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A polynomial generalization of the power-compositions determinant†

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Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$, and $\mathbf{x} = (x_1, \dots, x_p)$ a vector of indeterminates. For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ two p-compositions of n, define $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1+\alpha_1)^{\beta_1} \cdots (x_p+\alpha_p)^{\beta_p}$. In this paper we prove an explicit formula for the determinant $\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)}((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}})$. In the case $x_1 = \cdots = x_p$ the formula gives a proof of a conjecture by C. Krattenthaler.

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1 Introduction

Let us start with some notation. If $\mathbf{u} = (u_1, \dots, u_\ell)$ and $\mathbf{v} = (v_1, \dots, v_\ell)$ are two vectors of the same length, we define $\mathbf{u}^{\mathbf{v}} = u_1^{v_1} \cdots u_\ell^{v_\ell}$ (where, to be consistent $0^0 = 1$). In our case, the entries u_i and v_i of $\mathbf{u}$ and $\mathbf{v}$ will be nonnegative integers or polynomials. We use $\mathbf{x} = (x_1, \dots, x_p)$ to denote a vector of indeterminates and $\mathbf{1} = (1, \dots, 1)$. The lengths of $\mathbf{x}$ and $\mathbf{1}$ will be clear from the context. If $\mathbf{u} = (u_1, \dots, u_\ell)$, then $s(\mathbf{u})$ denotes the sum of the entries of $\mathbf{u}$, i.e. $s(\mathbf{u}) = u_1 + \cdots + u_\ell$, and $\bar{\mathbf{u}}$ denotes the vector obtained from $\mathbf{u}$ by deleting the last coordinate, $\bar{\mathbf{u}} = (u_1, \dots, u_{\ell-1})$.

Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)$ are two p-compositions of n, using the above notation, we have $\boldsymbol{\alpha}^{\boldsymbol{\beta}} = \alpha_1^{\beta_1} \cdots \alpha_p^{\beta_p}$. In [1] the following explicit formula

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for the determinant $\Delta(n,p) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left(\boldsymbol{\alpha}^{\boldsymbol{\beta}}\right)$ was proved:

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}}. \qquad (1)$$

In a complement [4] to his impressive *Advanced Determinant Calculus* [3], C. Krattenthaler mentions this determinant, and after giving the alternative formula

$$\Delta(n,p) = n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}} \qquad (2)$$

he states as a conjecture a generalization to univariate polynomials. Namely, let x be an indeterminate and

$$\Delta(n,p,x) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left((x\cdot\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} \right).$$

Note that $(x\cdot\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x+\alpha_1)^{\beta_1}\cdots(x+\alpha_p)^{\beta_p}$.

Conjecture [C. Krattenthaler]:

$$\Delta(n,p,x) = (px+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}. \qquad (3)$$

As $(n-i+1)\binom{n+p-i-1}{p-2} = (p-1)\binom{n+p-i-1}{p-1}$, formula (2) can be written in the form

$$\Delta(n,p) = n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}$$

and Krattenthaler's Conjecture (3) in the form

$$\Delta(n,p,x) = (px+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}. \qquad (4)$$

The main goal of this paper is to prove a generalization of formula (4) for p indeterminates. For this, let $\mathbf{x} = (x_1,\ldots,x_p)$ be a vector of indeterminates, and let

$$\Delta(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

(Recall that $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1+\alpha_1)^{\beta_1}\cdots(x_p+\beta_p)^{\beta_p}$). Then, we prove the following formula (Theorem 5.1):

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}. \tag{5}$$

As $s(\mathbf{x}) = x_1+\cdots+x_p$, if $x_1=\cdots=x_p=x$, then $s(\mathbf{x}) = px$ and the conjectured identity (4) follows.

