Software for computing the Canonical Gröbner Cover of a parametric ideal

The Singular grobcov.lib library

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Introduction

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Gröbner Cover algorithm

Representations


Software download (beta version):
http://www-ma2.upc.edu/~montes/

Standard software version will be distributed with the next Singular release.
The problem

Goal

**Given**: Parametric polynomial system of equations

\[
\begin{align*}
    p_1(a_1, \ldots, a_m, x_1, \ldots, x_n) &= 0 \\
    \cdots \\
    p_r(a_1, \ldots, a_m, x_1, \ldots, x_n) &= 0
\end{align*}
\]

**Goal**: describe the different kind of solutions \((x_1, \ldots, x_n)\) in dependence of the parameters \(a_1, \ldots, a_m\).
Some notations

Let:
$K$ be a computable field (in practice $\mathbb{Q}$).
$\overline{K}$ be an algebraically closed extension of $K$ (in practice $\mathbb{C}$).

$K[\bar{a}]$ the polynomial ring in the parameters $\bar{a} = a_1, \ldots, a_m$ over $K$.
$K[\bar{a}][\bar{x}]$ the polynomial ring in the variables $\bar{x} = x_1, \ldots, x_n$ over $K[\bar{a}]$.
$\overline{K}^m$ is the parameter space.

Fix: $\succ_{\bar{x}}$ monomial ordering wrt $\bar{x}$ and the ideal
$I = \langle p_1(\bar{a}, \bar{x}), \cdots, p_r(\bar{a}, \bar{x}) \rangle \subset K[\bar{a}][\bar{x}]$
$lpp(G) = \text{set of leading power products wrt } \succ_{\bar{x}} \text{ of the polynomials in } G$.

Specialization:
$a = (a_0^0, \cdots, a_m^0) \in \overline{K}^m$
$I_a = \langle p_1(a, \bar{x}), \cdots, p_r(a, \bar{x}) \rangle \subset \overline{K}[\bar{x}]$
Gröbner bases are the computational method par excellence for studying polynomial systems.

The set of \text{lpp} of the reduced Gröbner basis determines the type of solutions of the system.

In the case of parametric polynomial systems the goal is to describe the reduced Gröbner basis of $I_a \subset \overline{K}[\bar{x}]$ (with respect to $\prec_\bar{x}$) in dependence of $a \in \overline{K}^m$. 
Weispfenning (1992)

Given $I = \langle p_1, \ldots, p_r \rangle \subset K[\bar{a}][\bar{x}] = K[\bar{a}, \bar{x}]$ and $\succ_{\bar{x}}$

A Comprehensive Gröbner System (CGS) for $I$ and $\succ_{\bar{x}}$ is a finite set of pairs \{$(S_1, B_1), \ldots, (S_s, B_s)$\} (Segments: $S_i$, Bases: $B_i$) such that

1. The $S_i$’s are constructible subsets of $\overline{K}^m$ such that $\overline{K}^m = \bigcup S_i$.
2. The $B_i$’s are finite subsets of $K(\bar{a})[\bar{x}]$ and $B_i(a) = \{p(a, \bar{x}) : p \in B_i\}$ is a Gröbner basis of $I_a$ with respect to $\succ_{\bar{x}}$ for every $a \in S_i$.

Faithful: $B_i \subset I$. Leads to a Comprehensive Gröbner Basis

Non-faithful: $B_i$ reduced.
Historical development

Two directions:


Our goal:

- best output for applications,
- disjoint segments,
- segments with constant lpp,
- minimal number of segments,
- canonical output,
- locally closed segments.
Theorem (Wibmer)

Given a parametric ideal \( I \subset K[\overline{a}][\overline{x}] \) homogeneous in the variables \( \overline{x} = x_1, \ldots, x_n \) and a monomial order \( \succ_{\overline{x}} \), there exists a unique canonical Gröbner cover with the following properties:

It consists of a set of triplets \( \{(S_1, B_1, \text{lpp}_1), \ldots, (S_r, B_r, \text{lpp}_r)\} \) such that:

- the \( S_i \) are locally closed, disjoint segments, (that can be given in canonical prime-representation (\( P \)-representation)),
- the \( B_i \) are a set of monic \( I \)-regular functions having constant \( \text{lpp} \) on \( S_i \), such that for every point \( a \in S_i \) determine the reduced Gröbner basis of \( I_a \) and are called the reduced Gröbner basis of \( I \) over \( S_i \). (They can be provided in full representation or optionally in generic representation).
- the \( \text{lpp} \) characterize the segments, as different segments have different \( \text{lpp} \)'s.
Homogenizing the ideal, then computing the canonical Gröbner cover and finally dehomogenizing and reducing the bases, produces the canonical Gröbner cover of the non-homogeneous ideal $I$.

It has the same properties as for homogeneous ideals except the third one, as now several segments can have the same $lpp$. 
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S53. Conditions for isosceles orthic triangle

Fix $B = (-1, 0)$, $C = (1, 0)$ and let $A = (a, b)$ be a parametric point. Construct the orthic triangle (i.e. the triangle through the feet of the heights).

The question is:

for which points $A$ the orthic triangle is isosceles at $A'$?
The construction corresponds to the following equations:

\[
G = (a - 1)y_2 - b(x_2 - 1),
(a - 1)(x_2 + 1) + by_2,
(a + 1)y_3 - b(x_3 + 1),
(a + 1)(x_3 - 1) + by_3.
\]

Add the condition for equal length of both sides

\[
H_1 = (x_2 - a)^2 + y_2^2 - (x_3 - a)^2 - y_3^2.
\]

Compute the Gröbner cover:
S53. Automatic theorems discovering: Isosceles orthic triangle

> ring R=(0,a,b),(x2,x3,y2,y3),dp;
> ideal S53=(-b)*x2+(a-1)*y2+(b),
  (a-1)*x2+(b)*y2+(a-1),
  (b)*x3+(-a-1)*y3+(b),
  (a+1)*x3+(b)*y3+(-a-1),
  -x2^2+x3^2-y2^2+y3^2+(2*a)*x2+(-2*a)*x3;
> grobcoov(S53);
\[ [1]:
\quad [1]:
\quad \quad _1[1]=1
\quad [2]:
\quad \quad _1[1]=1
\quad [3]:
\quad [1]:
\quad \quad [1]:
\quad \quad \quad _1[1]=0
\quad [2]:
\quad \quad [1]:
\quad \quad \quad _1[1]=(a^2-b^2-1)
\quad [2]:
\quad \quad \quad _1[1]=(a^2+b^2-1)
\quad [3]:
\quad \quad \quad _1[1]=(a) \]
\[ y_3 = a^2 + 2a + b^2 + 1 \]
\[ y_2 = a^2 - 2a + b^2 + 1 \]
\[ x_3 = a^2 + 2a + b^2 + 1 \]
\[ x_2 = a^2 - 2a + b^2 + 1 \]

\[ y_3 = (a^2 + 2a + b^2 + 1) \cdot y_3 + (-2ab - 2b) \]
\[ y_2 = (a^2 - 2a + b^2 + 1) \cdot y_2 + (2ab - 2b) \]
\[ x_3 = (a^2 + 2a + b^2 + 1) \cdot x_3 + (-a^2 - 2a + b^2 - 1) \]
\[ x_2 = (a^2 - 2a + b^2 + 1) \cdot x_2 + (a^2 - 2a - b^2 + 1) \]

\[ a^2 - b^2 - 1 \]
\[ b \]
\[ a - 1 \]
\[ b \]
\[ a + 1 \]
\[ _1 = (b^2 + 1) \]
\[ _2 = (a) \]

[2]:
  [1]:
    \[ _1 = (a^2 + b^2 - 1) \]
  [2]:
    [1]:
      \[ _1 = (b) \]
      \[ _2 = (a - 1) \]
    [2]:
      \[ _1 = (b) \]
      \[ _2 = (a + 1) \]

