overline{x}]$ it has to be understood that $f \in R[\overline{a}]_{\ker(\sigma)}[\overline{x}]$.

For obtaining k-quasi-canonical representations of the specification, as well as in other parts of the algorithms in this paper, we need to decompose polynomials into irreducible factors. We denote

$$\mathrm{FACVAR}(W, \succ_{\overline{a}}) = \{q_1, \ldots, q_s\}$$

the set of irreducible polynomials that are factors of the polynomials of W, normalized in a canonical form in order to be recognized when compared.

Definition (2) is clearly specialization-invariant when the leading coefficients of f and g are assumed different from zero.

The ideal generated by the set F in the polynomial ring $\mathcal{R}$ will be denoted $\langle F \rangle$ or explicitly $\langle F \rangle \mathcal{R}$, when some doubt about the polynomial ring can arise.

3. Pseudo-Division Algorithm and Specialization

We now study pseudo-division algorithm (see Cox *et al.*, 1992) under specialization and prove the following

Theorem 1. *Let $f, g_1, \ldots, g_s \in R[\overline{a}][\overline{x}]$, $W = \mathrm{FACVAR}(\mathrm{lc}(g_1), \ldots, \mathrm{lc}(g_s))$ and σ a specialization satisfying $\sigma(w) \neq 0$, for all $w \in W$, so that $\sigma(\mathrm{lc}(g_i)) \neq 0$, for $1 \leq i \leq s$. Let $q_1, \ldots, q_s \in R(\overline{a})[\overline{x}]$ and $r \in R(\overline{a})[\overline{x}]$ be the quotients and remainder of the pseudo-division of f by $g_1, \ldots, g_s$ in $R(\overline{a})[\overline{x}]$. Then $\sigma(q_1), \ldots, \sigma(q_s)$ and $\sigma(r)$ make sense, and they are the corresponding quotients and remainders for the pseudo-division of $\sigma(f)$ by $\sigma(g_1), \ldots, \sigma(g_s)$ in $K[\overline{x}]$.*

Proof. We prove the result by induction on the number of division steps. Call $(p^{(j)}, {q_1}^{(j)}, \ldots, {q_s}^{(j)})$ and $(p'^{(k)}, {q'_1}^{(k)}, \ldots, {q'_s}^{(k)})$ the partial (remainders, quotients) at steps j and k, respectively of the divisions in $R(\overline{a})[\overline{x}]$ and in $K[\overline{x}]$. We want to prove that for each j there is some k so that $\sigma(p^{(j)}) = p'^{(k)}$ and $\sigma(q_i^{(j)}) = q_i'^{(k)}$ for $i = 1, \ldots, s$. It is true, by construction, for $j = 0$ by picking $k = 0$. Suppose we are at step j in the division in $R(\overline{a})[\overline{x}]$, and let us assume the induction hypothesis. Let i_j be the first index for which $\mathrm{lpp}(g_{i_j})$ divides $\mathrm{lpp}(p^{(j)})$. The new partial quotient and the new partial remainder will be

$$\psi_{i_j} = \frac{\mathrm{lm}(p^{(j)})}{\mathrm{lm}(g_{i_j})}$$
$$p^{(j+1)} = p^{(j)} - \psi_{i_j} g_{i_j}. \tag{4}$$

Observe that by hypothesis, σ makes sense on ψ_{i_j} and $p^{(j+1)}$. So, in specializing, two cases can occur:

Case 1: $\sigma(\mathrm{lc}(p^{(j)})) = 0$. In this case $\sigma(\psi_{i_j}) = 0$ and $\sigma(p^{(j+1)}) = \sigma(p^{(j)})$, and the same is true for the partial quotients. So the same k applies to $j + 1$.

Case 2: $\sigma(\mathrm{lc}(p^{(j)})) \neq 0$. Then $\sigma(\psi_{i_j}) \neq 0$. As by hypothesis all $\mathrm{lpp}(g_i)$ remain stable under specialization, i_j is the first index for which $\mathrm{lpp}(\sigma(g_{i_j}))$ divides $\mathrm{lpp}(p'^{(k)})$, so

$$\psi'_{i_j} = \frac{\mathrm{lm}(p'^{(k)})}{\mathrm{lm}(\sigma(g_{i_j}))}$$

$$p'^{(k+1)} = p'^{(k)} - \psi'_{i_j}\sigma(g_{i_j}).$$

Using the induction hypothesis for j and specializing equations (4) yields:

$$\sigma(\psi_{i_j}) = \frac{\sigma(\mathrm{lm}(p^{(j)}))}{\sigma(\mathrm{lm}(g_{i_j}))} = \frac{\mathrm{lm}(p'^{(k)})}{\mathrm{lm}(\sigma(g_{i_j}))} = \psi'_{i_j}$$

$$\sigma(p^{(j+1)}) = p'^{(k)} - \psi'_{i_j}\sigma(g_{i_j}) = p'^{(k+1)},$$

so in this case we pick $k+1$. Note that the partial quotients also remain stable. □

The pseudo-division in $R(\overline{a})[\overline{x}]$ introduces coefficients with denominators in $R[\overline{a}]$. But these denominators arise only from leading coefficients of the divisors $g_1, \ldots, g_s$, and are divisors of a product of powers of them. Under the hypothesis of Theorem 1, we can still multiply f by a convenient factor μ in $R[\overline{a}]$ to eliminate non-vanishing denominators for the specification.† This will only change the normalization of f.

Writing $q'_i = \mu\, q_i$ and $r' = \mu\, r$, we have

$$\mu\, f = \sum_{i=1}^{s} q'_i g_i + r'. \tag{5}$$

The result will be a division with quotients q'_i, $(1 \le i \le s)$ and remainder r' in $R[\overline{a}][\overline{x}]$. If the condition on the leading monomials is satisfied, Theorem 1 guarantees that the result specializes properly except for the not-null factor $\sigma(\mu) \neq 0$. We denote this algorithm PDIV and the remainder

$$r' = \overline{f}^{[g_1,\ldots,g_s]}.$$

4. Buchberger's Algorithm and Specialization

We denote GB0 the ordinary Buchberger's algorithm (see Cox *et al.*, 1992), that only adds remainders of S-polynomials, acting on $k[\overline{x}]$, where k is any field (in particular it can be $R(\overline{a})$ or K).

We define PGB0 as the version of GB0 that uses equation (2) for computing S-polynomials and the normalization of the divisions described in Section 3 to avoid denominators when computing in $R(\overline{a})[\overline{x}]$. We have:

Theorem 2. *Let* $F \subset K[\overline{a}][\overline{x}]$, $G_0 = \mathrm{PGB0}(F, \succ_{\overline{x}})$ *and*

$$W = \mathrm{FACVAR}(\{\mathrm{lc}(g) : g \in G_0\}, \succ_{\overline{a}}).$$

Then

(i) G_0 *is a Gröbner basis of* $\langle F\rangle R(\overline{a})[\overline{x}]$ *whose polynomials belong to* $R[\overline{a}][\overline{x}]$.
(ii) *If* $\sigma(w) \neq 0$, *for all* $w \in W$, *then* $\sigma(G_0)$ *is a Gröbner basis of* $\langle\sigma(F)\rangle K[\overline{x}]$.
(iii) G_0 *can be minimized and reduced to a new basis* G *whose elements are in* $R[\overline{a}][\overline{x}]$. *Furthermore, under the above conditions on* σ, $\sigma(G)$ *is the reduced Gröbner basis of* $\langle\sigma(F)\rangle K[\overline{x}]$ (*apart from normalization*).

†There are several possible definitions of μ depending on the ring, field and implementation.

We write down the algorithm PGB outlined in Theorem 2 (iii). The procedures MINIMIZE and REDUCE to be used are those described in Cox *et al.* (1992) (with PDIV as division algorithm). The goal of NORMALIZE is ensuring the uniqueness of the reduced Gröbner basis. It takes the primitive part and chooses in a unique form the R-factor unity in the leading coefficient. The choice depends on the ring and the implementation.

PGB (Parametric Gröbner Basis)

Input: $F = \{f_1, \ldots, f_s\} \subset R[\overline{x}, \overline{a}]$, and $\succ_{\overline{xa}}$
Output: G the unique reduced Gröbner basis of $\langle F \rangle$, with respect to $[\overline{x}]$.
W: A singular variety outside of which the basis specializes.

