Software for computing the Canonical Gröbner Cover of a parametric ideal

The Singular grobcov.lib library

Antonio Montes

Universitat Politècnica de Catalunya

ISSAC-2011, San Jose

Outline

Index

References

- M. Wibmer, “Gröbner bases for families of affine or projective schemes". *Jour. Symb. Comp.*, **42**:8 (2007), 803–834.
- Antonio Montes, Michael Wibmer. “Gröbner Bases for Polynomial Systems with Parameters". Journal of Symbolic Computation **45** (2010) 1391 - 1425.

- Software download (beta version): http://www-ma2.upc.edu/∼montes/
- Standard software version will be distributed with the next Singular release.

The problem

Goal

Given: Parametric polynomial system of equations

$$\begin{cases} p_1(a_1,\ldots,a_m,x_1,\ldots,x_n) = 0 \\ \cdots \\ p_r(a_1,\ldots,a_m,x_1,\ldots,x_n) = 0 \end{cases}$$

Goal: describe the different kind of solutions $(x_1,\ldots,x_n)$ *in dependence of the parameters* $a_1,\cdots,a_m$.

Some notations

Let:
K be a computable field (in practice $\mathbb{Q}$).
$\overline{K}$ be an algebraically closed extension of K (in practice $\mathbb{C}$).

$K[\overline{a}]$ the polynomial ring in the parameters $\overline{a} = a_1, \ldots, a_m$ over K.
$K[\overline{a}][\overline{x}]$ the polynomial ring in the variables $\overline{x} = x_1, \ldots, x_n$ over $K[\overline{a}]$.
$\overline{K}^m$ is the parameter space.

Fix: $\succ_{\overline{x}}$ monomial ordering wrt $\overline{x}$ and the ideal
$I = \langle p_1(\overline{a}, \overline{x}), \cdots, p_r(\overline{a}, \overline{x}) \rangle \subset K[\overline{a}][\overline{x}]$
$\mathrm{lpp}(G)$ = set of leading power products wrt $\succ_{\overline{x}}$ of the polynomials in G.

Specialization:
$a = (a_1^0, \cdots, a_m^0) \in \overline{K}^m$
$I_a = \langle p_1(a, \overline{x}), \cdots, p_r(a, \overline{x}) \rangle \subset \overline{K}[\overline{x}]$

Goal

Gröbner bases are the computational method par excellence for studying polynomial systems.

The set of lpp of the reduced Gröbner basis determines the type of solutions of the system.

In the case of parametric polynomial systems the goal is to describe the reduced Gröbner basis of $I_a \subset \overline{K}[\overline{x}]$ (with respect to $\succ_{\overline{x}}$) in dependence of $a \in \overline{K}^m$.

Precedents

Weispfenning (1992)

Given $I = \langle p_1, \dots, p_r \rangle \subset K[\overline{a}][\overline{x}] = K[\overline{a}, \overline{x}]$ *and* $\succ_{\overline{x}}$

A Comprehensive Gröbner System (CGS) for I *and* $\succ_{\overline{x}}$ *is a finite set of pairs* $\{(S_1, B_1), \dots, (S_s, B_s)\}$ *(Segments:* S_i*, Bases:* B_i*) such that*

1. *The* S_i*'s are constructible subsets of* $\overline{K}^m$ *such that* $\overline{K}^m = \cup S_i$.
2. *The* B_i*'s are finite subsets of* $K(\overline{a})[\overline{x}]$ *and* $B_i(a) = \{p(a, \overline{x}) : p \in B_i\}$ *is a Gröbner basis of* I_a *with respect to* $\succ_{\overline{x}}$ *for every* $a \in S_i$.

Faithful: $B_i \subset I$. *Leads to a Comprehensive Gröbner Basis*

Non-faithful: B_i *reduced.*

Historical development

Two directions:

- Speed up. Duval (1995), Kapur (1995), M. Moreno-Maza (1997), Kalkbrenner (1997), Dellière (1999), Sato (2003), Suzuki & Sato (2006), Nabeshima (2006), Deepak Kapur & Yao Sun & Dingkang Wang (2010).
- Improve output. Montes (2002), Weispfenning (2003), Wibmer (2007), Manubens & Montes (2009), Montes & Wibmer (2010).

Our goal:

- best output for applications,
- disjoint segments,
- segments with constant lpp,
- minimal number of segments,
- canonical output,
- locally closed segments.

Canonical Gröbner cover for homogeneous ideals

Theorem (Wibmer)

Given a parametric ideal $I \subset K[\overline{a}][\overline{x}]$ homogeneous in the variables $\overline{x} = x_1, \ldots, x_n$ and a monomial order $\succ_{\overline{x}}$, there exists a unique canonical Gröbner cover with the following properties:

It consists of a set of triplets $\{(S_1, B_1, \mathrm{lpp}_1), \ldots, (S_r, B_r, \mathrm{lpp}_r)\}$ such that:

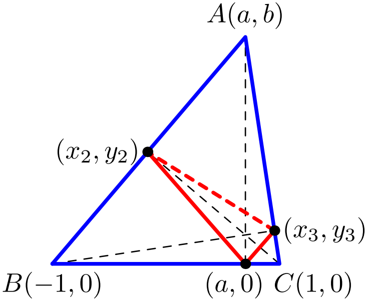
- the S_i are locally closed, disjoint segments, (that can be given in canonical prime-representation (P-representation)),
- the B_i are a set of monic I-regular functions having constant lpp on S_i, such that for every point $a \in S_i$ determine the reduced Gröbner basis of I_a and are called the reduced Gröbner basis of I over S_i. (They can be provided in full representation or optionally in generic representation).
- the lpp characterize the segments, as different segments have different lpp's.

Canonical Gröbner cover for non-homogeneous ideals

- Homogenizing the ideal, then computing the canonical Gröbner cover and finally dehomogenizing and reducing the bases, produces the canonical Gröbner cover of the non-homogeneous ideal I.
- It has the same properties as for homogeneous ideals except the third one, as now several segments can have the same lpp.

Index

S53. Conditions for isosceles orthic triangle



Fix $B=(-1,0)$, $C=(1,0)$ and let $A=(a,b)$ be a parametric point. Construct the orthic triangle (i.e. the triangle trough the feets of the heights).

The question is:

1. for which points A the orthic triangle is isosceles at A'?