[3]:
  [1]:
    \[ _1 = (a) \]
  [2]:
    [1]:
      \[ _1 = (b^2 + 1) \]
      \[ _2 = (a) \]
[3]:
  [1]:
    _[1]=y3
    _[2]=x3
    _[3]=x2^2
  [2]:
    _[1]=y3
    _[2]=x3-1
    _[3]=x2^2+y2^2-2*x2+1
  [3]:
    [1]:
      [1]:
        _[1]=(b)
        _[2]=(a-1)
    [2]:
      [1]:
        _[1]=1
[4]:
[1]:
  _[1]=1
[2]:
  _[1]=1
[3]:
  [1]:
    [1]:
      _[1]=(b^2+1)
      _[2]=(a)
    [2]:
      [1]:
        _[1]=1
\[ 1: \]
\[ y_2 \]
\[ x_2 \]
\[ x_3^2 \]

\[ 2: \]
\[ y_2 \]
\[ x_2 + 1 \]
\[ x_3^2 + y_3^2 + 2x_3 + 1 \]

\[ 3: \]
\[ 1: \]
\[ 1: \]
\[ b \]
\[ a + 1 \]

\[ 2: \]
\[ 1: \]
\[ 1 = 1 \]
The **generic segment with lpp = \{1\}** is:

\[ S_1 = \mathbb{C}^2 \setminus (\forall(a) \cup \forall(a^2 + b^2 - 1) \cup \forall(a^2 - b^2 - 1)) \]

\[ B_1 = \{1\} \]

The segment with lpp = \{x_2, y_2, x_3, y_3\} is:

\[ S_2 = (\forall(a) \setminus \forall(b^2 + 1, a)) \cup (\forall(a^2 + b^2 - 1) \setminus (\forall(b, a - 1)) \cup (\forall(b, a + 1))) \cup (\forall(a^2 - b^2 - 1) \setminus (\forall(b, a - 1)) \cup (\forall(b, a + 1))) \cup (\forall(b^2 + 1, a))) \]

\[ B_2 = (a^2 + 2a + b^2 + 1)y_3 + (-2ab - 2b), \]
\[ (a^2 - 2a + b^2 + 1)y_2 + (2ab - 2b), \]
\[ (a^2 + 2a + b^2 + 1)x_3 + (-a^2 - 2a + b^2 - 1), \]
\[ (a^2 - 2a + b^2 + 1)x_2 + (a^2 - 2a - b^2 + 1) \]

The Gröbner cover has 3 other segments corresponding to the points \(B(-1, 0), C(1, 0)\), and the pair of complex points \(\forall(b^2 + 1, a)\).
Locus of points for isosceles orthic triangle
Conjecture

If a polynomial of degree \( n \) in \( x \) has a common root which each of its \( n - 1 \) derivatives (not assumed to be the same), then it is of the form \( P(x) = k(x + a)^n \), i.e. the common roots must be all the same.

Let

\[
f(x) = x^n + \sum_{i=0}^{n-1} \binom{n}{i} a_i x^i.
\]

We have

\[
F_n(x,j) = \frac{j!}{n!} f^{(j)}(x) = x^{n-j} + \sum_{i=0}^{n-j-1} \binom{n-j}{i} a_{i+j} x^i
\]

The system of the hypothesis becomes

\[
\{ F_n(x_1, 0), F_n(x_1, 1), \ldots, F_n(x_n, 0), F_n(x_n, n - 1) \}\]
S92. Casas Alberó Conjecture

> ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
> proc Fn(poly x,int n,int j)
{   int i; poly f=x^n;
    for(i=0;i<=n-1;i++)
    {
        f=f+binomial(n,i)*par(i+1+j)*x^i;
    }
    return(f);
}
> int n=5; ideal F;
> for (i=1;i<=n-1;i++)
{   F[size(F)+1]=Fn(var(i),n,0);
    F[size(F)+1]=Fn(var(i),n-i,i);
}
> F;
F[1]=x1^5+(5*a4)*x1^4+(10*a3)*x1^3+(10*a2)*x1^2+(5*a1)*x1+(a0)
F[2]=x1^4+(4*a4)*x1^3+(6*a3)*x1^2+(4*a2)*x1+(a1)
F[3]=x2^5+(5*a4)*x2^4+(10*a3)*x2^3+(10*a2)*x2^2+(5*a1)*x2+(a0)
F[4]=x2^3+(3*a4)*x2^2+(3*a3)*x2+(a2)
F[5]=x3^5+(5*a4)*x3^4+(10*a3)*x3^3+(10*a2)*x3^2+(5*a1)*x3+(a0)
F[6]=x3^2+(2*a4)*x3+(a3)
F[7]=x4^5+(5*a4)*x4^4+(10*a3)*x4^3+(10*a2)*x4^2+(5*a1)*x4+(a0)
F[8]=x4+(a4)
> multigrobcov(F);
\[ \begin{align*}
_1[1] &= 1 \\
_2[1] &= 1 \\
_3[1] &= 1 \\
_1[1] &= 0 \\
_2[1] &= (a_3 - a_4^2) \\
_3[1] &= (a_2 - a_4^3) \\
_4[1] &= (a_1 - a_4^4) \\
_4[2] &= (a_0 - a_4^5)
\end{align*} \]
\[2\]:
\[1\]:
\[1\]:
\[1\] = x_4
\[2\] = x_3^2
\[3\] = x_2^3
\[4\] = x_1^4
\[2\]:
\[1\] = x_4 + (a_4)
\[2\] = x_3^2 + (2 \cdot a_4) \cdot x_3 + (a_4^2)
\[3\] = x_2^3 + (3 \cdot a_4) \cdot x_2^2 + (3 \cdot a_4^2) \cdot x_2 + (a_4^3)
\[4\] = x_1^4 + (4 \cdot a_4) \cdot x_1^3 + (6 \cdot a_4^2) \cdot x_1^2 + (4 \cdot a_4^3) \cdot x_1 + (a_4^4)
\[3\]:
\[1\]:
\[1\]:
\[1\] = (a_3 - a_4^2)
\[2\] = (a_2 - a_4^3)
\[3\] = (a_1 - a_4^4)
\[4\] = (a_0 - a_4^5)
\[2\]:
\[1\]:
\[1\] = 1
If we can solve the system for every \( n \) we are done. But for concrete values of \( n \) we can compute the Gröbner cover. For \( n = 5 \) we obtain two segments:

<table>
<thead>
<tr>
<th>Segment</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C}^5 \setminus \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5) )</td>
<td>{1}</td>
</tr>
<tr>
<td>( \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5) )</td>
<td>{x_4 + a_4, (x_3 + a_4)^2, (x_2 + a_4)^3, (x_1 + a_4)^4}</td>
</tr>
</tbody>
</table>

Thus the polynomial is \( F_5(x, 0) = (x + a_4)^5 \).

And the conjecture for the Gröbner cover for \( n \) becomes:

<table>
<thead>
<tr>
<th>Segment</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{C}^n \setminus \mathbb{V}(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n) )</td>
<td>{1}</td>
</tr>
<tr>
<td>( \mathbb{V}(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n) )</td>
<td>{x_{n-1} + a_{n-1}, \ldots, (x_1 + a_{n-1})^{n-1}}</td>
</tr>
</tbody>
</table>

Thus the polynomial is \( F_n(x, 0) = (x + a_{n-1})^n \).
If a triangle \( ABC \) has two (internal) angle-bisectors with the same length, i.e. \( |AA'| = |BB'| \), \( \alpha = \beta \), \( \gamma = \delta \), then the triangle must be isosceles with \( |AC| = |BC| \).

The converse is, obviously also true.
The theorem was first mentioned in 1840 in a letter by C. L. Lehmus to C. Sturm. Jakob Steiner was among the first to provide a solution.