$G_1 := \text{PGB0}(F, \succ_{\overline{x}})$
$W := \text{FACVAR}(\{\text{lc}(g, \succ_{\overline{x}}) : g \in G_1\})$
$G := \text{NORMALIZE}(\text{REDUCE}(\text{MINIMIZE}(G_1, \succ_{\overline{x}}), \succ_{\overline{x}}), \succ_{\overline{xa}})$.

Using Definition 2 and Theorem 1 the proof of the theorem above is easy.

The output of PGB is G and W. Nevertheless, W is a sufficient singular variety but not a minimal one. It is important to obtain a minimal solution. This objective will be approached in the next section and solved in Section 10.

5. GGE: Generalized Gaussian Elimination Algorithm

Using PGB produces the unique reduced Gröbner basis. Instead, the computed singular variety is algorithm depending on the development of PGB0 (see Example 5.1).

To reduce the variety as much as possible we introduce a GGE to transform the initial basis F of the ideal into a more convenient basis F' and use it as a pre-processor for PGB0.

The idea of the algorithm comes from ordinary Gaussian elimination, and produces a triangulation of the leading terms with respect to the order $\succ_{\overline{xa}}$.

GGE (Generalized Gaussian Elimination)

Input: $F = [f_1, \ldots, f_s] \subset R[\overline{x}, \overline{a}]$, ($\overline{x}$ and $\overline{a}$ as variables)
$\succ_{\overline{x}\,\overline{a}}$ the product elimination order.
Output: F' a new basis of $\langle F \rangle$ that specializes for any σ.

$G := F$; $G' = \phi$
WHILE $G \neq G'$
  FOR i from 1 to #G DO
    $G' := G$
    $G := \phi$
    FOR k from 1 to #G' DO
      IF $k \neq i$ THEN
        $f := \overline{g'_k}^{[g'_i]}$ respect to the order $\succ_{\overline{x}\,\overline{a}}$.
        IF $f \neq 0$ THEN $G := G \cup \{f\}$
      ELSE $G := G \cup \{g'_i\}$
$F' := \text{primpart}(G)$.

Proposition 3. *Let $F \subset R[\overline{a}][\overline{x}]$, and $F' := \mathrm{GGE}\,(F, \succ_{\overline{xa}})$. Then*

(i) *F' is a new basis of $\langle F\rangle \mathrm{Quot}(R)[\overline{x},\overline{a}]$ that has less or equal number of elements than F.*
(ii) *$\langle \mathrm{lpp}(F, \succ_{\overline{xa}})\rangle \subseteq \langle \mathrm{lpp}(F', \succ_{\overline{xa}})\rangle$, and also $\langle \mathrm{lpp}(F, \succ_{\overline{x}})\rangle \subseteq \langle \mathrm{lpp}(F', \succ_{\overline{x}})\rangle$.*
(iii) *F' specializes, i.e. $\sigma(F')$ is also a basis of $\langle \sigma(F)\rangle$.*
(iv) *The algorithm terminates.*
(v) *Each $p \in F'$, is the remainder of dividing p by $F' - \{p\}$ with respect to the given order $\succ_{\overline{xa}}$. We say that F' is reduced for this order.*

Proof. (i) At each division in the algorithm we substitute, in a given F-basis, two polynomials in $R[\overline{a}][\overline{x}]$, say $\{f, g\}$ by $\{g, r\}$, where $f = g\,h + r$. As f and g can be expressed as a linear combination of g and r with coefficients in $\mathrm{Quot}(R)$, then $\{g, r\}$ is a new basis of $\langle f, g\rangle \mathrm{Quot}(R)[\overline{x},\overline{a}]$ and, consequently, $\{g, r\}$ can replace $\{f, g\}$ to form a new basis of F. This is the only repeated action in computing the GGE. Observe also, that the final number of polynomials is less than or equal to the number of polynomials in F, as we replace at each step two polynomials by two or one (when the remainder is 0) polynomials.

(ii) The first part is obvious, as the new polynomials are remainders with respect to the order $\succ_{\overline{x}\,\overline{a}}$. When only $\overline{x}$ are taken as variables, the proposition is a consequence of $\succ_{\overline{x}\,\overline{a}}$ being a compatible elimination order.

(iii) As σ is a homomorphism, we have $\sigma(f) = \sigma(g)\sigma(h) + \sigma(r)$ and the argument given in (i) can be translated to $\{\sigma(f), \sigma(g)\}$ for any specialization. It must be noted that in specializing F', any coefficient (even the leading one) can become zero. Nevertheless the specialized basis $\sigma(F')$ is still a basis of $\langle \sigma(F)\rangle$.

(iv) We have to prove that the algorithm terminates. Let F_i be the base after the ith loop. Define

$$d_i = \sum_{f \in F_i} \mathrm{multideg}(f, \succ_{\overline{xa}}),$$

where $\mathrm{multideg}(f, \succ)$ is the list of the exponents of the variables of $\mathrm{lpp}(f, \succ)$. As $\#F_i \leq \#F_{i-1}$ and for every division in the algorithm, whenever the leading term is divisible, the multideg of the remainder is strictly smaller then that of the dividend: necessarily $d_i \preceq_{\overline{xa}} d_{i-1}$. If $d_i = d_{i-1}$ then no leading monomial has changed in loop i. So, no monomial of a $f \in F_i$ is divisible by any leading monomial of an element of F_i (with constant leading coefficient, when we only consider the $\overline{x}$ as variables), as all possible divisions have been performed in loop i. The algorithm stops in this case. As $\succ_{\overline{xa}}$ is a well-order, d_i necessarily stabilize, and the algorithm effectively terminates. GGE works in the direction of the reduced Gröbner basis, but may not reach it, as not all S-polynomials are considered and tested.

(v) As all leading coefficients are constants of R, all polynomials in F' have been considered as divisors of the others, and no polynomial will be divisible by the remainder. This property is the triangulation property of GGE. □

We include GGE as a pre-processing step for the PGB algorithm, before calling PGB0, in order to transform the initial basis. The resulting singular variety becomes smaller and, in simple examples, it suffices to obtain the minimal singular variety.

The GGE algorithm is remarkable for linear systems with parameters. Applying it by dividing only with respect to the $\overline{x}$ variables and divisors with constant leading coefficients, it reduces to the ordinary Gaussian elimination. But applying it, as defined, dividing with respect to variables and parameters, there is no restriction about the leading coefficients of the divisors, and then it provides a perfect first step for a reduced case discussion, as can be seen in the following example.

This algorithm is used in our general DISPGB algorithm. Nevertheless it is interesting by itself, as it provides a triangulation of the basis that is useful in applications.

5.1. EXAMPLE

Consider the following linear system in the variables (x, y, z, u) with the parameters (a, b):

$$\begin{aligned} F \quad := \quad & [a\,x + 2\,y + 3\,z + u - 6, x + 3\,y - z + 2\,u - b, \\ & 3\,x - a\,y + z - 2, 5\,x + 4\,y + 3\,z + 3\,u - 9]. \end{aligned}$$

We take F to be the basis of an ideal in $\mathbb{Z}[x, y, z, u, a, b]$ and compute a new basis with the GGE algorithm, relative to the order $\mathrm{lex}(x, y, z, u, a, b)$. The result is:

$$\begin{aligned} F' \quad := \quad & \mathrm{GGE}(F, \mathrm{lex}(x, y, z, u, a, b)) \\ = \quad & [756\,x - 39\,a\,b - 4\,b - 155 - 117\,a + (117\,a + 51)\,u, \\ & 189\,y + 6\,a\,b - 107 - 43\,b + 18\,a - (18\,a - 123)\,u, \\ & 756\,z - 1439 + 236\,b + 99\,a + 33\,a\,b - (99\,a - 15)\,u, \\ & (9\,a^2 - 30\,a + 21)\,u - 9\,a^2 - 3\,a^2\,b + 11\,a\,b + 22\,a - 49 + 28\,b]. \end{aligned}$$

As we see, the new basis is completely Gaussian reduced:

1. The new basis is triangular: each leading monomial is different when we consider as variables all x, y, z, u, a, b. In this case, it is also true if we consider only x, y, z, u as variables.
2. The three first pivots (leading coefficients) of the variables x, y, z are constants, and only the last leading coefficient in the variable u is a polynomial in the parameters. It is essentially the singular variety: $9\,a^2 - 30\,a + 21 = 3\,(a - 1)\,(3\,a - 7)$, precisely the value of the determinant.
3. Instead, the variable u is not eliminated in the equations giving x, y, z.
4. In this form, the system is very simple to discuss and to solve.
5. The result is, in this case, a minimal Gröbner basis, but not the reduced one.
6. Moreover, in this example, F' is itself a comprehensive Gröbner basis.

