Software for Discussing Parametric Polynomial Systems: The Gröbner Cover

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Abstract. We present the canonical Gröbner Cover method for discussing parametric polynomial systems of equations. Its objective is to decompose the parameter space into subsets (*segments*) for which it exists a *generalized reduced Gröbner basis* in the whole segment with fixed set of leading power products on it. Wibmer's Theorem guarantees its existence. The Gröbner Cover is designed in a joint paper of the authors, and the Singular grobcov.lib library [15] implementing it, is developed by Montes. The algorithm is canonic and groups the solutions having the same kind of properties into different disjoint segments. Even if the algorithms involved have high complexity, we show how in practice it is effective in many applications of medium difficulty. An interesting application to automatic deduction of geometric theorems is roughly described here, and another one to provide a taxonomy for exact geometrical loci computations, that is experimentally implemented in a web based application using the dynamic geometry software Geogebra, is explained in another session.

Keywords: Groebner cover, parametric polynomial, canonical algorithm, automatic theorem discovering.

1 The Gröbner Cover

The Gröbner Cover algorithm for discussing parametric polynomial ideals gives a canonical description, classifying the solutions by their characteristics (number of solutions, dimension, etc.).

The Gröbner Cover is the analog of the *reduced Gröbner basis* of an ideal for parametric ideals. Its existence was proved by Wibmer's Theorem [14], and the method and algorithms were developed in [8]. Montes implemented in Singular the grobcov.lib library [15], whose actual version incorporates Kapur-Sun-Wang algorithm [3] for computing the initial Gröbner System used in grobcov algorithm, as described in [6], and recently also the Locus algorithm used in Dynamical Geometry software as described in [1] and in another session.

Let $\mathbf{x} = x_1, \ldots, x_n$ be the set of variables and $\mathbf{a} = a_1, \ldots, a_m$ the set of parameters. Given a generating set $F = \{f_1, \cdots, f_s\} \subset \mathbb{Q}[\mathbf{a}][\mathbf{x}]$ of the parametric

H. Hong and C. Yap (Eds.): ICMS 2014, LNCS 8592, pp. 406–413, 2014.

ideal $I = \langle F \rangle$ and a monomial order $\succ_{\mathbf{x}}$ in the variables, the GROBCOV algorithm determines

- the unique *canonical partition* of the parameter space $\mathbb{C}^m$ into locally closed sets (*segments*) with associated *generalized reduced Gröbner basis*:

$$GC = \{(S_1, B_1, \text{lpp}_1), \ldots, (S_r, B_r, \text{lpp}_r)\}.$$

- The segments S_i are disjoint locally closed subsets of $\mathbb{C}^m$ and $\oplus_i S_i = \mathbb{C}^m$.
- The basis B_i of a segment S_i has *fixed set of leading power products* (lpp), who ensures that the type of solutions is the same over all points of the segment, and is the *generalized reduced Gröbner basis* of $\langle F \rangle$ over the segment S_i.
- The lpp's are included in the output, even if they they are given by the basis, to characterize the segments and facilitate the applications.
- Moreover, if the ideal is homogeneous, the lpp's are characteristic of the segment as no other segment has the same lpp's.

The generalized reduced Gröbner basis B_i of a segment S_i is formed by a set of monic I-regular functions over S_i. An I-regular function, representing an element of the basis, allows a full-representation in terms of a set of polynomials that specialize for every point $\mathbf{a}_0$ of the segment, either to the corresponding element of the reduced Gröbner basis of the specialized ideal $I_{\mathbf{a}_0}$ after normalization, or to zero. It also allows a generic representation given by a single polynomial that specializes well on an open subset of the segment and to zero on the remaining points of it. Usually the generic representation is sufficient, and we can, if needed, compute the full representation from it using the EXTEND algorithm.