The theorem became a rather popular topic in elementary geometry ever since, because of the difficulty to obtain a direct proof (P. Baptist, J. Conway, O. Bottema, V. Thebault.)

See references at:

http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus
Recently, its generalization, regarding internal as well as external angle bisectors, has been approached through automatic tools.

Bisectors for internal and external angles at vertex $A$ are constructed intersecting circle of center at $A$ and radius $|AC|$ with side $AB$ (and its prolongation) at $P(p, 0)$, then placing lines through $A$ and the midpoints $Q$ of $C$ and $P$. The two bisector lines intersect the opposite side $CB$ at $M(a, b)$.

The segments from $A$ to $M(a, b)$ are the two bisectors at $A$. 


Already difficult proof for the standard statement.

Impossibility to deal (in Wang’s approach) with the case of three bisectors, because it involves two thesis: bisectors at $A = \text{bisectors at } B = \text{bisectors at } C$
We fix $A(0, 0), B(1, 0)$.

$$C(x, y)$$ is free.

We look for the locus of $C$ such that some pair of bisectors (at $A$ and $B$, at $A$ and $C$, at $B$ and $C$) have equal length.

Given a parametric ideal $I \subset \mathbb{Q}[a_1, \ldots, a_m][x_1, \ldots, x_n]$ there exists a unique "Canonical Gröbner Cover" consisting in a set of triplets 
\{(S_1, B_1, lpp_1), \ldots, (S_r, B_r, lpp_r)\} with the following properties:

- the $S_i$ are locally closed, disjoint subsets of $\mathbb{C}^m$ (called segments),
- the $B_i$ are a set of monic $I$-regular functions having constant leading power products (lpp) on $S_i$, such that for every point $a \in S_i$ determine the reduced Gröbner basis of $I_a$ and are called the reduced Gröbner basis of $I$ over $S_i$.
- for homogeneous ideals, the lpp characterize the segments, as different segments have different lpp’s.
- for non-homogeneous ideals it can happen that more than one segment corresponds to the same lpp, but even though, the split is canonical and corresponds to solutions that are different at infinity.
The equations for the bisectors at $A$

\[ x^2 + y^2 = p^2, \quad \begin{vmatrix} 0 & 0 & 1 \\ (x+p)/2 & y/2 & 1 \\ a & b & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0, \]
The equations for the bisectors at $B$

\begin{align*}
(1 - x)^2 + y^2 &= (1 - r)^2, \\
\begin{vmatrix}
1 & 0 & 1 \\
(x + r)/2 & y/2 & 1
\end{vmatrix} &= 0, \\
\begin{vmatrix}
0 & 0 & 1 \\
m & n & 1
\end{vmatrix} &= 0,
\end{align*}
One bisector of $A$ is equal to one bisector of $B$

$$a^2 + b^2 = (1 - m)^2 + n^2$$
All the equations:

\[
\begin{align*}
    x^2 + y^2 - p^2, \\
    (a - 1)y + b(1 - x), \\
    -ay + b(x + p), \\
    (1 - x)^2 + y^2 - (1 - r)^2, \\
    my - xn, \\
    (1 - m)y + (x + r - 2)n, \\
    a^2 + b^2 = (1 - m)^2 + n^2.
\end{align*}
\]

Parameters: \( x, y \)  
Variables: \( a, b, m, n, p, r \)

Solutions:

<table>
<thead>
<tr>
<th></th>
<th>+</th>
<th>-</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( i_A )</td>
<td>( e_A )</td>
</tr>
<tr>
<td>( 1 - r )</td>
<td>( i_B )</td>
<td>( e_B )</td>
</tr>
</tbody>
</table>
All the equations:

\[
\begin{align*}
    x^2 + y^2 - p^2, \\
    (a - 1)y + b(1 - x), \\
    -ay + b(x + p), \\
    (1 - x)^2 + y^2 - (1 - r)^2, \\
    my - xn, \\
    (1 - m)y + (x + r - 2)n, \\
    a^2 + b^2 = (1 - m)^2 + n^2.
\end{align*}
\]