Now we compute $\mathrm{PGB}(F', \mathrm{lex}(x, y, z, u), \mathrm{lex}(a, b))$ to obtain the reduced Gröbner basis and the singular variety. We obtain:

$$\begin{aligned} G = & (9\,a^2 - 30\,a + 21)\,x - 6\,a\,b + 9\,a - 2\,b - 1, \\ & (9\,a^2 - 30\,a + 21)\,y + 3\,a\,b + 20 - 23\,b, \\ & (9\,a^2 - 30\,a + 21)\,z + 53\,a - 5\,a\,b - 18\,a^2 + 3\,a^2\,b - 39 + 6\,b, \\ & (9\,a^2 - 30\,a + 21)\,u - 9\,a^2 - 3\,a^2\,b + 11\,a\,b + 22\,a - 49 + 28\,b] \\ W = & \{a - 1, 3\,a - 7\}. \end{aligned}$$

In this example, W determines the minimal singular variety, as it includes just the factors of the determinant of the system, which we know by linear algebra to be the singular condition. The general discussion of all cases can be done with DISPGB described in the next sections. The complete output provided by DISPGB for this example can be found in Section 11.5.

If we compute the Gröbner basis using the PGB algorithm without previous reduction of F to F' by GGE, the basis G turns out to be the same, but the variety becomes $W_1 = \{a, a^2 + 35, 3a - 2, a - 1, 3a - 7\}$, showing the reduction obtained by GGE.

6. Specification of Specializations

Following Weispfenning (1992), we consider the following families of specializations (conditions in the paper of reference):

Definition 4. *Let σ be a specialization and $N = \{p_1, \ldots, p_s\} \subset R[\overline{a}]$ and $W = \{q_1, \ldots, q_r\} \subset R[\overline{a}]$ be sets of polynomials. We say that $\sigma \in \Sigma(N, W)$, or that (N, W) is the actual specification of σ if*

(i) *All the p_i's specialize to zero: $\sigma(p_i) = 0$, for $p_i \in N$.*
(ii) *All the q_i's specialize to non-zero: $\sigma(q_i) \neq 0$, for $q_i \in W$.*

We say that N are the null conditions and W the not-null conditions of the specification of all $\sigma \in \Sigma(N, W)$.

Proposition 5. *Let $\sigma \in \Sigma(N, W)$ be a specialization. If $f \in \sqrt{\langle N \rangle}$,* (the radical of the ideal $\langle N \rangle$)*, then $\sigma(f) = 0$. Equivalently:*

$$\sigma \in \Sigma(N, W) \Longrightarrow \sigma \in \Sigma(\sqrt{\langle N \rangle}, W).$$

Proof. Let $\sigma \in \Sigma(N, W)$ and $f \in \langle \sqrt{N} \rangle$. Then for some $n \in \mathbb{N}$ is $f^n \in \langle N \rangle$. Thus $\sigma(f^n) = (\sigma(f))^n = 0$, and the result follows. □

Proposition 6. *Let K be an algebraically closed field and $\Lambda = \Sigma(N, W)$. Set $h = \prod_{q \in W} q$. Then $\Lambda \neq \phi$ iff $\langle N \rangle \neq K[\overline{a}]$ and $h \notin \sqrt{\langle N \rangle}$.*

Proof. [⇒]: By Proposition 5.

[⇐]: By Hilbert's Nullstellensatz, $\mathbb{I}\mathbb{V}(N) = \sqrt{\langle N \rangle}$, and if $h \notin \sqrt{\langle N \rangle}$, it exists that $a_0 \in \mathbb{V}(N)$ for which $h(a_0) \neq 0$. Thus $\sigma_{a_0} \in \Lambda$, and $\Lambda \neq \phi$. □

The set of values of the parameters described by $\Sigma(N, W)$ is given by

$$\mathbb{V}(N) - [\mathbb{V}(N) \cap [\cup_{q \in W} \mathbb{V}(q)]] .$$

Proposition 7. *Let $W = \{q_1, \ldots, q_r\}$, N be a Gröbner basis relative to the order $\succ_{\overline{a}}$ and $W' = \mathrm{FACVAR}(\{\overline{q_1}^N, \ldots, \overline{q_r}^N\}$. Then*

$$\Sigma(N, W) = \Sigma(N, W').$$

Proof. We have $q_i = n_i + \overline{q_i}^N$, where $n_i \in \langle N \rangle$ and $\overline{q_i}^N$ is the remainder of the division of q_i by the basis N. As σ is a homomorphism,

$$\sigma(q_i) = \sigma(n_i) + \sigma(\overline{q_i}^N) = \sigma(\overline{q_i}^N).$$

So $\sigma(q_i) \neq 0$ iff $\sigma(\overline{q_i}^N) \neq 0$, and we can finally decompose into irreducible factors. □

Definition 8. (Quasi-canonical representation of a specification) *A representation (N, W) is said to be k-quasi-canonical, if it satisfies*

(i) *N is the reduced Gröbner basis in* $\mathrm{Quot}(R)[\overline{a}]$ *for the order $\succ_{\overline{a}}$ of the set of polynomials that specialize to zero.*
(ii) *The polynomials in W specializing to non-zero are reduced modulo N and are irreducible over $k[\overline{a}]$, where k is some intermediate field between* $\mathrm{Quot}(R)$ *and the algebraic closure $\overline{K}$. They are normalized in a canonical form in order to be recognized when compared.* (*Normalization depends on the field k, but is easy to define for $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.*)
(iii) $\prod_{q \in W} q \notin \sqrt{\langle N \rangle}$.
(iv) *The polynomials in N are square-free.*
(v) *If some $p \in N$ factors, then no factor of p belongs to W.*

Proposition 9. *Any specification that is non-empty in the algebraic closure has a* k-quasi-canonical *representation (N, W).*

It seems that imposing N to be a Gröbner basis of $\sqrt{\langle N \rangle}$, can provide a unique representation. Nevertheless, for practical computation reasons, we do not impose N to be radical.

Proof. (i) We can choose a Gröbner basis to define the ideal $\langle N \rangle$. This will report benefits in the algorithms.
(ii) By Proposition 7 we can reduce the polynomials in W modulo N, decompose into irreducible factors and normalize them (depending on the field).
(iii) By Proposition 6, $\prod_{q \in W} q \notin \sqrt{\langle N \rangle}$ is a necessary and sufficient condition for N and W to be compatible conditions in the algebraic closure, i.e. to determine a non-empty $\Sigma(N, W)$.
(iv) We can drop any multiple factor in N as a consequence of Proposition 5.
(v) We can also drop any factor of a polynomial $p \in N$ belonging to W as $p = gh$ and $g \in W$ imply $\sigma(p) = 0$ and $\sigma(g) \neq 0$, which leads to $\sigma(h) = 0$.

If in the development of the algorithms a polynomial $p' \in R[\overline{a}]$ such that $p' \in \sqrt{\langle N \rangle}$ and $p' \notin \langle N \rangle$ is detected, we can refine the representation of the specification adding p' to the base N, and re-computing the new Gröbner basis N'. We then have:

$$\langle N \rangle \subseteq \langle N' \rangle \subseteq \sqrt{\langle N \rangle}.$$

Then (N', W), (N, W) and $(\sqrt{\langle N \rangle}, W)$ are representations of the same specification $\Sigma(\sqrt{\langle N \rangle}, W)$, but (N', W) is a finer representation than (N, W). Even if some $p \in N'$ has a factor in W, it can also be dropped, as it specializes to non-zero.

Definition 10. *Let $f_1, f_2 \in R[\overline{a}][\overline{x}]$. We say that $f_1 \approx_\sigma f_2$ (are σ-equivalents) iff $\sigma(f_1) = \sigma(f_2)$.*

When a specification is assumed, then we can substitute the polynomials in a set by σ-equivalent polynomials. This will be done all along the algorithm. It must be noticed, that doing so, the new σ-equivalent set is no more a basis of the original $F \subset R[\overline{a}][\overline{x}]$. This is an important observation for the understanding of the algorithm. Doing so, we abandon computing a CGB, but the algorithm is simplified towards our objective. Observe that by doing so, unnecessary polynomials specializing to zero in the actual specification are also dropped.

Definition 11. *Given two specifications* (N_1, W_1) *and* (N_2, W_2) *we say that* $(N_2, W_2) \geq (N_1, W_1)$, *if* $\sqrt{\langle N_1 \rangle} \subseteq \sqrt{\langle N_2 \rangle}$ *and* $W_1 \subseteq W_2$.