We also prove a variant of this result for proper compositions. A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_p)$ of n such that $\alpha_i \geq 1$ for all $i = 1,\ldots,n$. Denote by $C^*(n,p)$ the set of proper p-compositions of n and define

$$\Delta^*(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)} \left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

The determinant $\Delta^*(n,p,\mathbf{x})$ has the following factorization (Theorem 6.1):

$$\Delta^*(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n-1}{p}} \left(\prod_{j=1}^{p}\prod_{i=1}^{n-p+1}(x_j+i)^{\binom{n-i-1}{p-2}}\right) \prod_{i=1}^{n-p+1} i^{(p-1)\binom{n-i-1}{p-1}}. \tag{6}$$

The paper is organized as follows. In the next section we collect some combinatorial identities for further reference. In Section 3 we prove the equivalence between the formula (2) given by Krattenthaler and (1). In Section 4 we prove two lemmas. The first one is a generalization of the determinant $\Delta(n,2,\mathbf{x})$. The second lemma uses the first and corresponds to a property of a sequence of rational functions which appear in the triangulation process of the determinant $\Delta(n,p,\mathbf{x})$. Section 5 contains the proof of the main result, Theorem 5.1. Finally, Section 6 is devoted to proving (6).

2 Auxiliary summation formulas

LEMMA 2.1 *Let a,b,c,d,m and n be nonnegative integers. Then, the following equalities hold.*

(i) $\sum_{k\in\mathbb{Z}}\binom{a}{c+k}\binom{b}{d-k} = \binom{a+b}{c+d}$;

(ii) $\sum_{k\leq n}\binom{a+k}{a} = \sum_{k\leq n}\binom{a+k}{k} = \binom{n+a+1}{a+1}$;
(iii) $\sum_{r=1}^{n} r\binom{n+a-r}{a} = \binom{n+a+1}{a+2}$;

Proof (i) is the well known Vandermonde's convolution, see [2, p. 169]. The formulas in (ii) are versions of the parallel summation [2, p. 159]. Part (iii) follows from

$$\sum_{r=1}^{n} r\binom{n+a-r}{a} = \sum_{r=1}^{n} r\binom{n+a-r}{n-r} = \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{a+i}{a}$$
$$= \sum_{k=0}^{n-1}\binom{a+k+1}{a+1} = \binom{a+n+1}{a+2}.$$

□

3 Equivalence between the two formulas for x = 0

Here we prove the equivalence beetween the formulas (1) and (2) for $\Delta(n,p)$. Obviously, the result of substituting $x = 0$ in formula (3) of the Conjecture gives formula (2) for $\Delta(n,p)$.

PROPOSITION 3.1 *Formulas* (1) *and* (2) *are equivalent.*

Proof We derive formula (2) from (1), which was already proved in [1]. First, note that if $p < k \leq n$, the binomial coefficient $\binom{p}{k}$ is zero. Thus, we can replace $\min\{p,n\}$ by n in formula (1). Analogously, if $n-k+1 < i \leq n$, the binomial coefficient $\binom{n-i-1}{k-2}$ is zero, and we can replace the upper value $n-k+1$ by n in the inner product. Second, the case $a = n-1$, $b = d = p$ and $c = 0$ of Lemma 2.1 (i) yields

$$\sum_{k=1}^{n}\binom{n-1}{k}\binom{p}{k} = -1 + \sum_{k=0}^{n}\binom{n-1}{k}\binom{p}{p-k} = \binom{n+p-1}{p} - 1,$$

and, if $i \geq 1$, by taking $a = n-i-1$, $b = d = p$ and $c = -2$ in Lemma 2.1 (i), we obtain

$$\sum_{k=1}^{n-1}\binom{n-i-1}{k-2}\binom{p}{k} = \sum_{k}^{n-1}\binom{n-i-1}{k-2}\binom{p}{p-k} = \binom{n+p-i-1}{p-2}.$$