Parameters: \(x, y\)  
Variables: \(a, b, m, n, p, r\)

Solutions:

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>(i_A)</td>
<td>(e_A)</td>
</tr>
<tr>
<td>(1 - r)</td>
<td>(i_B)</td>
<td>(e_B)</td>
</tr>
</tbody>
</table>
We restrict to non-degenerate triangles
Select only real values of the parameters.
We use `grevlex(a, b, m, n, p, r)` order for the variables in the call to the Gröbner cover algorithm. The parameters are \((x, y)\).
The result has the following geometrical interpretation.
The Gröbner cover of the Steiner-Lehmus system

\[ \begin{align*}
    \mathbf{P}_1 & \quad \mathbf{P}_2 \\
    \mathbf{P}_3 & \quad \mathbf{P}_4 \\
    \mathbf{P}_5 & \quad \mathbf{P}_6 \\
    \mathbf{P}_7 & \quad \mathbf{P}_8 \\
    \mathbf{P}_9 & \\
\end{align*} \]

\[ \begin{align*}
    i_A = i_B, \quad e_A = e_B \\
    e_A = e_B, \quad i_A = e_B \\
\end{align*} \]

Antonio Montes (UPC) Canonical Gröbner Cover ISSAC-2011 San Jose 30 / 64
The Gröbner cover of the Steiner-Lehmus system

Algebraic description: The following curves appear:

\[ C_1 = \mathbb{V}(8x^{10} + 41x^8y^2 + 84x^6y^4 + 86x^4y^6 + 44x^2y^8 + 9y^{10} - 40x^9 - 164x^7y^2 - 252x^5y^4 - 172x^3y^6 - 44xy^8 + 76x^8 + 246x^6y^2 + 278x^4y^4 + 122x^2y^6 + 14y^8 - 64x^7 - 164x^5y^2 - 136x^3y^4 - 36xy^6 + 16x^6 + 31x^4y^2 + 14x^2y^4 - y^6 + 8x^5 + 20x^3y^2 + 12xy^4 - 4x^4 - 10x^2y^2 - 6y^4 + y^2), \]

\[ C_2 = \mathbb{V}(2x - 1). \]

\[ C_3 = \mathbb{V}(y), \]
The Gröbner cover of the Steiner-Lehmus system

as well as the following varieties:

<table>
<thead>
<tr>
<th>Varieties</th>
<th>Real points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1 = \mathbb{V}(y, x)$</td>
<td>$P_1 = (0, 0)$</td>
</tr>
<tr>
<td>$V_2 = \mathbb{V}(y, x - 1)$</td>
<td>$P_2 = (1, 0)$</td>
</tr>
<tr>
<td>$V_3 = \mathbb{V}(y, 2x - 1)$</td>
<td>$P_3 = (\frac{1}{2}, 0)$</td>
</tr>
<tr>
<td>$V_4 = \mathbb{V}(y, 2x^2 - 2x - 1)$</td>
<td>$P_{4,12} = \left(\frac{1\pm\sqrt{3}}{2}, 0\right)$</td>
</tr>
<tr>
<td>$V_5 = \mathbb{V}(12y^2 - 1, 2x - 1)$</td>
<td>$P_{5,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right)$</td>
</tr>
<tr>
<td>$V_6 = \mathbb{V}(4y^2 - 3, 2x - 1)$</td>
<td>$P_{6,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$</td>
</tr>
<tr>
<td>$V_7 = \mathbb{V}(4y^4 + 5y^2 + 2, 2x - 1)$</td>
<td>$P_{8,12} = \left(2 - \sqrt{5}, \pm\frac{\sqrt{-22 + 10\sqrt{5}}}{2}\right)$</td>
</tr>
<tr>
<td>$V_8 = \mathbb{V}(y^4 + 11y^2 - 1, 5x + 2y^2 + 1)$</td>
<td>$P_{9,12} = \left(-1 + \sqrt{5}, \pm\frac{\sqrt{-22 + 10\sqrt{5}}}{2}\right)$</td>
</tr>
<tr>
<td>$V_9 = \mathbb{V}(y^4 + 11y^2 - 1, 5x - 2y^2 - 6)$</td>
<td>$P_{9,12} = \left(-1 + \sqrt{5}, \pm\frac{\sqrt{-22 + 10\sqrt{5}}}{2}\right)$</td>
</tr>
</tbody>
</table>
The Gröbner cover of the Steiner-Lehmus system