If $(N_2, W_2) \geq (N_1, W_1)$ *and* $\sqrt{\langle N_1 \rangle} \neq \sqrt{\langle N_2 \rangle}$ *or* $W_1 \neq W_2$, *then we say that* $(N_2, W_2) > (N_1, W_1)$.

Definition 12. *The specifications where* $N = \{0\}$ *are denoted* generic*, in the sense that* $\Sigma(\{0\}, W)$ *contains all specializations except a set in the* $m-1$ *dimensional variety* $\cup_{q \in W} \mathbb{V}(q)$ *defined by* W.

6.1. canspec algorithm

The algorithms will only use quasi-canonical representations of specifications.

Proposition 13. *Given the specification* (N, W)*, the algorithm* CANSPEC *produces a* k*-quasi-canonical representation* (N', W') *if* test = true. *If* test = false *then* (N, W) *are incompatible, i.e.* $\Sigma(N, W) = \phi$.

CANSPEC

Input:	(N, W):	The specification of σ.
	$\succ_{\overline{a}}$:	Monomial order for the parameters.
Output:	*test*:	Has the value *true* if $\Sigma(N, W) \neq \phi$ in the algebraic closure and *false* if they are incompatible.
	(N', W'):	A k-quasi-canonical representation of the specification.

$W' :=$ FACVAR($\{\overline{q}^N : q \in W\}$)
$test := true$; $t := true$; $N' := N$
$h := \prod_{q \in W} q$
IF $h \in \sqrt{\langle N' \rangle}$ THEN $test := false$; $N' := \{1\}$; STOP
WHILE t DO
  $t := false$
  $N'' :=$ Drop any factor of a polynomial in N' that belongs
    to W', as well as multiple factors.
  IF $N'' \neq N'$ THEN
    $t := true$
    $N' :=$ GB($N'', \succ_{\overline{a}}$)
    $W' :=$ FACVAR($\{\overline{q}^N : q \in W'\}$).

Proof. The first step reduces W' modulo N using Proposition 7. Then one uses Proposition 6 to test compatibility. It is sufficient to test it only once as no transformation to a new representation (N', W') will alter it.

Then the loop intends to simplify N' eliminating any factor in N' belonging to W'. If one factor is found then we recompute the Gröbner basis for N' and reduces W' modulo N'. The loop is finite by the ascending chain condition (ACC).

Thus all the conditions for (N', W') to be a quasi-canonical representation are met. □

7. DISPGB Discussion Algorithm

We are now ready to give the general description of the main algorithm DISPGB (discussing parametric Gröbner bases), which is very similar to Weispfenning's construction of a Gröbner system. Its goal is to build up a binary tree structure beginning at the root. Let us give the algorithm:

DISPGB

Input: $F \subset R[\overline{x}][\overline{a}]$, and the orders $\succ_{\overline{x}}$ and $\succ_{\overline{a}}$.
Output: The table $T[v]$ having tree structure.

$B := \text{GGE}(F, \succ_{\overline{xa}})$
$T[\phi] := (\phi, B, \phi, \phi)$ (Creates and stores the root vertex)
BRANCH(ϕ, B, ϕ, ϕ) (Begins the recursive building of the tree).

At each vertex v the following objects are stored in the table variable

$$T[v] = (c_v, B_v, N_v, W_v).$$

1. The new condition c_v of the form $p(\overline{a}) = 0$ (for type 0 vertices), or $p(\overline{a}) \neq 0$ (for type 1 vertices), that is assumed when the algorithm gains access to vertex v in order to increase the specification of σ.
2. The pair (N_v, W_v) determining a quasi-canonical representation of the specification of σ at this vertex (see Definition 8).
3. $B_v \subset R[\overline{a}][\overline{x}]$, a set of polynomials that specializes to a basis of the ideal $\langle \sigma(F) \rangle$ for any $\sigma \in \Sigma(N_v, W_v)$.

DISPGB begins by computing GGE(F). As discussed in Section 5 this reduces the number of cases to be discussed, reducing at the same time the singular variety.

At the root, the initial basis transformed by GGE is stored, and the remaining arguments are empty. So $T[\phi] := (\phi, B, \phi, \phi)$.

Then the recursive algorithm BRANCH, described in Section 9, is called: bifurcation takes place making the dichotomic decision about some leading coefficient of a polynomial in the actual basis to become zero or not-zero in the specialization, as in dynamical evaluation (Duval, 1995). The recursive algorithm BRANCH, combined with NEWVERTEX, also described in Section 9, will be responsible for the control flow of the procedure. As the algorithm progresses in depth, the specification of the specialization is refined at the branches, allowing the algorithm CONDPGB, described in Section 8, to advance in the computation of the specializing Gröbner basis.

A vertex is terminal whenever B_v becomes a set of polynomials specializing to a Gröbner basis of $\langle \sigma(F) \rangle$ for $\sigma \in \Sigma(\langle N_v \rangle, W_v)$. Specialization of σ is then finished. The algorithm terminates when all branches arrive at terminal vertices.

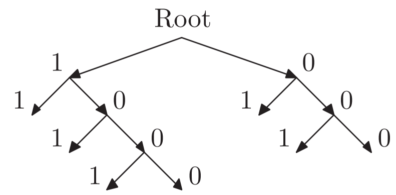

Figure 1. Tree structure of the algorithm in the example of Section 11.4.

Terminal vertices contain the complete information about the distinct specialized Gröbner basis and specifications. Tracing the tree provides a dichotomic discussion of the decisions leading to the corresponding cases.

Even though a unique canonical discussion does not exist—as it depends on the order in which the decisions are taken—a minimum disjoint set of cases is obtained.

To clarify the description, we give in Figure 1 the tree corresponding to the example in Section 11.4, with four variables, four parameters and four equations (second degree in the variables). It is a typical discussion and gives seven distinct final cases.

An important observation is that DISPGB can be completely parallelized. When a decision at some vertex is taken, producing a branching of the algorithm, only the information of the actual vertex is needed to follow the branches below it, and no other information about the upper or lateral vertices is needed. So the algorithm can be parallelized.† In the example four processors would be useful as that is the maximal width of the tree.

In order to arrive at a terminal vertex following the direct path, only some more lateral computations than for the generic case are needed. This is the reason why the parallelized algorithm has the same time-complexity.

We now need to describe two kinds of algorithms that are used by the main DISPGB algorithm: the conditional algorithms and the control algorithms.

8. Conditional Algorithms

DISPGB uses two conditional algorithms: NEWCOND and CONDPGB. The goal of NEWCOND is, given $f \in R[\overline{a}][\overline{x}]$ and a specification (N, W) of σ, to test if a new undecided condition must be assumed in order to be able to decide about the specialization of the leading coefficient to zero or not. The goal of CONDPGB is, given the specification and $B \subset R[\overline{a}][\overline{x}]$, where $\sigma(B)$ is a basis of $\langle \sigma(F) \rangle$, to advance in the direction of a specializing Gröbner basis as much as possible.

8.1. NEWCOND ALGORITHM

PROPOSITION 14. *Let* (N, W) *be the specification of* σ, *and* $f \in R[\overline{a}][\overline{x}]$. *Then the following algorithm* NEWCOND *determines the quadruplet* (cd, f', N', W') *where* $f' \approx_\sigma f$.

If $cd = \phi$, *then* $\sigma(\mathrm{lc}(f', \succ_{\overline{x}})) \neq 0$ *for* $\sigma \in \Sigma(N, W)$. *If* $cd \neq \phi$, *set* $c = \prod_{w \in cd} w$. *Then* $\sigma(\mathrm{lc}(f', \succ_{\overline{x}}))$ *vanishes or not depending on the nullity or not of* $\sigma(c)$.

† *Maple V*, release 6 does not allow parallel computations at yet. But a parallel implementation is desirable as, even if no theoretical proof is provided that the generic case is the almost complex case, it is reasonable and in practice observed. Thus the parallel algorithm will present essentially the complexity of the generic case, plus the vertex computations of quasi-canonical representations and reductions.

(N', W') is a refinement of (N, W), when some polynomial in $\sqrt{\langle N\rangle}$ is detected.