S53. Conditions for isosceles orthic triangle

1. The construction corresponds to the following equations:

$$\begin{aligned} G = \; & (a-1)y_2 - b(x_2-1), \\ & (a-1)(x_2+1) + by_2, \\ & (a+1)y_3 - b(x_3+1), \\ & (a+1)(x_3-1) + by_3. \end{aligned}$$

2. Add the condition for equal length of both sides $H_1 = (x_2-a)^2 + y_2^2 - (x_3-a)^2 - y_3^2$.
3. Compute the Gröbner cover:

S53. Automatic theorems discovering: Isosceles orthic triangle

```
> ring R=(0,a,b),(x2,x3,y2,y3),dp;
> ideal S53=(-b)*x2+(a-1)*y2+(b),
            (a-1)*x2+(b)*y2+(a-1),
            (b)*x3+(-a-1)*y3+(b),
            (a+1)*x3+(b)*y3+(-a-1),
           -x2^2+x3^2-y2^2+y3^2+(2*a)*x2+(-2*a)*x3;
> grobcov(S53);
```

```
[1]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(a^2-b^2-1)
            [2]:
               _[1]=(a^2+b^2-1)
            [3]:
               _[1]=(a)
```

```
[2]:
   [1]:
      _[1]=y3
      _[2]=y2
      _[3]=x3
      _[4]=x2
   [2]:
      _[1]=(a^2+2*a+b^2+1)*y3+(-2*a*b-2*b)
      _[2]=(a^2-2*a+b^2+1)*y2+(2*a*b-2*b)
      _[3]=(a^2+2*a+b^2+1)*x3+(-a^2-2*a+b^2-1)
      _[4]=(a^2-2*a+b^2+1)*x2+(a^2-2*a-b^2+1)
   [3]:
      [1]:
         [1]:
            _[1]=(a^2-b^2-1)
         [2]:
            [1]:
               _[1]=(b)
               _[2]=(a-1)
            [2]:
               _[1]=(b)
               _[2]=(a+1)
            [3]:
```

```
          _[1]=(b^2+1)
          _[2]=(a)
[2]:
   [1]:
      _[1]=(a^2+b^2-1)
   [2]:
      [1]:
         _[1]=(b)
         _[2]=(a-1)
      [2]:
         _[1]=(b)
         _[2]=(a+1)
[3]:
   [1]:
      _[1]=(a)
   [2]:
      [1]:
         _[1]=(b^2+1)
         _[2]=(a)
```

```
[3]:
   [1]:
      _[1]=y3
      _[2]=x3
      _[3]=x2^2
   [2]:
      _[1]=y3
      _[2]=x3-1
      _[3]=x2^2+y2^2-2*x2+1
   [3]:
      [1]:
         [1]:
            _[1]=(b)
            _[2]=(a-1)
         [2]:
            [1]:
               _[1]=1
```

```
[4]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=(b^2+1)
            _[2]=(a)
         [2]:
            [1]:
               _[1]=1
```

```
[5]:
   [1]:
      _[1]=y2
      _[2]=x2
      _[3]=x3^2
   [2]:
      _[1]=y2
      _[2]=x2+1
      _[3]=x3^2+y3^2+2*x3+1
   [3]:
      [1]:
         [1]:
            _[1]=(b)
            _[2]=(a+1)
         [2]:
            [1]:
               _[1]=1
```

S53. Orthic triangle is isosceles

The generic segment with $\mathrm{lpp} = \{1\}$ is:

$$S_1 = \mathbb{C}^2 \setminus \left(\mathbb{V}(a) \cup \mathbb{V}(a^2+b^2-1) \cup \mathbb{V}(a^2-b^2-1)\right)$$
$$B_1 = \{1\}$$

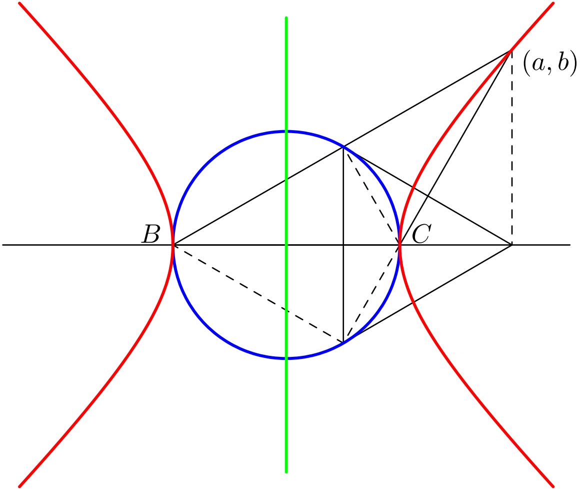
The segment with $\mathrm{lpp} = \{x_2, y_2, x_3, y_3\}$ is:

$$\begin{aligned} S_2 = &\ (\mathbb{V}(a) \setminus \mathbb{V}(b^2+1, a)) \cup \\ &(\mathbb{V}(a^2+b^2-1) \setminus (\mathbb{V}(b, a-1)) \cup (\mathbb{V}(b, a+1))) \cup \\ &(\mathbb{V}(a^2-b^2-1) \setminus (\mathbb{V}(b, a-1)) \cup (\mathbb{V}(b, a+1)) \cup (\mathbb{V}(b^2+1, a))). \end{aligned}$$

$$\begin{aligned} B_2 = &\ (a^2+2a+b^2+1)y_3 + (-2ab-2b), \\ &(a^2-2a+b^2+1)y_2 + (2ab-2b), \\ &(a^2+2a+b^2+1)x_3 + (-a^2-2a+b^2-1), \\ &(a^2-2a+b^2+1)x_2 + (a^2-2a-b^2+1) \end{aligned}$$

The Gröbner cover has 3 other segments corresponding to the points $B(-1,0)$, $C(1,0)$, and the pair of complex points $\mathbb{V}(b^2+1, a)$.

Locus of points for isosceles orthic triangle



S92. Casas Alberó conjecture

Conjecture

If a polynomial of degree n in x has a common root which each of its $n-1$ derivatives (not assumed to be the same), then it is of the form $P(x) = k(x+a)^n$, i.e. the common roots must be all the same.

Let

$$f(x) = x^n + \sum_{i=0}^{n-1} \binom{n}{i} a_i x^i.$$

We have

$$F_n(x,j) = \frac{j!}{n!} f^{(j)}(x) = x^{n-j} + \sum_{i=0}^{n-j-1} \binom{n-j}{i} a_{i+j} x^i$$