The segments S_i are expressed in canonical P-representation, given by a set of prime ideals of the form

$$\text{Prep}(S) = \{\{\mathfrak{p}_i, \{\mathfrak{p}_{ij} : 1 \leq j \leq r_i\}\} : 1 \leq i \leq s\}$$

representing the set:

$$S = \bigcup_{i=1}^{s} \left(\mathbb{V}(\mathfrak{p}_i) \setminus \bigcup_{j=1}^{r_i} \mathbb{V}(\mathfrak{p}_{ij}) \right).$$

Each $\mathbb{V}(\mathfrak{p}_i) \setminus \bigcup_{j=1}^{r_i} \mathbb{V}(\mathfrak{p}_{ij})$ is a *component* of the segment, and its representative $\{\mathfrak{p}_i, \{\mathfrak{p}_{ij} : 1 \leq j \leq r_i\}\}$, by abuse of language, is also denoted a component when there is no ambiguity. $\mathfrak{p}_i$ is called the *top* of the component, and $\{\mathfrak{p}_{ij} : 1 \leq j \leq r_i\}$ the *holes*.

1.1 Historical Development of the Theory of Gröbner Bases for Parametric Polynomial Ideals

The first steps in the algebraic study of parametric polynomial ideals where made by V. Weispfenning (1992) in [12], who proved the existence of a Comprehensive Gröbner System (CGS) and a Comprehensive Gröbner Basis (CGB). Progress were made in two directions:

1. Improving the output: Montes (2002) [5], Weispfenning (2003) [13], Manubens & Montes (2009) [4], Montes & Wibmer (2010) [8], Montes (2012) [6].
2. Speed up the algorithms: Kapur (1995), Kalkbrenner (1997), Sato (2005), Suzuki & Sato (2006) [11], Nabeshima (2007) [9], Kapur & Sun & Wang (2010) [3].

The Gröbner Cover [8] is the final state of the research of point 1., and the actual implementation of the GC algorithm incorporates the best speed up algorithm [3] of point 2. as described in [6].

1.2 The Gröbner Cover Algorithm

The algorithm for computing the Gröbner Cover has the following steps:

1. Homogenize the input ideal wrt the variables.
2. Compute a disjoint reduced Comprehensive Gröbner System (DRCGS).[1]
3. Compute the P-representation of the segments.
4. Add together the segments with common lpp using LCUnion algorithm, knowing that the union is locally closed by Wibmer's Theorem.
5. Dehomogenize the bases.
6. For every GC-segment, compute the generic representation of the generalized reduced Gröbner basis using Combine algorithm.
7. Optionally, one can also compute the full representation of the bases using Extend algorithm after computing the generic GC

When the GC algorithm [8] was introduced in 2010, the DRCGS used for step 2. in the implementation was our own algorithm buildtree [8]. But its use is not strictly necessary. We only need to compute a DRCGS. In the new 2012 implementation of the GC the DRCGS used in step 2. was Kapur-Sun-Wang algorithm [3] because it is simpler and generally faster. This is described in [6].

1.3 Example

To fix ideas on the use of the grobcov algorithm of the Singular "grobcov.lib" library [15], let us consider a very simple example: the inverse kinematic problem of the robot arm of Figure 1. The problem consist of determining the angles θ_1 and θ_2 and the length ℓ to reach the point of coordinates (r, z). Setting $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$ the equations are obviously:

$$F = s_1 s_2 \ell - c_1 c_2 \ell - c_1 + r, s_1 c_2 \ell - s_1 - c_1 s_2 \ell + z, s_1^2 + c_1^2 - 1, s_2^2 + c_2^2 - 1.$$

The call for solving the problem using Singular grobvcov is:

[1] A DRCGS is a CGS whose segments are dijoint and the bases specialize to the reduced Gröbner basis and have fixed lpp over the whole segment.

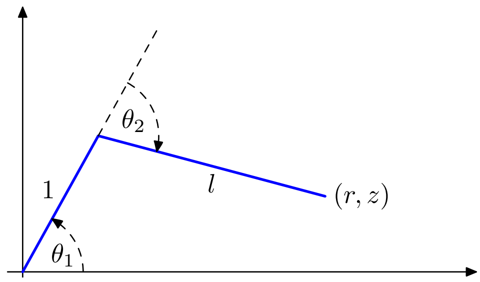

Fig. 1. Simple robot arm

Input:

```
LIB "grobcov.lib";
ring R=(0,r,z),(s1,c1,s2,c2,l),lp;
ideal F= s1*s2*l-c1*c2*l-c1+(r), s1*c2*l-s1-c1*s2*l+(z),
        s1^2+c1^2-1, s2^2+c2^2-1;
def G=grobcov(F);
"grobcov(F)=" G;
```