NEWCOND

Input:	f:	$\in R[\overline{a}][\overline{x}]$
	(N, W):	Null and not-null conditions of the specification of σ.
Output:	cd:	The new conditional set.
	f':	The σ-equivalent to f with deciding leading coefficient.
	(N', W'):	A specialization refinement of (N, W).

```
f' := f;
test := true
N' := N
WHILE test DO
    IF lc(f') ∈ √⟨N'⟩ THEN
        f' := f' − lm(f')
        N' := GB(N' ∪ lc(f'), ≻_a)
        W' := {w̄^{N'} : w ∈ W}
    ELSE test := false
f' := f̄'^{N'}
cd := FACVAR(lc(f')) − W'.
```

Proof. If the specification of σ implies that $\sigma(\text{lc}(f)) = 0$, NEWCOND begins eliminating iteratively the leading term of f. In this process, some polynomial in $\sqrt{\langle N\rangle}$ can be detected. This is used to improve N and a new Gröbner basis N' is recomputed. This is then used to also improve W. When no more leading monomials of f' can be dropped, then f' is reduced with respect to N' by dividing by it. (N', W') is now insufficient to decide if the actual $\text{lc}(f')$ specialize to zero as a consequence of the null condition N'.

The algorithm now computes $cd := \text{FACVAR}(\text{lc}(f')) - W'$. Obviously, if $cd = \phi$ then $\sigma(\text{lc}(f')) \neq 0$ for $\sigma \in \Sigma(N', W')$. If not, $\sigma(\text{lc}(f', \succ_{\overline{x}}))$ vanishes or not depending on the nullity or not of $\sigma(c)$. □

When NEWCOND has concluded and has detected a new condition, in order to continue the specification of σ, DISPGB will have to decide either

(0) The product of all polynomials in cd become zero by specialization ($c = 0$).
(1) All polynomials in cd become different from zero by specialization ($c \neq 0$).

8.2. condpgb: conditional parametric Gröbner basis algorithm

At a given point DISPGB needs to use a Buchberger's type algorithm that takes into account the specification and intends to determine a specializing Gröbner basis. We denote it CONDPGB (conditional parametric Gröbner basis).[†]

[†]A new release of the implementation is in the process of development. It introduces improvements in the algorithm CONDPGB, to accelerate Buchberger's algorithm, that are not described here.

CONDPGB

Input:	B:	The actual specializing basis.
	(N,W):	The actual specification of σ.
Output:	*test*:	If $test = true$ then $\sigma(B')$ is yet the Gröbner basis.
	B':	The new completed specializing basis.
	(N',W'):	The specification (N,W) can be refined in the process.

```
test := true;  s := #B;  J := {(i,j) : 1 ≤ i < j ≤ s}
B' := B; N' := N; W' := W; t := s
WHILE J ≠ φ and test DO
    Select (i,j) ∈ J
    J := J − {(i,j)}
    IF Buchberger's reductibility criterions are false THEN
        S := overline{SPOL(B'[i], B'[j], ≻_x̄)}^{B'}_{≻_x̄}
        S := overline{S}^{N'}_{≻_x̄ā}
        IF S ≠ 0 THEN
            (c_dec, S, N', W') := NEWCOND(S, N', W')
            IF c_dec = φ THEN
                IF S ≠ 0 THEN
                    t := t + 1
                    B' := B' ∪ {S}
                    J := J ∪ {(i,t) : 1 ≤ i < t}
            ELSE
                test := false
                B' := B' ∪ {S}
IF test THEN
    B' := REDGB(MINGB(B', ≻_x̄), ≻_x̄, ≻_ā).
```

Proposition 15. *Let (N,W) be the specification of σ and let $B \subset R[\overline{a}][\overline{x}]$ be a set of polynomials for which:*

(i) *$\sigma(B)$ is a basis of $\langle\sigma(F)\rangle$.*
(ii) *$\sigma(\mathrm{lc}(g,\succ_{\overline{x}})) \neq 0$ for $g \in B$ and $\sigma \in \Sigma(N,W)$.*

The algorithm CONDPGB *determines the quadruplet* $(test, B', N', W')$.

If test = true, then $\sigma(B')$ is (except for normalization), the reduced Gröbner basis of $\langle\sigma(F)\rangle$ for $\sigma \in \Sigma(N',W')$.

If *test = false*, then B' is an extended set of B for which $\langle \mathrm{lpp}(B,\succ_{\overline{x}})\rangle \subsetneq \langle \mathrm{lpp}(B',\succ_{\overline{x}})\rangle$, and B' contains at least one polynomial for which the actual specification (N',W') cannot decide if its leading coefficient specializes to zero or not.

Proof. CONDPGB is essentially Buchberger's algorithm (PGB0) with the following differences:

(i) Step $S := \overline{S}^{N'}_{\succ_{\overline{x}\overline{a}}}$.
After computing the usual remainder of the S-polynomial, the result is reduced with

respect to N' for the order $\succ_{\overline{xa}}$. Obviously, both polynomials are σ-equivalents, as the difference is a polynomial with coefficients in $\langle N' \rangle$ that specializes to 0.

(ii) Step $(c_{dec}, S, N', W') :=$ NEWCOND(S, N', W').
If $S \neq 0$ then, before continuing PGB0, CONDPGB applies NEWCOND to test if $\mathrm{lc}(S, \succ_{\overline{x}})$ specializes to zero or not. This can reduce S to a σ-equivalent polynomial, say S', that can eventually be 0. Three cases can occur:

(a) If $S' = 0$ then there is nothing new to do and the algorithm continues with PGB0.
(b) If $S' \neq 0$ and $\sigma(\mathrm{lc}(S, \overline{x})) \neq 0$ then S' is adjoined to the base and PGB0 continues.
(c) If $S' \neq 0$ and $c_{dec} \neq \phi$ then S' is adjoined to the base and CONDPGB stops, returning the new base B'. It is important to note that, in that case, at least one new polynomial has been adjoined to the base, whose leading power product cannot be divided by the others in B'. Thus the ideal of leading power products is strictly greater than before.

If CONDPGB never go through the sub-case (c), then it continues until a Gröbner basis with non-zero specializing leading coefficients is reached. In this case, CONDPGB determines first the minimal and then the reduced Gröbner base. Then $c_{dec} = \phi$ is returned. □

9. Control Algorithms

We now give the algorithms BRANCH and NEWVERTEX that are recursive procedures calling one another. The control flow of the main DISPGB algorithm is governed by them.

BRANCH

Input:	v:	Label of the vertex.
	B:	Specializing basis at the vertex v.
	(N, W):	Specification of σ at vertex v (not necessarily canonical).
Output:		Recursive algorithm. It stores the refined (B', N', W') (basis and quasi- canonical representation) at the vertex v, creating two new hanging vertices when necessary or marking the vertex as terminal. It manages the control flow.

```
cd := φ
FOR i TO #B WHILE cd = φ DO
      f := B[i]
      (cd, f', N', W') := NEWCOND(f, N, W)
      Substitute the i-th element f of B' by f'
pivot := i − 1
T[v] := (−, B, N', W') (cond is already stored in T(v). Refinement of data)
IF cd = φ THEN
      (test, B', N', W') := CONDPGB(B, N', W') G.B.)
      IF test THEN
            T[v] = (−, B', N', W', terminal vertex)
            STOP (B' is already the Gröbner basis)
```

ELSE
    BRANCH(v, B', N', W') (Further refinement is needed)
ELSE (A pair of new hanging vertices is created)
    NEWVERTEX($1, v, cd, B', N', W', pivot$)
    NEWVERTEX($0, v, cd, B', N', W', pivot$).

BRANCH begins by testing the polynomials in B using NEWCOND. This can refine the data at the vertex.

If all have leading coefficients specializing to non-zero ($cd = \phi$), then CONDPGB is called in order to find the specializing Gröbner basis. If this objective is reached, the results are stored at the vertex and the procedure stops. Else, at least one new polynomial with non-decided leading coefficient has been adjoined by CONDPGB (see Proposition 15). Thus BRANCH is recursively called in order to increase the refinement and advance in the Gröbner basis direction.

Otherwise some polynomial in B with non-decided leading coefficient is detected ($cd \neq 0$). Then it is taken as pivot and is used to bifurcate the tree (NEWVERTEX) with two new branches of type 0 and 1, increasing the specification.

NEWVERTEX

Input:	n	0 or 1. If $n = 1$, the new condition is taken not-null. If $n = 0$ then it is taken null.
	u:	Label of previous vertex. Current vertex becomes $v := (u, n)$.
	cd:	Irreducible factors of the new condition.
	B:	The current base at previous vertex.
	(N, W):	Specification of σ at previous vertex.
	$pivot$:	The index of pivot polynomial of B whose leading coefficient is responsible for the new condition.
Output:		NEWVERTEX creates the new vertex with label v and stores the new values of $(cond_v, B_v, N_v, W_v)$ in $T(v)$. Then it calls BRANCH to continue the process.