Therefore,

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}}\left(n^{\binom{n-1}{k}}\prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}}\right)^{\binom{p}{k}}$$
$$= \prod_{k=1}^{n}\left(n^{\binom{n-1}{k}}\prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}}\right)^{\binom{p}{k}}$$

$$= \left(\prod_{k=1}^{n} n^{\binom{n-1}{k}\binom{p}{k}}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}\binom{p}{k}}\right)$$

$$= n^{\binom{n+p-1}{p}-1}\left(\prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}}\right) n^{\sum_{k=1}^{n}\binom{-1}{k-2}\binom{p}{k}}$$

$$= n^{\binom{n+p-1}{p}+p-1}\prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}}$$

$$= n^{\binom{n+p-1}{p}}\prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}.$$

□

4 A recurrence

The next lemma evaluates the determinant

$$D_r(n,y,z) = \det_{0\le i,j\le r}\left((y-i)^{n-j}(z+i)^j\right),$$

by reducing it to a Vandermonde determinant. Note that $D_n(n, x_1+n, x_2) = \Delta(n, 2, \mathbf{x})$.

Lemma 4.1

$$D_r(n,y,z) = (y+z)^{\binom{r+1}{2}}\left(\prod_{i=0}^{r}(y-i)^{n-r}\right)\left(\prod_{i=1}^{r} i^{r-i+1}\right).$$

Proof

$$D_r(n,y,z) = \begin{vmatrix} (y-0)^n(z+0)^0 & (y-0)^{n-1}(z+0)^1 & \cdots & (y-0)^{n-r}(z+0)^r \\ (y-1)^n(z+1)^0 & (y-1)^{n-1}(z+1)^1 & \cdots & (y-1)^{n-r}(z+1)^r \\ \vdots & \vdots & & \vdots \\ (y-r)^n(z+r)^0 & (y-r)^{n-1}(z+r)^1 & \cdots & (y-r)^{n-r}(z+r)^r \end{vmatrix}$$

$$= \left(\prod_{i=0}^{r}(y-i)^n\right)\begin{vmatrix} 1 & (z+0)/(y-0) & \cdots & (z+0)^r/(y-0)^r \\ 1 & (z+1)/(y-1) & \cdots & (z+1)^r/(y-1)^r \\ \vdots & \vdots & & \vdots \\ 1 & (z+r)/(y-r) & \cdots & (z+r)^r/(y-r)^r \end{vmatrix}$$

$$= \left(\prod_{i=0}^{r}(y-i)^n\right) \prod_{0\le i<j\le r}\left(\frac{z+j}{y-j}-\frac{z+i}{y-i}\right)$$
$$= \left(\prod_{i=0}^{r}(y-i)^n\right) \prod_{0\le i<j\le r}\frac{(y+z)(j-i)}{(y-j)(y-i)}$$
$$= \left(\prod_{i=0}^{r}(y-i)^n\right) (y+z)^{\binom{r+1}{2}}\frac{\prod_{i=1}^{r} i^{r-i+1}}{\prod_{i=0}^{r}(y-i)^r}$$
$$= (y+z)^{\binom{r+1}{2}}\left(\prod_{i=0}^{r}(y-i)^{n-r}\right)\left(\prod_{i=1}^{r} i^{r-i+1}\right).$$

□

LEMMA 4.2 *Define* $f_r\colon \mathbb{N}_0\times\mathbb{N}_0\to\mathbb{Q}(y,z)$ *recursively by*

$$f_0(i,j) = (z+i)^j;$$
$$f_{r+1}(i,j) = f_r(i,j) \quad \textit{if} \quad j\le r;$$
$$f_{r+1}(i,j) = f_r(i,j) - \left(\frac{y-i}{y-r}\right)^{j-r}\frac{f_r(i,r)f_r(r,j)}{f_r(r,r)} \quad \textit{if} \quad j>r.$$

Then

(i) $f_{r+1}(r,j)=0$ *for* $j\ge r+1$*;*

(ii) $f_r(r,r) = (y+z)^r\dfrac{r!}{\prod_{i=0}^{r-1}(y-i)}$.

Proof Part (i) is trivial using induction. To obtain $f_r = f_r(r,r)$, we take $n\ge r$ and calculate $D(n,y,z)=D_n(n,y,z)$ by Gauss triangulation method.