Output: We summarize the output in the following table

S.	lpp	Basis	Segment
1	$c_2\ell, s_2^2$, c_1, s_1.	$2c_2\ell+\ell^2+(-r^2-z^2+1),\quad s_2^2+c_2^2-1$, $(2r^2+2z^2)c_1+(-2z)s_2\ell+r\ell^2+(-r^3-rz^2-r)$, $(2r^2+2z^2)s_1+(2r)s_2\ell+(z)\ell^2+(-r^2z-z^3-z)$.	$\mathbb{C}^2\setminus\mathbb{V}(r^2+z^2)$
2	$c_2\ell s_2$, $c_1\ell^2$, c_1c_2, s_1.	$2c_2\ell+\ell^2+1, (z)s_2+(-r)c_2+(-r)l$, $(4z^2)c_1\ell^2+(-4z^2)c_1+(-r)\ell^4+(2r)\ell^2+(4rz^2-r)$, $(8z^2)c_1c_2+(8z^2)c_1\ell+(-8rz^2+2r)c_2+(-r)\ell^3+$ $+(-4rz^2+3r)\ell,\quad (2z)s_1+(2r)c_1+\ell^2-1$.	$\mathbb{V}(r^2+z^2)\setminus\mathbb{V}(z,r)$
3	ℓ^2, c_2, s_2, s_1^2	$\ell^2-1, c_2+1, s_2, s_1^2+c_1^2-1$.	$\mathbb{V}(z,r)$

There are 3 segments, and for each segment there are 4 arguments: 1) the lpp, 2) the basis, 3) the P-representation of the segment, 4) the lpp of the homogenized ideal. The fourth argument is purely informative to verify that each segment has a characteristic lpp of the homogenized ideal. It can be discarded, and we deleted it form the output. The output is to be read as follows:

1) The first segment represents the generic case: the solution is valid for every values of the parameters r, z, except when $r^2+z^2=0$. We have one-degree of freedom in the variables. One can choose ℓ free. For each value of $\ell\neq 0$ there are two angle solutions with opposite value of θ_2. For fixed ℓ we have

$$c_2=\frac{r^2+z^2-\ell^2-1}{2\ell},\qquad s_2=\pm\sqrt{1-c_2^2},$$

$$c_1=\frac{2zs_2\ell+r(r^2+z^2+1-\ell^2)}{2(r^2+z^2)},\ s_1=\frac{-2rs_2\ell+z(r^2+z^2+1-\ell^2)}{2(r^2+z^2)}.$$

As we want real solutions, we must choose ℓ such that $|c_2| \leq 1$. We set $\ell > 0$. The limits for $\cos(\theta_2)$ imply $|\ell - 1| \leq \sqrt{r^2 + z^2} \leq \ell + 1$. With this choice the angles are real.

2) The second segment is purely complex and can be discarded in practice.

3) There is only one special position $(r, z) = (0, 0)$ for which necessarily $\ell = \pm 1$ and in practice $\ell = 1$. Then $\theta_2 = \pi$, and θ_1 free.

These results correspond accurately to the geometry.

2 Applications

The Gröbner Cover has many applications. Let us highlight one interesting problem that can be solved using it: automatic discovering of geometrical theorems. In the "Parametric Polynomial Systems" session we show its use for determining and classifying geometrical loci that can be used by Dynamical Geometry software [1].

2.1 Automatic Deduction of Geometrical Theorems

Consider a generally false geometrical statement depending on some variable points for which we want to find the conditions in order to make the statement to hold true. Consider the coordinates of the free points of a construction as parameters and the remaining coordinates or values as variables. Then apply grobcov to the system defining the statement and the construction, and find the conditions over the parameters that makes the statement hold true. We show an interesting example: the generalization of the classical XIX-century known Steiner-Lehmus Theorem [10] that is described in [7]. Let us summarize here the results.