$v := (u, n)$
$c := \prod cd$
IF $n = 0$ THEN
    $cond := (c = 0)$
    $W' := W$
    $N' := \mathrm{GB}(cd \cup N, \succ_{\overline{a}})$
ELSE ($n = 1$)
    $cond := (c \neq 0)$
    $W' := W \cup cd$
    $N' := N$
    $B' :=$ Substitute every polynomial g in B whose $\mathrm{lpp}(g, \succ_{\overline{x}})$ can be divided by $\mathrm{lpp}(g_{pivot}, \succ_{\overline{x}})$, by the S-polynomial of both.
$(test, N', W') := \mathrm{CANSPEC}(N', W')$
IF $test$ THEN (Specification is compatible)

$B' := \mathrm{GGE}(\{\overline{g}^{N'} : g \in B'\} - \{0\}, \succ_{\overline{xa}})$ (Optional†)
$T[v] := (cond, B', N', W')$ (Create vertex and store)
BRANCH(v, B', N', W') (Begin the analysis)
ELSE STOP (Incompatible specification has been detected).

THEOREM 16. *Given $F \subset R[\overline{a}][\overline{x}]$ and the monomial orders $\succ_{\overline{x}}$, $\succ_{\overline{a}}$,* DISPGB *constructs a table T with binary tree structure that contains at each terminal vertex the quadruple*

$$T_v = (cd, B, N, W)$$

where, either the vertex is marked as incompatible, or

(i) *(N, W) is a quasi-canonical specification of a specialization.*
(ii) *For $\sigma \in \Sigma(N, W)$, $\sigma(B)$ is the reduced Gröbner basis of $\sigma(F)$ (except for normalization).*
(iii) *Any specialization corresponds to one and only one of the specifications of the terminal vertices.*
(iv) *The algorithm terminates.*

Moreover, traversing the tree from the root and considering the condition cd *at the vertices, a dichotomic discussion of the cases is provided.*

PROOF. (i) Whenever an extended specialization is computed (in NEWVERTEX), the algorithm CANSPEC is called, producing either a quasi-canonical representation of the specification, or detecting incompatible conditions. In that case it marks the vertex as terminal and with incompatible conditions. Thus, only quasi-canonical representations of specializations are used.

(ii) A vertex is marked as terminal only if conditions are incompatible or if CONDPGB has concluded with a basis specializing to the reduced Gröbner basis of $\sigma(F)$ (except for normalization).

(iii) Is an immediate consequence of the dichotomic character of the decisions taken at each vertex ($\sigma(c) = 0$ or $\sigma(c) \neq 0$).

(iv) Is a consequence of the ACC of the ideals in Noetherian rings. When BRANCH is called, the leading coefficients in B are analysed. If all these leading coefficients specialize to non-zero in the specification, then CONDPGB is called. Proposition 15 ensures that $\langle \mathrm{lpp}(B, \succ_{ox}) \rangle$ strictly increases in the call.

If the specialization to zero or not of some leading coefficient cannot be decided, then bifurcation takes place calling NEWVERTEX twice. The branch for which the null-condition is assumed, increases the ideal $\langle N \rangle$. The branch for which the not-null condition is assumed increases the number of assumed not-null leading coefficients under specialization, without modifying the base. If in the next decisions none of the cited cases occur, then a moment will arrive where all leading coefficients will be assumed not-null in the specification, and then, CONDPGB will be called, increasing $\langle \mathrm{lpp}(B, \succ_{ox}) \rangle$.

So in any case we have ascending chains of ideals that stabilize. Consequently the algorithm terminates. □

†The use of GGE at this point of the algorithm is optional. Nevertheless we observed experimentally that in some examples (like in 11.3) it can drastically reduce the size of the tree, intermediate computations and the number of cases. See the footnote of Section 8.2.

10. Minimal Singular Variety: GENCASE Algorithm

In Section 4, we proved that PGB determines the unique reduced Gröbner basis G and a singular variety W such that outside W, G is specialization-invariant. This is the only m-dimensional case. All other cases are contained in the $(m-1)$-dimensional variety W.

Let us describe a case C obtained by DISPGB by the 5-tuple $C = (l, p, B, N, W)$, where l is the label, p is the the list of inequalities and equalities in the order that they have been taken (i.e. the path of the case), B is the reduced Gröbner basis, and (N, W) the quasi-canonical representation of the specification. Let L be the list of cases obtained by DISPGB in the form described earlier.

Except for the trivial problems where the reduced Gröbner basis specializes everywhere, DISPGB always obtains the case C_0, labelled $[1, \ldots ,1]$, specified by $(N_0 = \phi, W_0)$, corresponding to the PGB solution. The inequalities W_0 are sufficient to ensure proper specialization of B_0. But not all the conditions in W_0 are always necessary: some other cases can also specialize properly.

A case $C = (l, p, B, N, W)$ in L is said to be *special* if $\text{lpp}(B) \neq \text{lpp}(B_0)$ and *normal* if $\text{lpp}(B) = \text{lpp}(B_0)$. Let L_s denote the set of special cases in L and L_n the set of normal cases in L (including C_0). Using these notations we can define the minimal variety and state a theorem about it.

DEFINITION 17. *Given a set of polynomials $F \subset R[\overline{a}][\overline{x}]$, the* minimal singular variety $V_{\min}$ *is the minimal variety of K^m containing all special cases L_s provided by* DISPGB.

THEOREM 18. *Given $F \subset R[\overline{a}][\overline{x}]$, let L be the list of cases determined by* DISPGB*, C_0 the case corresponding to the label* $[1, \ldots, 1]$*, $L_n \subseteq L$ the set of normal cases, $L_s \subseteq L$ the set of special cases and*

$$W_{0m} = \text{FACVAR}(\text{lc}(B_0, \succ_{\overline{x}})).$$

Then

(i) *The minimal singular variety $V_{\min} = \bigcup_{w \in W_{\min}} \mathbb{V}(w)$, determined by its irreducible components, verifies: $W_{0m} \subseteq W_{\min} \subseteq W_0$.*

(ii) *For all normal cases, the Gröbner basis is specialization-invariant.*

PROOF. (i) Conditions W_0 are sufficient for proper specialization of the algorithms, but some of them may be unnecessary. On the other hand, the conditions in W_{0m} are obviously necessary in order to have a normal case. Thus the components of $V_{\min}$ are those of W_{0m} plus, at most, some of the irreducible components in W_0 not in W_{0m}.

(ii) Let G be the reduced Gröbner basis of the ideal $I = \langle F \rangle R(\overline{a})[\overline{x}]$. Under the hypothesis W_0, we have $\sigma_0(G) = B_0$ (except for normalization) as all steps of the computation of G specialize properly to B_0. So

$$\langle \text{lpp}(I) \rangle = \langle \text{lpp}(G) \rangle = \langle \text{lpp}(B_0) \rangle = \langle \text{lpp}(\langle \sigma_0(F) \rangle) \rangle.$$

But by the definition, $\langle \text{lpp}(B_{L_{ni}}) \rangle = \langle \text{lpp}(B_0) \rangle$ for all cases in L_n. As $B_{L_{ni}}$ is the reduced Gröbner basis for the case L_{ni}, we have $\langle \text{lpp}(B_{L_{ni}}) \rangle = \langle \text{lpp}(\langle \sigma_{L_{ni}}(F) \rangle) \rangle$. Thus $\langle \text{lpp}(I) \rangle = \langle \text{lpp}(\langle \sigma_{L_{ni}}(F) \rangle) \rangle$, so that, for all cases in L_n, proper specialization holds. □

Using part (ii) of Theorem 18 we define the *generic case* C_g by the specification $(N_g, W_g) = (\phi, W_{\min})$. It contains all the normal cases L_n that are completely outside $V_{\min}$. There can exist cases in L_n that have a non-empty intersection with $V_{\min}$. The parts of these cases outside $V_{\min}$ are in the generic case, and the parts inside $V_{\min}$ are considered as special cases, even if they specialize properly.

This theorem does not determine completely W_g, although it limits quite strictly the candidates. In practice, we observe that in most examples $W_{\min} = W_{0m}$, as was pointed out by V. Weispfenning in the discussions. Nevertheless there are examples where $W_{\min}$ is strictly greater than W_{0m}.[†]

This induces the GENCASE algorithm to improve the output of DISPGB.