The entry (i,j) of $D(n,y,z)$ is $(y-i)^{n-j}(z+i)^j = (y-i)^{n-j}f_0(i,j)$. If $j\ge 1$, add to the column j the column 0 multiplied by

$$-\frac{1}{(y-0)^{j-0}}\frac{f_0(0,j)}{f_0(0,0)}.$$

Then, the entry (i,j) with $j\ge1$ is modified to

$$(y-i)^{n-j}f_0(i,j) - (y-i)^{n-0}f_0(i,0)\frac{1}{(y-0)^{j-0}}\frac{f_0(0,j)}{f_0(0,0)}$$
$$= (y-i)^{n-j}\left\{f_0(i,j) - \left(\frac{y-i}{y-0}\right)^{j-0}\frac{f_0(i,k)f_0(k,j)}{f_0(0,0)}\right\}$$

$$= (y-i)^{n-j} f_1(i,j).$$

Therefore, $D(n,y,z) = \det_{0\le i,j\le r}\big((y-i)^{n-j} f_1(i,j)\big)$ and $f_1(0,j) = 0$ for $j \geq 1$.

Now, assume that $D(n,y,z) = \det_{0\le i,j\le n}\big((y-i)^{n-j} f_k(i,j)\big)$ for $k \geq 1$ with $f_k(i,j) = 0$ for $k, j > i$. Add to the column $j \geq k+1$ the column k multiplied by

$$-\frac{1}{(y-k)^{j-k}} \frac{f_k(k,j)}{f_k(k,k)}.$$

The entry (i,j) is modified to

$$\begin{aligned}
&(y-i)^{n-j} f_k(i,j) - (y-i)^{n-k} f_k(i,k) \cdot \frac{1}{(y-k)^{j-k}} \cdot \frac{f_k(k,j)}{f_k(k,k)} \\
&= (y-i)^{n-j} \left\{ f_k(i,j) - \left(\frac{y-i}{y-k}\right)^{j-k} \frac{f_k(i,k) f_k(k,j)}{f_k(k,k)} \right\} \\
&= (y-i)^{n-j} f_{k+1}(i,j).
\end{aligned}$$

Clearly $f_{k+1}(k,j) = 0$ for $j > k$. After n iterations, we get the determinant of a triangular matrix. Hence

$$D(n,y,z) = \det_{0\le k\le n}\Big((y-k)^{n-k} f_k(k,k)\Big) = \prod_{r=0}^{n} (y-k)^{n-k} f_k.$$

The principal minor of order $r+1$ is $D_r(n,y,z) = \prod_{k=0}^{r} (y-k)^{n-k} f_k$. Therefore,

$$\frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} = (y-r)^{n-r} f_r. \qquad (7)$$

On the other hand, by Lemma 4.1 we obtain

$$\begin{aligned}
\frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} &= \frac{(y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r} (y-i)^{n-r}\right) \left(\prod_{i=1}^{r} i^{r-i+1}\right)}{(y+z)^{\binom{r}{2}} \left(\prod_{i=0}^{r-1} (y-i)^{n-r-1}\right) \left(\prod_{i=1}^{r-1} i^{r-i}\right)} \\
&= (y+z)^r \cdot r! \cdot \frac{(y-r)^{n-r}}{\prod_{i=0}^{r-1} (y-i)}.
\end{aligned}$$

Comparing with (7), we have arrived at

$$f_r = (y+z)^r \frac{r!}{\prod_{i=0}^{r-1}(y-i)}.$$

□

5 Proof of the main theorem

We sort $C(n,p)$ in lexicographic order. For instance, for $n=5$, and $p=3$, we obtain

$$\begin{aligned} C(5,3) = \{ & (5,0,0),(4,1,0),(3,2,0),(2,3,0),(1,4,0),(0,5,0),\\ & (4,0,1),(3,1,1),(2,2,1),(1,3,1),(0,4,1),\\ & (3,0,2),(2,1,2),(1,2,2),(0,3,2),\\ & (2,0,3),(1,1,3),(0,2,3),\\ & (1,0,4),(0,1,4),\\ & (0,0,5)\}. \end{aligned}$$

Let $M(n,p,\mathbf{x})$ be the matrix with rows and columns labeled by the p-compositions of n in lexicographic order and with the entry $(\boldsymbol{\alpha},\boldsymbol{\beta})$ equal to $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$. We have $\Delta(n,p,\mathbf{x}) = \det M(n,p,\mathbf{x})$.