Classical Theorem states that the length of the inner bisectors of a triangle are equal if and only if the triangle is isosceles. Consider the triangle ABC of

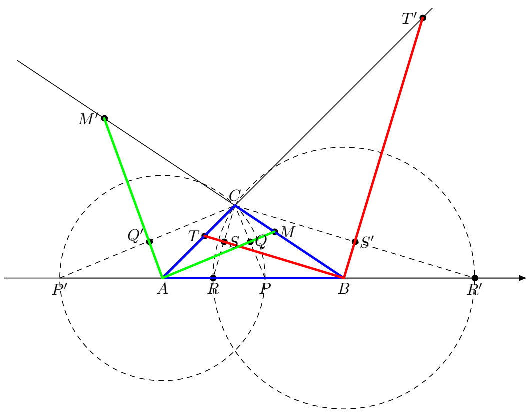

Fig. 2. Bisectors of the triangle ABC

Figure 2, and take coordinates $A(-1,0)$, $B(1,0)$ and $C(a,b)$. Trace the circles with center A and radius AC that intersects line AB (i.e the x-axis) at $P(p,0)$ and P', and the circle with center B and radius BC intersecting line AB at $R(r,0)$ and R'. The equation $(a+1)^2+b^2-(p+1)^2$ determining p in terms of a,b does not distinguish between P and P'. The same happens for the points R and R' and for the equations determining M and T (or T') and M (or M'), so that the statement $\overline{AM}^2 = \overline{BT}^2$, i.e. $(x_1+1)^2+y_1^2 = (x_2-1)^2+y_2^2$, does not distinguish between inner and outer bisectors. System F only implies that one bisector (inner or outer) of A is equal to one bisector of B. The system is

$$\begin{aligned} F = {} & (a+1)^2+b^2-(p+1)^2,\ (a-1)^2+b^2-(r-1)^2, \\ & ay_1-bx_1-y_1+b,\ ay_2-bx_2+y_2-b, \\ & -2y_1+bx_1-(a+p)y_1+b,,\ 2y_2+bx_2-(a+r)y_2-b, \\ & (x_1+1)^2+y_1^2-(x_2-1)^2-y_2^2. \end{aligned}$$

Applying GROBCOV in the `ring R=(0,a,b),(x1,y1,x2,y2,p,r),dp` to the ideal generated by F it outputs 9 segments. Table 1 gives the 3 curves and 9 point varieties representing real and complex points in the parameter space appearing in the description of the GROBCOV(F). We do not detail the complex points as we are not interested in. Table 2 summarizes the relevant characteristics of the output of GROBCOV(F) for our purposes.

Table 1. Curves and point varieties appearing in GROBCOV(F)

Curves
$\mathcal{C}_1 = \mathbb{V}((8a^2+9b^2)(a^2+b^2)^4 - 4(14a^4+13a^2b^2-3b^4)(a^2+b^2)^2$ $+2(72a^6+43a^4b^2-74a^2b^4-37b^6) - 4(44a^4-39a^2b^2+43b^4)+104a^2+137b^2-24)$,
$\mathcal{C}_2 = \mathbb{V}(a)$,
$\mathcal{C}_3 = \mathbb{V}(b)$.

Point varieties	real points	numerical values
$V_1 = \mathbb{V}(b.a+1)$	P_1	$=(-1,0)$
$V_2 = \mathbb{V}(b,a-1)$	P_2	$=(1,0)$
$V_3 = \mathbb{V}(a,b)$	P_4	$=(0,0)$
$V_4 = \mathbb{V}(b,a^2-3)$	P_{42}, P_{41}	$=(\pm\sqrt{3},0)$
$V_5 = \mathbb{V}(3b^2-1,a)$	P_{52}, P_{51}	$=(0,\pm\sqrt{3}/3)$
$V_6 = \mathbb{V}(b^2-3,a)$	P_{62}, P_{61}	$=(0,\pm\sqrt{3})$
$V_7 = \mathbb{V}(b^4+5b^2+8,a)$		no real roots
$V_8 = \mathbb{V}(b^4+44b^2-16,5a+b^2+7)$	P_{82}, P_{81}	$=(3-2\sqrt{5},\pm\sqrt{-22+10\sqrt{5}})$
$V_9 = \mathbb{V}(b^4+44b^2-16,5a-b^2-7)$	P_{92}, P_{91}	$=(-3+2\sqrt{5},\pm\sqrt{-22+10\sqrt{5}})$

Curves and points can be visualized on Figure 3. The fourth column in Table 2 is direct consequence of the lpp in column 3. We need the basis to determine the fifth column, who indicates which bisectors (internal i or external e) are