GENCASE

Input:	T	The table constructed by DISPGB.
	$\succ_{\overline{x}}$:	$\overline{x}$-termorder.
	$\succ_{\overline{a}}$:	$\overline{a}$-termorder
Output:	L'	The list of cases containing: the generic case $C_g = (\text{'Generic case'}, p_g, B_g, N_g = \phi, W_g)$, and the normal and special cases inside W_g.

$L := \{C(l) : l \in \text{terminal vertices of } T\}$
$C_0 :=$ Select from L the case labelled [1..1] specified by $(N_0 = \phi, W_0)$
$L_1 := L - C_0$
$lp_0 := \{\text{lpp}(f) : f \in B_0\}$
$L_0 :=$ Select from L_1 the cases having lp_0 as set of lpp(f)
$L_s := L_1 - L_0$
$W_g :=$ FACVAR$(\text{lc}(B_0, \succ_{\overline{x}}))$
$V_g = \cup_{w \in W_g} \mathbb{V}(w)$
WHILE not all cases in L_s belong to V_g DO
  Warning!
  Add convenient elements of $W_0 - W_g$ to V_g
$C_g := (\text{'Generic case'}, \{w \neq 0 : w \in W_g\}, B_0, \phi, W_g)$
$L_{ns} :=$ Select all cases in L_0 with non-empty intersection with V_g and
  intersect them with V_g
$L' := C_g \cup L_{ns} \cup L_s$.

GENCASE begins testing if all special cases obtained by DISPGB are inside W_{0m}. If so, the generic case substitutes all cases in L_n. Those being completely outside $V_{\min}$ will disappear from the list, and those having a non-empty intersection with $V_{\min}$ will be restricted to their intersection with $V_{\min}$ and, consequently, this condition will be added to their specification. Those cases will be considered special. If not, GENCASE gives a warning and adds convenient polynomials of $W_0 - W_{0m}$ to $V_{\min}$ to reach the goal.

For linear systems the system determinant gives the generic case condition. The corresponding generalized value for general polynomial systems is thus

$$\Delta = \prod_{w \in W_g} w.$$

[†]At the end of the tutorial included in the implemented software, such an example is presented.

It must be pointed out that the given definition of minimal singular variety is algorithm dependent. Although this concept seems to be an intrinsic object, no proof of its algorithm independence is provided here. This remains an open question.

11. Applications

Many interesting applications can be examined using DISPGB and GENCASE with very satisfactory results.

11.1. LINEAR SYSTEMS

A particularity of DISPGB is its exceptional behaviour for linear systems. Let us give an example. Consider the following linear system with the variables (x, y, z) and the parameters (a, b, c):

$$x + cy + bz + a = 0$$
$$cx + y + az + b = 0$$
$$bx + ay + z + c = 0.$$

Applying DISPGB and GENCASE with respect to the monomial orders $\mathrm{lex}(x, y, z)$ and $\mathrm{lex}(a, b, c)$, the following polynomial appears in the discussion:

$$\Delta \equiv a^2 + b^2 + c^2 - 2\,a\,b\,c - 1,$$

that is nothing other than the value of the determinant of the system. Table 1 resumes the discussion of the cases.

GENCASE summarizes the [1,1] case and the [0,1] cases obtained by DISPGB into the generic case $\Delta \neq 0$, as it corresponds to the discussion by determinants. The [1,1] case corresponds to a larger singular variety, namely $\mathbb{V}(\Delta) \cup \mathbb{V}(c^2 - 1)$, and the [0,1] case corresponds to $c^2 - 1 = 0, a - bc \neq 0$. But the final use of GENCASE, reduces both cases to only one, and the singular variety to the minimal one. All the others are special cases corresponding to $\Delta = 0$.

One can see that the discussion provided by the algorithm cannot be improved: only strictly different cases corresponding to different sets of leading power products of the reduced Gröbner basis appear. It is hard to obtain such a reduced discussion using only determinants.

11.2. SINGULAR POINTS OF A CONIC

We study the singular points of the conic

$$x^2 + b\,y^2 + 2\,c\,x\,y + 2\,d\,x + 2\,e\,y + f = 0.$$

The polynomial system is:

$$[x^2 + b\,y^2 + 2\,c\,x\,y + 2\,d\,x + 2\,e\,y + f,\;\; x + c\,y + d,\;\; b\,y + c\,x + e].$$

The following “discriminant” appears in the discussion

$$\Delta = b\,d^2 - b\,f + c^2\,f - 2\,e\,c\,d + e^2.$$

Table 1. Discussion of the linear system. (Section 11.1)

Case	Conditions	Gröbner basis
Generic case	$\Delta \neq 0$	$[\ \Delta\, z + c^3 - b^2c - a^2c + 2\,a\,b - c,$ $\Delta\, y + b^3 - c^2b - a^2b + 2\,c\,a - b,$ $\Delta\, x + a^3 - c^2a - b^2a + 2\,b\,c - a\ ]$
Special cases	$\Delta = 0$	
[1, 0, 1]	$c^2 - 1 \neq 0,$ $\Delta = 0,$ $a\,b - c \neq 0$	$[\ 1\ ]$
[1, 0, 0]	$c^2 - 1 \neq 0,$ $\Delta = 0,$ $a\,b - c = 0$	$[\ (c^2-1)y + (b\,c - a)z - b^3 + c^2b,$ $(c^2-1)x + (c\,a - b)z + cb^3 - a + c^2a - c^3b\ ]$
[0, 0, 1]	$c^2 - 1 = 0,$ $a - b\,c = 0,$ $b^2 - 1 \neq 0$	$[\ z + c,$ $x + c\,y\ ]$
[0, 0, 0]	$c^2 - 1 = 0,$ $a - b\,c = 0,$ $b^2 - 1 = 0$	$[\ x + cy + bz + b\,c\ ]$

Table 2. Discussion of singular points of a conic. (Section 11.2)

Case	Conditions	Gröbner basis
Generic case	$\Delta \neq 0$	$[1]$
Special cases	$\Delta = 0$	
$[0,1]$	$cd - e = 0, d^2 - f \neq 0$	$[1]$
$[1,0]$	$cd - e \neq 0$	$[\ d^2 - f + (cd-e)y, -de + cf + (cd-e)x\ ]$
$[0,0,1]$	$cd - e = 0, d^2 - f = 0,$ $b - c^2 \neq 0$	$[\ y, x + d\ ]$
$[0,0,0]$	$cd - e = 0, d^2 - f = 0,$ $b - c^2 = 0$	$[\ x + cy + d\ ]$

Table 2 describes the discussion. The generic case corresponds to non-degenerate conics. Case [0, 1] corresponds to two parallel lines. This is an interesting case, as even if the Gröbner basis specializes properly, it is inside the singular variety, corresponding to a special case (degenerate conic without singular points). Finally, cases [1, 0] and [0, 0, 1] correspond to two incident lines, and case [0, 0, 0] to two coincident lines.

11.3. THE INVERSE KINEMATICS PROBLEM FOR A SIMPLE ROBOT

We now consider the simple plane robot of Figure 2 with two arms of lengths 1 and l, respectively.
The inverse kinematics problem is provided by the solution of the following system:

$$\begin{aligned} &r - c_1 + l(s_1\, s_2 - c_1\, c_2) = 0 \\ &z - s_1 - l(s_1\, c_2 + s_2\, c_1) = 0 \\ &s_1^2 + c_1^2 - 1 = 0 \\ &s_2^2 + c_2^2 - 1 = 0. \end{aligned}$$

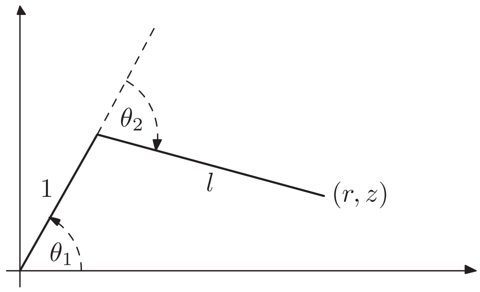

Figure 2. Simple two arms robot. (Section 11.3)

Table 3. Discussion of the two arms robot.