An entry $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$ in $M(n,p,\mathbf{x})$ can be written in the form $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}(x_p+\alpha_p)^{\beta_p}$. For $0\le i,j\le n$, let S_{ij} be the matrix with entries $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $\alpha_p = i$ and $\beta_p = j$. Thus, the submatrix of $M(n,p,\mathbf{x})$ formed by the entries labeled $(\boldsymbol{\alpha},\boldsymbol{\beta})$ with $\alpha_p = i$ and $\beta_p = j$ can be written $\big(S_{ij}(x_p+i)^j\big)$. Note that

$$S_{kk} = M(n-k, p-1, \bar{\mathbf{x}}).$$

Define $f_0(i,j) = (x_p+i)^j$. Therefore, $M(n,p,\mathbf{x})$ admits the block decomposition

$$M(n,p,\mathbf{x}) = (S_{ij}f_0(i,j))_{0\le i,j\le n}.$$

The idea is to put $M(n,p,\mathbf{x})$ in block triangular form in such a way that at each step only the last factor of each block is modified.

Theorem 5.1

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}.$$

Proof The proof is by induction on p. For $p=1$, $\Delta(n,p,x)$ is the determinant of the 1×1 matrix $((x+n)^n)$. Hence $\Delta(n,p,x)=(x+n)^n$. This value coincides with the right hand side of the formula for $p=1$.

Consider now the case $p=2$. Any 2-composition of n is of the form $(n-i,i)$ for some i, $0\le i\le n$. The determinant to be calculated is $\Delta(n,2,\mathbf{x}) = \det_{0\le i,j\le n}\left((x_1+n-i)^{n-j}(x_2+i)^j\right)$. By taking $r=n$, $y=x_1+n$ and $z=x_2$ in Lemma 4.1, we get

$$\Delta(n,2,\mathbf{x}) = D_n(n,x_1+n,x_2) = (x_1+x_2+n)^{\binom{n+1}{2}}\prod_{i=1}^{n} i^{n-i+1}.$$

Therefore, the formula holds for $p=2$.

Now, let $p>2$ and assume that the formula holds for $p-1$. Begin with the block decomposition of the matrix $M(n,p,\mathbf{x}) = (S_{ij}f_0(i,j))_{0\le i,j\le n}$.

Assume $\Delta(n,p,\mathbf{x}) = \det(S_{ij}f_r(i,j))$ where $S_{ij} = ((\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}})$, with $\alpha_p=i$, $\beta_p=j$, and $f_r(i,j)=0$ for $i<r$ and $j>i$.

Fix a column $\boldsymbol{\beta}$ with $\beta_p=j>r$. For each $\boldsymbol{\gamma}\in C(n,p)$ with $\gamma_p=r$ and $\gamma_k\ge\beta_k$ for $k\in[p-1]$, add to the column $\boldsymbol{\beta}$ the column $\boldsymbol{\gamma}$ multiplied by

$$-\frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\binom{j-r}{\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}}\frac{f_r(r,j)}{f_r(r,r)}.$$

The differences $\bar{\boldsymbol{\delta}}=\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}$ are exactly the $(p-1)$-compositions of $j-r$. Also note that by the multinomial theorem,

$$(s(\bar{\mathbf{x}})+n-i)^{j-r} = ((x_1+\alpha_1)+\cdots+(x_{p-1}+\alpha_{p-1}))^{j-r} = \sum_{\bar{\boldsymbol{\delta}}}\binom{j-r}{\bar{\boldsymbol{\delta}}}(s(\bar{\mathbf{x}})+\bar{\boldsymbol{\alpha}})^{\bar{\delta}}.$$