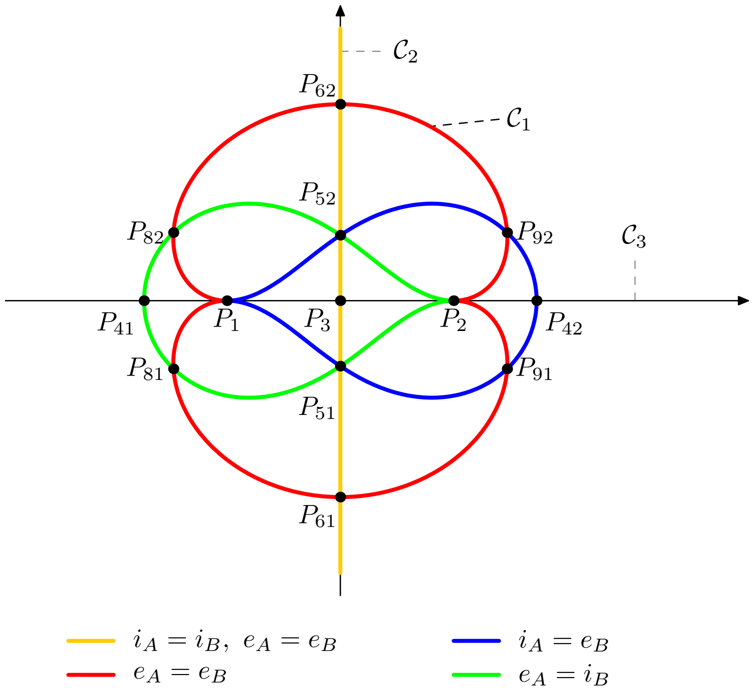

Fig. 3. Generalized Steiner-Lehmus Theorem

equal for the different solutions. From the basis we can determine the signs of $p+1$ and of $r-1$ for each point of the solution.

$p > -1$ corresponds to the inner bisector i_A and $p < -1$ to the external e_A,
$r < 1$ corresponds to the inner bisector i_B and $r > 1$ to the external e_B.

The line $\mathcal{C}_3$, i.e. the x-axis, corresponds to degenerate triangles, and so the segments 4,7,8 can be discarded. The remaining segments give the whole information on the generalized Theorem. The curve $\mathcal{C}_1$ has different colors, that can change only at the special self intersecting points. To determine its color it suffices to evaluate p and r on an intermediate point of the interval. The curve $\mathcal{C}_2$ corresponds to the classical Theorem and with the Gröbner Cover we can appreciate also more details on it. On the whole line (isosceles triangles) we have $i_A = i_B$ and also $e_A = e_B$ except for special points P_{51} and P_{52} where all bisectors are equal and special points P_{61} and P_{62} where the external bisectors become infinity. The Gröbner Cover reveals the generalized Theorem over the curve $\mathcal{C}_1$ with all the details.

Table 2. Segments of GROBCOV(F). (Bases are not explicitely given)

Nr.	Segment	lpp	Num. S.	Bisectors
1	$\mathbb{C}^2 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$	$\{1\}$	0	-
2	$\mathcal{C}_1 \setminus \left(\left(\bigcup_{i=1}^{2} V_i\right) \cup \left(\bigcup_{i=4}^{9} V_i\right)\right)$	$\}\{r, p, y_2, x_2, y_1, x_1\}$	1	depends on sector
3	$(\mathcal{C}_2 \setminus (V_3 \cup V_5 \cup V_6)) \cup$ $\cup\ V_8$	$\{p, y_2, x_2, y_1, x_1, r^2\}$	2	$i_A = i_B,\ e_A = e_B$ $e_A = e_B = i_B$
4	$\mathcal{C}_3 \setminus (V_1 \cup V_2)$	$\{y_2, y_1, r^2, p^2, x_1^2\}$	∞	
5	V_5	$\{y_2, x_2, y_1, x_1, r^2, p^2\}$	4	$i_A = i_B = e_A = e_B$
6	V_6	$\{r, p, y_2, x_2, y_1, x_1\}$	1	$i_A = i_B$
7	V_1	$\{y_1, r^2, y_2 r, p^2 x_1^2\}$	∞	
8	V_2	$\{y_2, r^2, p^2, y_1 p, x1^2\}$	∞	
9	V_9	$\{r, y_2, x_2, y_1, x_1, p^2\}$	2	$e_A = e_B = i_A$

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