The system of the hypothesis becomes

$$\{F_n(x_1,0), F_n(x_1,1), \ldots, F_n(x_n,0), F_n(x_n,n-1)\}$$

S92. Casas Alberó Conjecture

```
> ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
> proc Fn(poly x,int n,int j)
  {
    int i; poly f=x^n;
    for(i=0;i<=n-1;i++)
    {
      f=f+binomial(n,i)*par(i+1+j)*x^i;
    }
    return(f);
  }
> int n=5;  ideal F;
> for (i=1;i<=n-1;i++)
  {
    F[size(F)+1]=Fn(var(i),n,0);
    F[size(F)+1]=Fn(var(i),n-i,i);
  }
```

```
> F;
F[1]=x1^5+(5*a4)*x1^4+(10*a3)*x1^3+(10*a2)*x1^2+(5*a1)*x1+(a0)
F[2]=x1^4+(4*a4)*x1^3+(6*a3)*x1^2+(4*a2)*x1+(a1)
F[3]=x2^5+(5*a4)*x2^4+(10*a3)*x2^3+(10*a2)*x2^2+(5*a1)*x2+(a0)
F[4]=x2^3+(3*a4)*x2^2+(3*a3)*x2+(a2)
F[5]=x3^5+(5*a4)*x3^4+(10*a3)*x3^3+(10*a2)*x3^2+(5*a1)*x3+(a0)
F[6]=x3^2+(2*a4)*x3+(a3)
F[7]=x4^5+(5*a4)*x4^4+(10*a3)*x4^3+(10*a2)*x4^2+(5*a1)*x4+(a0)
F[8]=x4+(a4)
> multigrobcov(F);
```

```
[1]:
   [1]:
      [1]:
         _[1]=1
      [2]:
         _[1]=1
      [3]:
         [1]:
            [1]:
               _[1]=0
            [2]:
               [1]:
                  _[1]=(a3-a4^2)
                  _[2]=(a2-a4^3)
                  _[3]=(a1-a4^4)
                  _[4]=(a0-a4^5)
```

```
[2]:
   [1]:
      [1]:
         _[1]=x4
         _[2]=x3^2
         _[3]=x2^3
         _[4]=x1^4
      [2]:
         _[1]=x4+(a4)
         _[2]=x3^2+(2*a4)*x3+(a4^2)
         _[3]=x2^3+(3*a4)*x2^2+(3*a4^2)*x2+(a4^3)
         _[4]=x1^4+(4*a4)*x1^3+(6*a4^2)*x1^2+(4*a4^3)*x1+(a4^4)
      [3]:
         [1]:
            [1]:
               _[1]=(a3-a4^2)
               _[2]=(a2-a4^3)
               _[3]=(a1-a4^4)
               _[4]=(a0-a4^5)
            [2]:
               [1]:
                  _[1]=1
```

S92. Casas Alberó conjecture

If we can solve the system for every n we are done.
But for concrete values of n we can compute the Gröbner cover.
For $n = 5$ we obtain two segments:

Segment	Basis
$\mathbb{C}^5 \setminus \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{1\}$
$\mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{x_4 + a_4, (x_3 + a_4)^2, (x_2 + a_4)^3, (x_1 + a_4)^4\}$

Thus the polynomial is $F_5(x, 0) = (x + a_4)^5$.

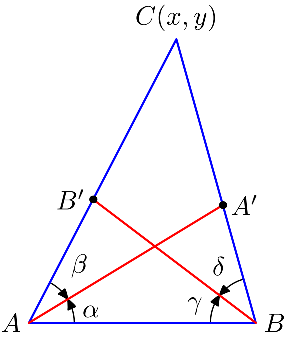
And the conjecture for the Gröbner cover for n becomes:

Segment	Basis
$\mathbb{C}^n \setminus \mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{1\}$
$\mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{x_{n-1} + a_{n-1}, \ldots, (x_1 + a_{n-1})^{n-1}\}$

Thus the polynomial is $F_n(x, 0) = (x + a_{n-1})^n$.

Classical Steiner-Lehmus theorem (circa 1840)

- If a triangle ABC has two (internal) angle-bisector with the same length, i.e. $|AA'| = |BB'|$, $\alpha = \beta$, $\gamma = \delta$,



then the triangle must be isosceles with $|AC| = |BC|$.

- The converse is, obviously also true.

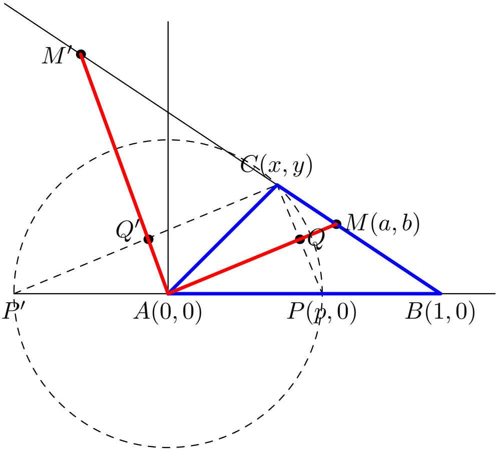
Classical Steiner-Lehmus theorem (circa 1840)

- The theorem was first mentioned in 1840 in a letter by C. L. Lehmus to C. Sturm. Jakob Steiner was among the first to provide a solution.
- The theorem became a rather popular topic in elementary geometry ever since, because of the difficulty to obtain a direct proof (P. Baptist, J. Conway, O. Bottema, V. Thebault.)

See references at:

http://www.mathematik.uni-bielefeld.de/∼sillke/PUZZLES/steiner-lehmus

Recently, its generalization, regarding internal as well as external angle bisectors, has been approached through automatic tools.



Bisectors for internal and external angles at vertex A are constructed intersecting circle of center at A and radius $|AC|$ with side AB (and its prolongation) at $P(p,0)$, then placing lines through A and the midpoints Q of C and P. The two bisector lines intersect the opposite side CB at $M(a,b)$.

The segments from A to $M(a,b)$ are the two bisectors at A.

- A booklet on the topic, by the “father" of the algebraic geometry approach to geometry theorem proving: W.-t.Wu, X.-l. Lü: “Triangles with equal bisectors". *People’s Education Press*, Beijing, (1985) [in Chinese]
- Recent contributions: D. Wang (2004), F. Botana (2007)
- Already difficult proof for the standard statement.
- Impossibility to deal (in Wang’s approach) with the case of three bisectors, because it involves two thesis: bisectors at A = bisectors at B = bisectors at C

- We fix $A(0,0)$, $B(1,0)$.

  $C(x,y)$ is free.

  We look for the locus of C such that some pair of bisectors (at A and B, at A and C, at B and C) have equal length.

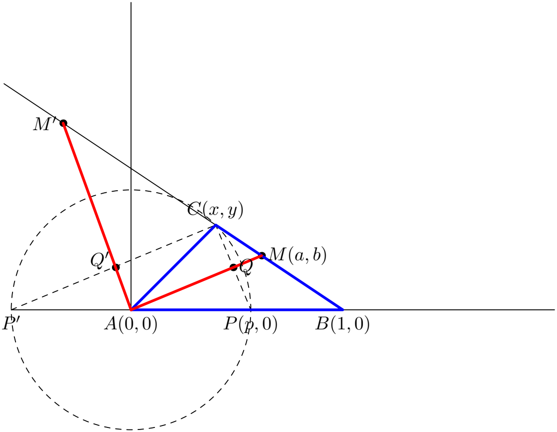
- Method: G. Dalzotto, T. Recio: On protocols for the automated discovery of theorems in elementary geometry. *J. Automated Reasoning*, (2009) no. 43, pp. 203-236.
- R. Losada, T. Recio, J. L. Valcarce, On the automatic discovery of Steiner-Lehmus generalizations. *Proceedings ADG* 2010, (J. Richter-Gebert, P. Schreck, editors), pp.171-174, Munich, 2010.