Then, a term of column $\boldsymbol{\beta}$ is modified to

$$\begin{aligned}
&(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}f_r(i,j) - \sum_{\bar{\gamma}}\frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\binom{j-r}{\bar{\boldsymbol{\gamma}}-\bar{\boldsymbol{\beta}}}\frac{f_r(r,j)}{f_r(r,r)}(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\gamma}}f_r(i,r)\\
&= (\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}\left\{f_r(i,j) - \frac{1}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\left(\sum_{\bar{\boldsymbol{\delta}}}\binom{j-r}{\bar{\boldsymbol{\delta}}}(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\delta}}}\right)\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}\right\}\\
&= (\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}\left\{f_r(i,j) - \frac{(s(\bar{\mathbf{x}})+n-i)^{j-r}}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}\right\}.
\end{aligned}$$

Now, define $f_{r+1}(i,j)=f_r(i,j)$ for $j\le r$ and

$$f_{r+1}(i,j) = f_r(i,j) - \frac{(s(\bar{\mathbf{x}})+n-i)^{j-r}}{(s(\bar{\mathbf{x}})+n-r)^{j-r}}\frac{f_r(r,j)f_r(i,r)}{f_r(r,r)}$$

for $j > r$. Note that $f_{r+1}(r,j) = 0$ for $j > r$. After n iterations, we arrive at the block matrix $(S_{ij} f_n(i,j))_{0\leq i,j\leq n}$ where $f(i,j) = 0$ for $j > i$. Thus, the determinant $\Delta(n,p,\mathbf{x})$ is the product of the determinants of the diagonal blocks:

$$\Delta(n,p,\mathbf{x}) = \prod_{r=0}^{n} \det(S_{rr} f_r(r,r)).$$

Now, $S_{rr} = M(n-r, p-1, \bar{\mathbf{x}})$, a square matrix of order $\binom{n-r+p-2}{p-2}$. Therefore

$$\Delta(n,p,\mathbf{x}) = \prod_{r=0}^{n} \left(\Delta(n-r, p-1, \bar{\mathbf{x}}) f_r(r,r)^{\binom{n-r+p-2}{p-2}} \right).$$

Now, observe that the rational funcions f_r satisfy the hypothesis of Lema 4.2 with $y = s(\bar{\mathbf{x}}) + n = x_1 + \cdots + x_{p-1} + n$ and $z = x_p$. Thus,

$$f_r = f_r(r,r) = (s(\mathbf{x}) + n)^r \cdot \frac{r!}{\prod_{i=0}^{r-1}(s(\bar{\mathbf{x}}) + n - i)}.$$

By the induction hypothesis,

$$\Delta(n,p,\mathbf{x}) = \prod_{r=0}^{n} \left((s(\bar{\mathbf{x}}) + n - r)^{\binom{n-r+p-2}{p-1}} \prod_{i=1}^{n-r} i^{(p-2)\binom{n-r+p-i-2}{p-2}} \right)$$
$$\cdot \prod_{r=0}^{n} \left((s(\mathbf{x}) + n)^r \cdot r! \cdot \frac{1}{\prod_{i=0}^{r-1}(s(\bar{\mathbf{x}}) + n - i)} \right)^{\binom{n-r+p-2}{p-2}}$$

It remains to count how many factors of each type there are in the above product.

The number of factors $(s(\mathbf{x}) + n)$ is $\sum_{r=1}^{n} r\binom{n+p-r-2}{p-2}$. From Lemma 2.1 (iii) for $a = p-2$ this coefficient is $\binom{n+p-1}{p}$.