Case	Conditions	Gröbner basis
Generic case	$l \neq 0$, $r^2 + z^2 \neq 0$,	$[\ 2lc_2 + l^2 - r^2 - z^2 + 1, l^4 - 2r^2l^2 - 2z^2l^2 - 2l^2$ $+4l^2s_2^2 - 2r^2 + r^4 - 2z^2 + 2r^2z^2 + z^4 + 1$, $rl^2 - r - r^3 - z^2r + (2r^2 + 2z^2)c_1 - 2lzs_2$, $zl^2 - zr^2 - z^3 - z + (2r^2 + 2z^2)s_1 + 2rls_2\ ]$
Singular cases		
[1, 1, 0, 1, 1]	$l \neq 0,\ r \neq 0$, $r^2 + z^2 = 0, z \neq 0$ $l^2 - 1 \neq 0$	$[\ 2lc_2 + l^2 + 1,\ 2zls_2 + r - rl^2$, $4r(l^2 - 1)c_1 + l^4 + 1 - 4z^2 - 2l^2,\ -4z^4 + z^2 + 2r^2$ $+4zr^2(l^2 - 1)s_1 + l^4z^2 + 2r^2l^4 - 4r^2l^2 - 2l^2z^2\ ]$
[1, 1, 0, 1, 0]	$l \neq 0,\ r \neq 0$, $r^2 + z^2 = 0, z \neq 0$ $l^2 - 1 = 0$	[1]
[1, 0, 0, 1]	$l \neq 0, r = 0$, $z = 0, l^2 - 1 \neq 0$	[1]
[1, 0, 0, 0]	$l \neq 0, r = 0, z = 0$, $l^2 - 1 = 0$	$[\ lc_2 + 1, s2, c_1^2 + s_1^2 - 1\ ]$
[0, 1]	$l = 0, r^2 + z^2 - 1 \neq 0$	[1]
[0, 0]	$l = 0, r^2 + z^2 - 1 = 0$	$[\ {c_2}^2 + {s_2}^2 - 1, c_1 - r, s_1 - z\ ]$.

where (s_1, c_1, s_2, c_2) are the sines and cosines of the two angles θ_1 and θ_2 defined in Figure 2. In the inverse kinematics problem these are the variables to be determined for the parameter values (r, z, l). Applying DISPGB plus GENCASE to this system, using $\text{lex}(s_1, c_1, s_2, c_2)$ and $\text{lex}(r, z, l)$ orders, the discussion provided in Table 3 is obtained.

DISPGB provides $V = \mathbb{V}(l) \cup \mathbb{V}(r) \cup \mathbb{V}(r^2 + z^2) \cup \mathbb{V}(z) \cup \mathbb{V}(-l^2 + r^2 + 1)$ as a singular variety, plus four other cases that are, in fact, special cases of the generic case. GENCASE obtains the minimal singular variety $V_g = \mathbb{V}(l) \cup \mathbb{V}(r^2 + z^2)$ in only one generic case containing all five previous cases.

Moreover, the minimal singular variety has a simple geometrical interpretation: the special cases correspond either to the end of the robot being at the origin, or to the length of the second arm being 0.

The special cases [1, 1, 0, 1, 1] and [1, 1, 0, 1, 0] are only compatible for complex values of the parameters ($r = \pm i\, z \neq 0$) and have no interest for the real configuration.

For the cases [1, 0, 0, 1] and [1, 0, 0, 0] the end of the robot is at the origin. For [1, 0, 0, 1] the system is inconsistent as the length of the second arm is different from ± 1. For [1, 0, 0, 0] the angle θ_1 is free whether θ_2 is π for $l = 1$ or 0 for $l = -1$, corresponding to the same physical solution.

The cases [0, 1] and [0, 0] correspond to the degenerate robot of length $l = 0$. For [0, 1] the system is inconsistent as the position of the end of the robot arm is not at distance 1 from the origin. For [0, 0] the angle θ_1 is determined, whether θ_2 is free, corresponding to the degeneration of the configuration.

GENCASE reveals itself to be very useful for this problem.

11.4. Load-flow problem for a three nodes electrical network

In Montes (1995, 1998) we studied the load-flow problem for electrical networks. The equations are polynomial systems with parameters. The following system concerning a concrete three nodes electrical network is considered:

$$
\begin{aligned}
&14 - 12\, e_2 - 110\, f_2 - 2\, e_3 - 10\, f_3 - P_1 = 0,\\
&2397 - 2200\, e_2 + 240\, f_2 - 200\, e_3 + 40\, f_3 - 20\, Q_1 = 0,\\
&16\, e_2^2 - 4\, e_2\, e_3 - 20\, e_2\, f_3 + 20\, e_3\, f_2 + 16\, f_2^2 - 4\, f_2\, f_3 - 12\, e_2 + 110\, f_2 - P_2 = 0,\\
&2599\, e_2^2 - 400\, e_2\, e_3 + 80\, e_2\, f_3 - 80\, e_3\, f_2 + 2599\, f_2^2 - 400\, f_2\, f_3\\
&\qquad\qquad - 2200\, e_2 - 240\, f_2 - 20\, Q_2 = 0.
\end{aligned}
$$

Here e_2, f_2, e_3, f_3 are the variables, representing real and imaginary parts of voltages, and P_1, Q_1, P_2, Q_2 are the parameters representing the real and imaginary components of power. We use the orders $\mathrm{lex}(e_2, f_2, e_3, f_3)$ and $\mathrm{lex}(P_1, Q_1, P_2, Q_2)$. As a result of the application of DISPGB the following polynomials appear:

$$
\begin{aligned}
h_1 &= P_1 - 20\\
h_2 &= 20\, Q_1 - 3497,\\
h_3 &= 6999\, P_2 - 800\, Q_2\\
g_1 &= 7786876\, h_1 - 79955\, h_2,\\
g_2 &= 400\, h_1^2 + h_2^2,\\
g_3 &= 7995500\, h_1 + 1946719\, h_2\\
g_4 &= h_2\, Q_2\, (1814947407168\, h_2 + 20\,778249588225\, h_1) + 1392556035\, h_3^2.
\end{aligned}
$$

The discussion provided by DISPGB and GENCASE is given in Table 4.

In this example, DISPGB directly gives the generic case, and GENCASE does not need to improve the result. We only give the sets of leading power products of the different bases, in order to illustrate how the initial goal has been reached (the bases themselves have large coefficients and would confuse the results). Except for the inconsistency cases [1,0,0,1] and [0, 0, 1], all others have distinct sets of leading power products which cannot be reduced. Obviously the minimal singular variety in this example is $\mathbb{V}(g_1, g_2)$.†

†In the tutorial, the analysis on how the conditions can be used to simplify the output is described.

Table 4. Discussion of the load-flow problem.

Case	Conditions	Leading power products
[1, 1] = Generic case	$g_1 \neq 0,\ g_2 \neq 0$	f_3^2, e_3, f_2, e_2
Special cases		
[1, 0, 1]	$g_1 \neq 0,\ g_2 = 0,\ g_3\, h_3 \neq 0$	f_3, e_3, f_2, e_2
[1, 0, 0, 1]	$g_1 \neq 0,\ g_2 = 0,\ g_3\, h_3 = 0,\ g_4 \neq 0$	1
[1, 0, 0, 0]	$g_1 \neq 0,\ g_2 = 0,\ g_3\, h_3 = 0,\ g_4 = 0$	e_3, f_2, e_2
[0, 1]	$g_1 = 0,\ h_2 \neq 0$	f_3, e_3^2, f_2, e_2
[0, 0, 1]	$g_1 = 0,\ h_2 = 0,\ h_3 \neq 0$	1
[0, 0, 0]	$g_1 = 0,\ h_2 = 0, h_3 = 0$	e_3^2, f_2, e_2

Table 5. Discussion of Example 5.1.

Case	Conditions	Gröbner basis
[1] = Generic case	$\Delta \neq 0$	G
Special cases	$\Delta = 0$	
[0,1]	$\Delta = 0$, $ba - 8a - 28 + 35b \neq 0$	[1]
[0,0]	$\Delta = 0$, $ba - 8a - 28 + 35b = 0$	[$-696 + 339b + (-176b + 148)u + 252z$, $-75 + 33b + 63y + (-32b + 67)u$, $-309b + 204 + 252x + (208b - 152)u$]

11.5. Application to Example 5.1

We present here the complete discussion for Example 5.1 (see Table 5) given by DISPGB, where

$$\Delta = (a - 1)(3a - 7) = 3a^2 - 10a + 7$$

and G is the basis of the generic case given there.

Finally, we give the time evaluation of applying DISPGB to the examples described in the paper (and two others from the tutorial), using a PC with a Pentium(r) II, Intel MMX(TM) tech., 128 MB.

Example	Time(sec)
11.1	8.8
11.2	5.2
11.3	115.9
11.4	33.0

Example	Time(sec)
5.1	8.4
Sing. points cubic	1.8
2×2 gen. lin. sys.	8.2

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