The Canonical Gröbner cover (A.Montes, M.Wibmer)

Given a parametric ideal $I \subset \mathbb{Q}[a_1, \ldots, a_m][x_1, \ldots, x_n]$ there exists a unique "*Canonical Gröbner Cover"* consisting in a set of triplets $\{(S_1, B_1, \text{lpp}_1), \ldots, (S_r, B_r, \text{lpp}_r)\}$ with the following properties:

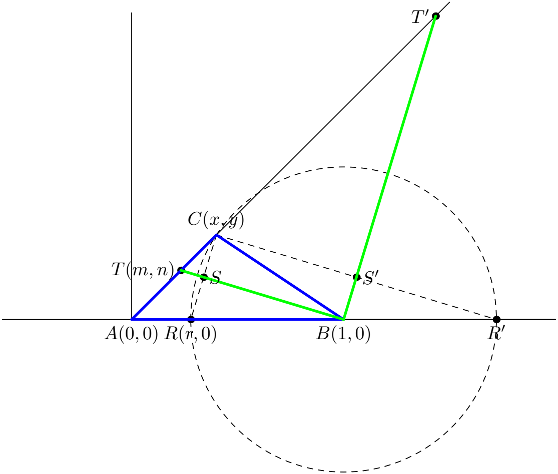
- the S_i are locally closed, disjoint subsets of $\mathbb{C}^m$ (called segments),
- the B_i are a set of monic I-regular functions having constant leading power products (lpp) on S_i, such that for every point $a \in S_i$ determine the reduced Gröbner basis of I_a and are called the reduced Gröbner basis of I over S_i.
- for homogeneous ideals, the lpp characterize the segments, as different segments have different lpp's.
- for non-homogeneous ideals it can happen that more than one segment corresponds to the same lpp, but even though, the split is canonical and corresponds to solutions that are different at infinity.

The equations for the bisectors at A



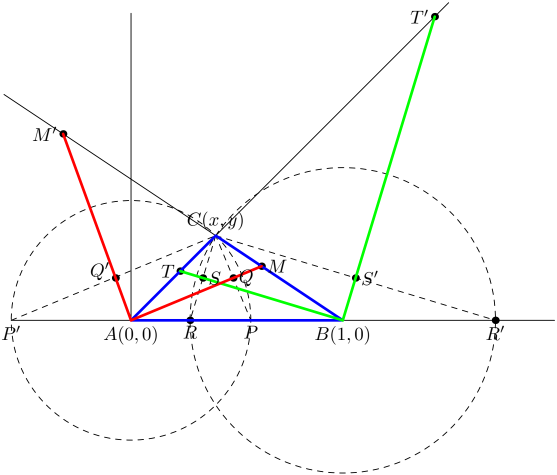
$$x^2+y^2=p^2, \begin{vmatrix} 0 & 0 & 1 \\ (x+p)/2 & y/2 & 1 \\ a & b & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0,$$

The equations for the bisectors at B



$$(1-x)^2+y^2=(1-r)^2, \begin{vmatrix} 1 & 0 & 1 \\ (x+r)/2 & y/2 & 1 \\ m & n & 1 \end{vmatrix} = 0, \begin{vmatrix} 0 & 0 & 1 \\ m & n & 1 \\ x & y & 1 \end{vmatrix} = 0,$$

One bisector of A is equal to one bisector of B



$$a^2 + b^2 = (1 - m)^2 + n^2$$

All the equations:

$$\begin{cases} x^2+y^2-p^2, \\ (a-1)y+b(1-x), \\ -ay+b(x+p), \\ (1-x)^2+y^2-(1-r)^2, \\ my-xn, \\ (1-m)y+(x+r-2)n, \\ a^2+b^2=(1-m)^2+n^2. \end{cases}$$

Parameters: x, y

Variables: a, b, m, n, p, r

Solutions:

	$+$	$-$
p	i_A	e_A
$1-r$	i_B	e_B

All the equations:

$$\begin{cases} x^2+y^2-p^2, \\ (a-1)y+b(1-x), \\ -ay+b(x+p), \\ (1-x)^2+y^2-(1-r)^2, \\ my-xn, \\ (1-m)y+(x+r-2)n, \\ a^2+b^2=(1-m)^2+n^2. \end{cases}$$

Parameters: x, y

Variables: a, b, m, n, p, r

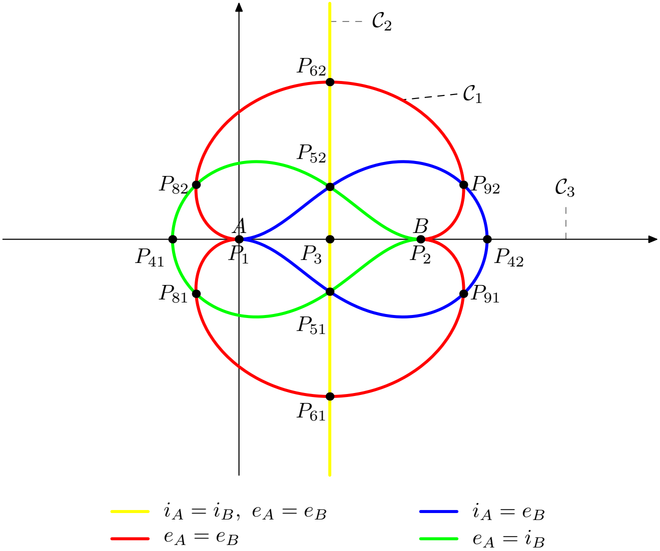
Solutions:

	$+$	$-$
p	i_A	e_A
$1-r$	i_B	e_B

Using the Gröbner cover

- We restrict to non-degenerate triangles
- Select only real values of the parameters.
- We use $\mathrm{grevlex}(a, b, m, n, p, r)$ order for the variables in the call to the Gröbner cover algorithm. The parameters are (x, y).
- The result has the following geometrical interpretation.

The Gröbner cover of the Steiner-Lehmus system



The Gröbner cover of the Steiner-Lehmus system

- Algebraic description: The following curves appear:

$$
\begin{aligned}
\mathcal{C}_1 &= \mathbb{V}(8x^{10} + 41x^8y^2 + 84x^6y^4 + 86x^4y^6 + 44x^2y^8 + 9y^{10} - 40x^9 \\
&\quad -164x^7y^2 - 252x^5y^4 - 172x^3y^6 - 44xy^8 + 76x^8 + 246x^6y^2 \\
&\quad +278x^4y^4 + 122x^2y^6 + 14y^8 - 64x^7 - 164x^5y^2 - 136x^3y^4 \\
&\quad -36xy^6 + 16x^6 + 31x^4y^2 + 14x^2y^4 - y^6 + 8x^5 + 20x^3y^2 + 12xy^4 \\
&\quad -4x^4 - 10x^2y^2 - 6y^4 + y^2), \\
\mathcal{C}_2 &= \mathbb{V}(2x - 1). \\
\mathcal{C}_3 &= \mathbb{V}(y),
\end{aligned}
$$

The Gröbner cover of the Steiner-Lehmus system

- as well as the following varieties:

Varieties	Real points
$V_1 = \mathbb{V}(y, x)$	$P_1 = (0, 0)$
$V_2 = \mathbb{V}(y, x-1)$	$P_2 = (1, 0)$
$V_3 = \mathbb{V}(y, 2x-1)$	$P_3 = (\frac{1}{2}, 0)$
$V_4 = \mathbb{V}(y, 2x^2 - 2x - 1)$	$P_{4,12} = \left(\frac{1 \pm \sqrt{3}}{2}, 0\right)$
$V_5 = \mathbb{V}(12y^2 - 1, 2x - 1)$	$P_{5,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right)$
$V_6 = \mathbb{V}(4y^2 - 3, 2x - 1)$	$P_{6,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
$V_7 = \mathbb{V}(4y^4 + 5y^2 + 2, 2x - 1)$	
$V_8 = \mathbb{V}(y^4 + 11y^2 - 1, 5x + 2y^2 + 1)$	$P_{8,12} = \left(2 - \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$
$V_9 = \mathbb{V}(y^4 + 11y^2 - 1, 5x - 2y^2 - 6)$	$P_{9,12} = \left(-1 + \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$

The Gröbner cover of the Steiner-Lehmus system

1. Segment with $\mathrm{lpp} = \{1\}$ Generic segment
Segment: $\mathbb{C}^2 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$
Description: The whole parameter space except the curves $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$.
Basis: $B_1 = \{1\}$

2. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$
Segment: $\mathcal{C}_2 \setminus (V_3 \cup V_5 \cup V_6))$
Description: Line $\mathcal{C}_2$ minus the intersecting points with $\mathcal{C}_1$ and $\mathcal{C}_2$
Basis: $B_2 =$

$$\left\{p + r - 1, (4y^2 - 3)n + (4y)r, (4y^2 - 3)m + 2r, (4y^2 - 3)b + (4y)r, (4y^2 - 3)a - 2r + (-4y^2 + 3), 4r^2 - 8r + (-4y^2 + 3)\right\}.$$

The Gröbner cover of the Steiner-Lehmus system

3. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: $\mathcal{C}_1 \setminus (V_1 \cup V_2 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8 \cup V_9)$
Description: The curve $\mathcal{C}_1$ except the real points P_1, P_2, P_{41}, P_{42}, P_{51}, P_{52}, P_{61}, P_{62}, $P_{81}, P_{82}, P_{91}, P_{92}$ and some other complex points on it.
Basis: $B_3 =$

$$\begin{aligned}
&\{(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 + 3y^4 + 5y^2 - 1)r \\
&+(x^5 - 10x^4 + 2x^3y^2 + 17x^3 - 18x^2y^2 - 10x^2 + xy^4 + 17xy^2 - x - 8y^4 - 10y^2 + 2), \\
&(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 - 4x + 3y^4 + 5y^2 + 1)p \\
&+(x^5 + 2x^4 + 2x^3y^2 - 7x^3 + 6x^2y^2 + 4x^2 + xy^4 - 7xy^2 - x + 4y^4 + 4y^2), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)n \\
&+(-3x^4y + 6x^3y - 6x^2y^3 - 5x^2y + 6xy^3 - 3y^5 - 5y^3 + y), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)m \\
&+(-3x^5 + 6x^4 - 6x^3y^2 - 5x^3 + 6x^2y^2 - 3xy^4 - 5xy^2 + x), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)b \\
&+(3x^4y - 6x^3y + 6x^2y^3 + 5x^2y - 6xy^3 - 4xy + 3y^5 + 5y^3 + y), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)a \\
&+(2x^5 - 8x^4 + 4x^3y^2 + 12x^3 - 12x^2y^2 - 8x^2 + 2xy^4 + 12xy^2 + 2x - 4y^4 - 4y^2)\}
\end{aligned}$$

The Gröbner cover of the Steiner-Lehmus system

4. Segment with $\mathrm{lpp} = \{n, m, b, a, r^2, p^2\}$
Segment: V_5
Description: Points P_{51}, P_{52}
Basis: $B_4 =$

$$\{2n - 3yr, 4m - 3r, 2b + 3yp - 3y, 4a - 3p - 1, 3r^2 - 6r + 2, 3p^2 - 1\}$$

5. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: V_6
Description: Points P_{61}, P_{62}
Basis: $B_5 = \{r, p - 1, 2n - y, 4m - 1, 2b - y, 4a - 3\}$

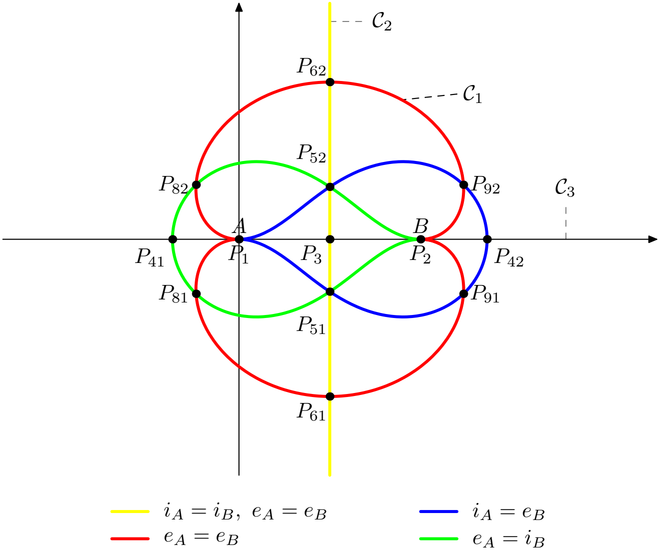
6. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$
Segment: $V_8(\cup V_7)$
Description: 2 real points P_{81}, P_{82} (and other complex points)
Basis:

$$B_6 = \{(7284y^6 + 88197y^4 - 15633y^2 - 3849)p + (8820y^6 + 97285y^4 \\ -5905y^2 - 265)r + (-11380y^6 - 103045y^4 + 1425y^2 - 1015), \\ (116y^6 + 1493y^4 + 2403y^2 + 179)n + (660y)r, \\ (116y^6 + 1493y^4 + 2403y^2 + 179)m + (-72y^6 - 866y^4 - 1006y^2 - 58)r, \\ (87932y^6 + 779351y^4 + 109221y^2 - 31747)b + (-35280y^7 - 389140y^5 \\ +23620y^3 + 1060y)r + (16384y^7 + 59392y^5 + 56832y^3 + 19456y), \\ (87932y^6 + 779351y^4 + 109221y^2 - 31747)a + (17640y^6 + 194570y^4 \\ -11810y^2 - 530)r + (-51068y^6 - 786519y^4 - 157349y^2 + 5123), \\ 660r^2 - 1320r + (-116y^6 - 1493y^4 - 2403y^2 - 179)\}.$$

7. Segment with $\mathrm{lpp} = \{r, n, m, b, a, p^2\}$
Segment: V_9
Description: Points P_{91}, P_{92}
Basis:

$$B_7 = \left\{(23y^2 - 1)r + (-83y^2 + 6), (134y^2 - 13)n + (83y^3 - 6y),\right.$$
$$(134y^2 - 13)m + (-268y^2 + 26),$$
$$(y^2 + 3)b + (-5y)p + (5y), (y^2 + 3)a + (-2y^2 - 1)p + (y^2 - 2),$$
$$\left. 5p^2 + (-y^2 - 8)\right\}.$$