The number of factors $s(\bar{\mathbf{x}}) + n - i$, for $0 \leq i \leq n-1$, is (by using Lemma 2.1 (ii) with $a = p-2$)

$$\binom{n-i+p-2}{p-1} - \sum_{r=i+1}^{n} \binom{n-r+p-2}{p-2} = \binom{n-i+p-2}{p-1} - \binom{n-i+p-2}{p-1} = 0.$$

Finally, for $1 \leq i \leq n$, the number of factors equal to i is

$$(p-2)\sum_{r=0}^{n-i} \binom{n+p-i-r-2}{p-2} + \sum_{r=i}^{n} \binom{n+p-r-2}{p-2} =$$

$$(p-2)\binom{n+p-i-r-1}{p-1}+\binom{n+p-r-1}{p-1}=(p-1)\binom{n+p-r-1}{p-1}.$$

□

6 Proper compositions

A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_p)$ of n such that $\alpha_i\geq 1$ for all $i=1,\ldots,n$. We denote by $C^*(n,p)$ the set of proper p-compositions of n. In [1] the following formula was given:

$$\Delta^*(n,p)=\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}(\boldsymbol{\alpha}^{\boldsymbol{\beta}})=n^{\binom{n-1}{p}}\prod_{i=1}^{n-p+1} i^{(n-i+1)\binom{n-i-1}{p-2}}.$$

Here, we study the corresponding generalization

$$\Delta^*(n,p,\mathbf{x})=\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}\left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

THEOREM 6.1 *If $p\leq n$, then*

$$\Delta^*(n,p,\mathbf{x})=(s(\mathbf{x})+n)^{\binom{n-1}{p}}\left(\prod_{i=1}^{n-p+1}\prod_{j=1}^{p}(x_j+i)^{\binom{n-i-1}{p-2}}\right)\prod_{i=1}^{n-p} i^{(p-1)\binom{n-i-1}{p-1}}.$$

Proof The mapping $C^*(n,p)\to C(n-p,p)$ defined by $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_p)\mapsto \boldsymbol{\alpha}-\mathbf{1}=(\alpha_1-1,\ldots,\alpha_p-1)$ is bijective. Thus, we have

$$\begin{aligned}
\Delta^*(n,p,\mathbf{x}) &= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}\left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}\left((\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha}-\mathbf{1})^{\boldsymbol{\beta}-\mathbf{1}+\mathbf{1}}\right)\\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C(n-p,p)}\left((\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}\right)\\
&= \Delta(n-p,p,\mathbf{x}+\mathbf{1})\prod_{\boldsymbol{\alpha}\in C(n-p,p)}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}.
\end{aligned}$$

The number of times that an integer i, $0\leq i\leq n-p$ appears as the first entry of p-compositions of $n-p$ is the number of solutions $(\alpha_2,\ldots,\alpha_{n-p})$ of $i+\alpha_2+\cdots+\alpha_p=n-p$, which is $\binom{n-p-i+p-2}{p-2}=\binom{n-i-2}{p-2}$. The count is the same for every coordinate. Then, in the product $\prod_{\boldsymbol{\alpha}\in C(n-p,p)}(\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}}$, the number of factors equal to x_j+1+i is $\binom{n-i-2}{p-2}$; equivalently, for $1 \leq i \leq n-p+1$, the number of factors equal to $x_j + i$ is $\binom{n-i-1}{p-2}$. Therefore,

$$\begin{aligned}\Delta^*(n, p, \mathbf{x}) &= \Delta(n-p, p, \mathbf{x}+\mathbf{1}) \prod_{\boldsymbol{\alpha} \in C(n-p,p)} (\mathbf{x}+\mathbf{1}+\boldsymbol{\alpha})^{\mathbf{1}} \\ &= (s(\mathbf{x})+n)^{\binom{n-1}{p}} \left(\prod_{i=1}^{n-p+1} \prod_{j=1}^{p} (x_j+i)^{\binom{n-i-1}{p-2}} \right) \prod_{i=1}^{n-p} i^{(p-1)\binom{n-i-1}{p-1}}.\end{aligned}$$

□

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