The Gröbner cover of the Steiner-Lehmus system



Solutions at the special points

Point	$(p, 1-r)$	Bisectors
P_{51}, P_{52}	$(0.5773502693, 0.5773502693)$	$i_A = i_B$
	$(0.5773502693, -0.577350269)$	$i_A = e_B$
	$(-0.5773502693, 0.5773502693)$	$e_A = i_B,$
	$(-0.5773502693, -0.5773502693)$	$e_A = e_B$
P_{61}, P_{62}	(1,1)	$i_A = i_B$
P_{81}, P_{82}	$(-0.3819659526, -1.272019650)$	$e_A = e_B$
	$(-0.3819659526, 1.272019650)$	$e_A = i_B$
P_{91}, P_{92}	$(-1.272019650, -0.381965976)$	$e_A = e_B$
	$(1.272019650, -0.381965976)$	$i_A = e_B$

Table: Coincidences of bisectors of A and B at the special points

The colors of the curve

Point	Branch	$(p, 1-r)$	Bisectors
$(0, .7013671986)$	P_{62}-P_{82}	$(-.7013, -1.2214)$	$e_A = e_B$
$(0, .4190287818)$	P_{52}-P_{82}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.4190287818)$	P_{51}-P_{81}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.7013671986)$	P_{61}-P_{81}	$(-.7013, -1.2214)$	$e_A = e_B$
$(1, .7013671986)$	P_{62}-P_{92}	$(-1.2215, -0.7013)$	$e_A = e_B$
$(1, .4190287818)$	P_{52}-P_{92}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.4190287818)$	P_{51}-P_{91}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.7013671986)$	P_{61}-P_{91}	$(-1.2215, -0.7013)$	$e_A = e_B$

Table: Coincidences of bisectors of A and B at some points of curve $\mathcal{C}_1$.

Generalized Steiner-Lehmus Theorem

Theorem (Generalized Steiner-Lehmus)

Let ABC be a triangle and i_A, e_A, i_B, e_B the lengths of the inner and outer bisectors of the angles A and B. Then, considering the conditions for the equality of some bisector of A and some bisector of B the following excluding situations occur:

- *the triangle ABC is degenerate (i.e. C is aligned with A and B);*
- *ABC is equilateral and then $i_A = i_B$ whereas e_A and e_B become infinite, (P_{61}, P_{62});*
- *point C is in the center of an equilateral triangle, and then $i_A = i_B = e_A = e_B$, (P_{51}, P_{52});*
- *the triangle is isosceles but not of the special form of cases 2) or 3) and then $i_A = i_B \neq e_A = e_B$, (ordinary Theorem);*

continues in the next slide ..

Generalized Steiner-Lehmus Theorem

Theorem (continues)

- $\frac{\overline{AC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, $\frac{\overline{BC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, *and then* $e_A = e_B = i_B$, (P_{81}, P_{82});
- $\frac{\overline{AC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, $\frac{\overline{BC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, *and then* $e_A = e_B = i_A$, (P_{91}, P_{92});
- *C lies in the curve of degree 10 relative to points* A *and* B *(case of curve* $\mathcal{C}_1$*) passing through all the special points above but is none of these points, and then only one of the following things arrive: either* $e_A = e_B$ *or* $i_A = e_B$ *or* $e_A = i_B$ *depending on the branch of the curve (see Figure, the color representing which of the situations occur);*
- *none of the above cases occur, and then no bisector of* A *is equal to no bisector of* B.

S10. Inverse kinematic problem of a simple robot

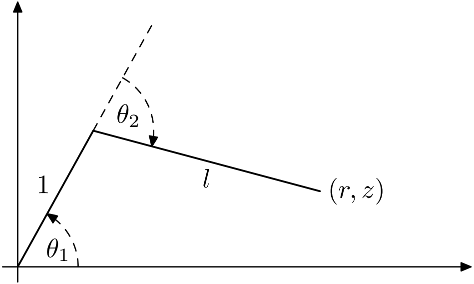


Figure: Simple two arms robot.

$$\begin{cases} c_1 = \cos(\theta_1); \\ s_1 = \sin(\theta_1); \\ c_2 = \cos(\theta_2); \\ s_2 = \sin(\theta_2); \end{cases} \qquad \begin{cases} r = c_1 + l(c_1c_2 - s_1s_2), \\ z = s_1 + l(s_1c_2 + c_1s_2) \\ c_1^2 + s_1^2 - 1, \\ c_2^2 + s_2^2 - 1, \end{cases}$$

Problem S10. Inverse kinematic problem of a simple robot

```
> LIB "grobcov.lib";
> ring R=(0,r,z),(s1,c1,s2,c2,l),lp;
> ideal S10=r-c1-l*c1*c2+l*s1*s2,
            z-s1-l*c1*s2-l*s1*c2,
            c1^2+s1^2-1,
            c2^2+s2^2-1;
> grobcov(S10);
```

```
[1]:
   [1]:
      _[1]=c2*l
      _[2]=s2^2
      _[3]=c1
      _[4]=s1
   [2]:
      _[1]=2*c2*l+l^2+(-r^2-z^2+1)
      _[2]=s2^2+c2^2-1
      _[3]=(2*r^2+2*z^2)*c1+(-2*z)*s2*l+(r)*l^2+(-r^3-r*z^2-r)
      _[4]=(2*r^2+2*z^2)*s1+(2*r)*s2*l+(z)*l^2+(-r^2*z-z^3-z)
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(r^2+z^2)
```

```
[2]:
   [1]:
      _[1]=c2*l
      _[2]=s2
      _[3]=c1*l^2
      _[4]=c1*c2
      _[5]=s1
   [2]:
      _[1]=2*c2*l+l^2+1
      _[2]=(z)*s2+(-r)*c2+(-r)*l
      _[3]=(4*r)*c1*l^2+(-4*r)*c1+l^4-2*l^2+(-4*z^2+1)
      _[4]=(8*r)*c1*c2+(8*r)*c1*l+(8*z^2-2)*c2+l^3+(4*z^2-3)*l
      _[5]=(2*z)*s1+(2*r)*c1+l^2-1
   [3]:
      [1]:
         [1]:
            _[1]=(r^2+z^2)
         [2]:
            [1]:
               _[1]=(z)
               _[2]=(r)
```

```
[3]:
   [1]:
      _[1]=l^2
      _[2]=c2
      _[3]=s2
      _[4]=s1^2
   [2]:
      _[1]=l^2-1
      _[2]=c2+l
      _[3]=s2
      _[4]=s1^2+c1^2-1
   [3]:
      [1]:
         [1]:
            _[1]=(z)
            _[2]=(r)
         [2]:
            [1]:
               _[1]=1
```

1. Generic segment:

lpp:	$c_2 l, s_2^2, c_1, s_1$.
Segment:	$\mathbb{C}^2 \setminus \mathbb{V}(r^2+z^2)$.
Basis:	$\left\{\begin{array}{l} 2c_2 l + l^2 + (-r^2 - z^2 + 1), \\ s_2^2 + c_2^2 - 1, \\ (2r^2+2z^2)c_1 + (-2z)s_2 l + (r)l^2 + (-r^3 - rz^2 - r), \\ (2r^2+2z^2)s_1 + (2r)s_2 l + (z)l^2 + (-r^2 z - z^3 - z). \end{array}\right.$

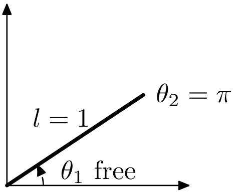
l is free. But in order to have real values with $-1 \leq c_2 \leq 1$, we must choose $|\sqrt{r^2+z^2} - 1| \leq l \leq \sqrt{r^2+z^2} + 1$, and then

2. Segment representing complex points:

lpp:	$c_2 l, s_2, c_1 l^2, c_1 c_2, s_1$.
Segment:	$\mathbb{V}(z^2+r^2) \setminus \mathbb{V}(z,r)$.
Basis:	$\begin{cases} 2c_2 l + l^2 + 1, \\ zs_2 - rc_2 - rl, \\ 4rc_1 l^2 - 4rc_1 + l^4 - 2l^2 + (-4z^2+1), \\ 8rc_1c_2 + 8rc_1 l + (8z^2-2)c_2 + l^3 + (4z^2-3)l, \\ 2zs_1 + 2rc_1 + l^2 - 1. \end{cases}$

3. Segment representing the origin:

lpp:	l^2, c_2, s_2, c_1^2.	
Segment:	$\mathbb{V}(z,r)$.	
Basis:	$\begin{cases} l^2 - 1, \\ c_2 + 1, \\ s_2, \\ s_1^2 + c_1^2 - 1. \end{cases}$	$l = 1$, $\theta_2 = \pi$, θ_1 free



S42. Need of sheaves

- Consider the following simple system:

$$\begin{cases} u_1x + u_2 = 0, \\ u_3x + u_4 = 0; \end{cases}$$

- Compute the Gröbner cover:

Problem S42: Need of sheaves.

```
> ring R=(0,u1,u2,u3,u4),(x),dp;
> short=0;
> ideal S42=u1*x+u2,
            u3*x+u4;
> grobcov(S42);
[1]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(u1*u4-u2*u3)
```

```
[2]:
   [1]:
      _[1]=x
   [2]:
      [1]:
         _[1]=(u3)*x+(u4)
         _[2]=(u1)*x+(u2)
   [3]:
      [1]:
         [1]:
            _[1]=(u1*u4-u2*u3)
         [2]:
            [1]:
               _[1]=(u3)
               _[2]=(u1)
```

```
[3]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=(u3)
            _[2]=(u1)
         [2]:
            [1]:
               _[1]=(u4)
               _[2]=(u3)
               _[3]=(u2)
               _[4]=(u1)
```

```
[4]:
   [1]:
      _[1]=0
   [2]:
      _[1]=0
   [3]:
      [1]:
         [1]:
            _[1]=(u4)
            _[2]=(u3)
            _[3]=(u2)
            _[4]=(u1)
         [2]:
            [1]:
               _[1]=1
```

S42. Need of sheaves

The result is:

lpp	Basis	Segment
1	1	$\mathbb{C}^4 \setminus \mathbb{V}(u_1u_4 - u_2u_3)$
x	$u_3x + u_4, u_1x + u_2$	$\mathbb{V}(u_1u_4 - u_2u_3) \setminus \mathbb{V}(u_3, u_1)$
1	1	$\mathbb{V}(u_3, u_1) \setminus \mathbb{V}(u_4, u_3, u_2, u_1)$
0	0	$\mathbb{V}(u_4, u_3, u_2, u_1)$

Index

Topological concepts on the parameter space

Definition

A subset $S \subset \overline{K}^m$ is locally closed, if it is difference of two varieties: $S = \mathbb{V}(M) \setminus \mathbb{V}(N)$.

Definition (Open subset)

A subset $U \subset S$ is said to be open on S if $\overline{S \setminus U} \subsetneq S$.

I-regular functions

Definition (I-Regular function)

Let S be a locally closed subset of $\overline{K}^m$. We call a function $f : S \longrightarrow \overline{K}[\overline{x}]$ I-regular, if $\forall a \in S$ it exists an open $U \subset S$ with $a \in U$ and

$$f(b) = \frac{P(b,\overline{x})}{Q(b)} \text{ for all } b \in U,$$

where $P \in I$ and $Q \in K[\overline{a}]$ and $Q(b) \neq 0$ for all $b \in U$.

Remark

Let P and Q be a polynomials as above, (they are not unique), $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$ and $p(b,\overline{x}) = P(b,\overline{x}) \mod \mathfrak{a}$. If f is monic and $\mathrm{lpp}(f)$ is constant on S, then, for all $b \in U$ is

- $\mathrm{lpp}_{\overline{x}}(p(b,\overline{x})) = \mathrm{lpp}_{\overline{x}}(f)$, *and*
- $\mathrm{lc}_{\overline{x}}(p(b,\overline{x})) = Q(b) \mod \mathfrak{a}$.

Parametric subsets

Definition (Parametric subset of $\overline{K}^m$)

A locally closed subset $S \in \overline{K}^m$ *is called parametric (wrt to* I *and* $\succ_{\overline{x}}$*) if there exist monic* I*-regular functions* $\{g_1, \dots, g_s\}$ *over* S *so that* $\{g_1(a, \overline{x}), \dots, g_s(a, \overline{x})\}$ *is the reduced Gröbner basis of* I_a *for all* $a \in S$*.*

Note

Note that the definition immediately implies that if a, b *lie in a parametric set* S*, then* $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$*.*
The amazing thing is that the converse also holds if we additionally assume that $I \subset K[\overline{a}][\overline{x}]$ *is homogeneous (wrt to the variables).*

Wibmer's Theorem

Theorem (M. Wibmer)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal and $a \in \overline{K}^m$. Then the set

$$S_a = \{b \in \overline{K}^m \ : \ \mathrm{lpp}_{\overline{x}}(I_b) = \mathrm{lpp}_{\overline{x}}(I_a)\}$$

is parametric.
In particular, S_a is locally closed.

Gröbner cover

Definition (Gröbner cover)

By a Gröbner cover of $\overline{K}^m$ wrt to I and $\succ_{\overline{x}}$ we mean a finite set of pairs $\{(S_1, B_1), \ldots, (S_r, B_r)\}$ such that

1. *the S_i's are parametric and so, $B_i \subset \mathcal{O}(S_i)[\overline{x}]$ is the reduced Gröbner basis of I over S_i for $i = 1, \ldots, r$, and*
2. *the union of all S_i's equals $\overline{K}^m$.*

Theorem (Canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.

Gröbner cover

Definition (Gröbner cover)

By a Gröbner cover of $\overline{K}^m$ wrt to I and $\succ_{\overline{x}}$ we mean a finite set of pairs $\{(S_1, B_1), \ldots, (S_r, B_r)\}$ such that

1. *the S_i's are parametric and so, $B_i \subset \mathcal{O}(S_i)[\overline{x}]$ is the reduced Gröbner basis of I over S_i for $i = 1, \ldots, r$, and*
2. *the union of all S_i's equals $\overline{K}^m$.*

Theorem (Canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.

Non-homogeneous ideals

Note (Homogenization and dehomogenization)

For homogenization introduce a new variable x_0 and extend $\succ_{\overline{x}}$ to the monomials in $\overline{x}, x_0$ by setting

$$\overline{x}^{\alpha} x_0^i \succ_{\overline{x},x_0} \overline{x}^{\beta} x_0^j \text{ iff } (\overline{x}^{\alpha} \succ_{\overline{x}} \overline{x}^{\beta}) \text{ or } (\overline{x}^{\alpha} = \overline{x}^{\beta} \text{ and } i > j)$$

Denote τ the dehomogenization consisting of substituting $x_0 = 1$.

Affine canonical Gröbner cover

Definition (Affine canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a non-homogeneous ideal and let $J \subset K[\overline{a}][\overline{x}, x_0]$ denote its homogenization. The disjoint Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$ obtained by dehomogenization and reduction will be called the canonical Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$.

Remark

The affine canonical Gröbner cover does not necessarily summarize in a unique segment all the points corresponding to the same lpp. Nevertheless it is canonical, and when two segments occur with the same lpp they correspond to different kind of solutions at infinity.

Index

Gröbner Cover algorithm

Algorithm (Homogeneous GröbnerCover)

GCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$

T :=**BuildTree**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$. *(Initial disjoint and reduced CGS)*

$G := \emptyset$

Group the segments of T *by* lpp*'s:* $T = \{T_i : 1 \leq i \leq s\}$.

where $T_i = \{(S_{i_j}, B_{i_j}) : 1 \leq j \leq s_i\}$ *with* $\mathrm{lpp}(B_{ij}) = \mathrm{lpp}(B_{ik})$

For *each* lpp*-segment* T_i

S_i :=**LCUnion**$(S_{i_j} : 1 \leq j \leq s_i)$. *(Summarizing* lpp*-segments)*

B_i :=**Basis**(S_i, T_i). *(Determining the generic basis for* S_i *using* T_i*.)*

$G := G \cup (S_i, B_i)$

end for

Return G

Gröbner Cover algorithm

Algorithm (Affine GröbnerCover)

GröbnerCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
If F *is homogeneous* **then** $G :=$ **GCover**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
else
$F' :=$ **Homogenize**(F, x_0), $\overline{y} := \overline{x}, x_0$, $\succ_{\overline{y}} = \succ_{\overline{x}, x_0}$
$G :=$ **GCover**$(F', \succ_{\overline{y}}, \succ_{\overline{a}})$
$\overline{y} := \overline{x}, 1$, *(***Dehomogenize** *the bases in* G*)*
Reduce *the bases in* G
end if
Extend *the bases in* G *(to obtain a full representation)*
Return G

Index

Representation of locally closed subsets

Proposition (Canonical representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist uniquely determined radical ideals $\mathfrak{a} \subset \mathfrak{b}$ of $K[\overline{a}]$, with $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$, such that

- $\overline{S} = \mathbb{V}(\mathfrak{a})$,
- $\overline{S} \setminus S = \mathbb{V}(\mathfrak{b})$.

The pair $(\mathfrak{a}, \mathfrak{b})$ -top,hole- is called the canonical representation of S.

Representation of locally closed sets

Proposition (Canonical prime representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist a unique canonical prime representation of S given the prime components of $\mathfrak{a}$, say $\mathfrak{p}_i$, and associated to each, a set of prime ideals $\mathfrak{p}_{ij}$ (holes) in the form $((\mathfrak{p}_1, (\mathfrak{p}_{11}, \dots, \mathfrak{p}_{1j_1})), \dots, (\mathfrak{p}_k, (\mathfrak{p}_{k1}, \dots, \mathfrak{p}_{kj_k})))$ so that

$$S = \bigcup_{i=1}^{k} \left(\mathbb{V}(\mathfrak{p}_i) \setminus (\bigcup_{j=1}^{j_i} \mathbb{V}(\mathfrak{p}_{ij})) \right).$$

and $\mathfrak{p}_i \subset \mathfrak{p}_{ij}$ for all i, j, such that

- $\overline{S} = \mathbb{V}(\mathfrak{p}_1) \cup \dots \cup \mathbb{V}(\mathfrak{p}_r)$ *and*
- $(\overline{S} \setminus S) \cap \mathbb{V}(\mathfrak{p}_i) = \mathbb{V}(\mathfrak{p}_{i1}) \cup \cdots \cup \mathbb{V}(\mathfrak{p}_{ir_i})$

are the minimal decompositions into irreducible closed sets.

Representation of I-regular functions

Definition (Generic representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \rightarrow \overline{K}[\overline{x}]$ a monic I-regular function. We say that $p \in K[\overline{a}][\overline{x}]$ generically represents f if

- $\mathrm{lpp}(f) = \mathrm{lpp}(p)$,
- $\mathrm{lc}(p)(a) \neq 0$ *on an open and dense set of points in* S,
- *if* $\mathrm{lc}(p)(a) \neq 0$ *then* $f(a, \overline{x}) = p(a, \overline{x}) / \mathrm{lc}(p)(a)$, *otherwise is* $p(a, \overline{x}) = 0$.

Proposition

Every monic I-regular function $f : S \rightarrow \overline{K}[\overline{x}]$ admits a generic representation.

Representation of I-regular functions

Definition (Full representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \to \overline{K}[\overline{x}]$ a monic I-regular function. We say that a the set of polynomials $\{p_1, \cdots, p_r\} \subset K[\overline{a}][\overline{x}]$ fully represents f if

- $\mathrm{lpp}(f) = \mathrm{lpp}(p_i)$, *for* $1 \leq i \leq r$,
- *for* $a \in S$ *and* $1 \leq i \leq r$ *either* $\mathrm{lc}(p_i)(a) \neq 0$ *or* $p_i(a, \overline{x}) = 0$,
- *for all* $a \in S$ *it exist al least one* i *and an open* $U \subset S$ *such that for every* $b \in U$ *is* $\mathrm{lc}(p_i)(a) \neq 0$ *and* $f(a, \overline{x}) = p(a, \overline{x}) / \mathrm{lc}(p)(a)$.

Proposition

Given a generic representation of a monic I-regular function $f : S \to \overline{K}[\overline{x}]$, the algorithm EXTEND *computes a full representation.*

Representation of I-regular functions

Example

Let $I = \langle ax + by, cx + dy \rangle$ and F be the monic I-regular function

$$
\begin{array}{rcl}
F: \quad S = \mathbb{V}(ad - bc) \setminus \mathbb{V}(a, c) \subset \mathbb{C}^4 & \to & \mathbb{C}[x, y] \\
(a, b, c, d) & \mapsto & \begin{cases} x + \frac{b}{a} y & \text{if } a \neq 0 \\ x + \frac{d}{c} y & \text{if } c \neq 0 \end{cases}
\end{array}
$$

Then

Generic representation of F: $p = ax + by$
Full representation of F: $\{p_1 = ax + by, p_2 = cx + dy\}$