1. Segment with \( \text{lpp} = \{1\} \)
   
   **Generic segment**

   **Segment:** \( C^2 \setminus (C_1 \cup C_2 \cup C_3) \)

   **Description:** The whole parameter space except the curves \((C_1 \cup C_2 \cup C_3)\).

   **Basis:** \( B_1 = \{1\} \)

2. Segment with \( \text{lpp} = \{p, n, m, b, a, r^2\} \)
   
   **Segment:** \( C_2 \setminus (V_3 \cup V_5 \cup V_6)\)

   **Description:** Line \( C_2 \) minus the intersecting points with \( C_1 \) and \( C_2 \)

   **Basis:** \( B_2 = \)

   \[
   \{ p + r - 1, (4y^2 - 3)n + (4y)r, (4y^2 - 3)m + 2r, (4y^2 - 3)b + (4y)r, \\
   (4y^2 - 3)a - 2r + (-4y^2 + 3), 4r^2 - 8r + (-4y^2 + 3) \} .
   \]
The Gröbner cover of the Steiner-Lehmus system

3. Segment with lpp = \{r, p, n, m, b, a\}
Segment: \(C_1 \setminus (V_1 \cup V_2 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8 \cup V_9)\)
Description: The curve \(C_1\) except the real points \(P_1, P_2, P_{41}, P_{42}, P_{51}, P_{52}, P_{61}, P_{62}, P_{81}, P_{82}, P_{91}, P_{92}\) and some other complex points on it.
Basis: \(B_3 = \)

\[
\{(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 + 3y^4 + 5y^2 - 1)r \\
+(x^5 - 10x^4 + 2x^3y^2 + 17x^3 - 18x^2y^2 - 10x^2 + xy^4 + 17xy^2 - x - 8y^4 - 10y^2 + 2),
(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 - 4x + 3y^4 + 5y^2 + 1)p \\
+(x^5 + 2x^4 + 2x^3y^2 - 7x^3 + 6x^2y^2 + 4x^2 + xy^4 - 7xy^2 - x + 4y^4 + 4y^2),
(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)n \\
+(-3x^4y + 6x^3y - 6x^2y^3 - 5x^2y + 6xy^3 - 3y^5 - 5y^3 + y),
(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)m \\
+(-3x^5 + 6x^4 - 6x^3y^2 - 5x^3 + 6x^2y^2 - 3xy^4 - 5xy^2 + x),
(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)b \\
+(3x^4y - 6x^3y + 6x^2y^3 + 5x^2y - 6xy^3 - 4xy + 3y^5 + 5y^3 + y),
(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)a \\
+(2x^5 - 8x^4 + 4x^3y^2 + 12x^3 - 12x^2y^2 - 8x^2 + 2xy^4 + 12xy^2 + 2x - 4y^4 - 4y^2)\} \}
4. Segment with $l_{pp} = \{n, m, b, a, r^2, p^2\}$
Segment: $V_5$
Description: Points $P_{51}, P_{52}$
Basis: $B_4 =$
\[\{2n - 3yr, 4m - 3r, 2b + 3yp - 3y, 4a - 3p - 1, 3r^2 - 6r + 2, 3p^2 - 1\}\]

5. Segment with $l_{pp} = \{r, p, n, m, b, a\}$
Segment: $V_6$
Description: Points $P_{61}, P_{62}$
Basis: $B_5 = \{r, p - 1, 2n - y, 4m - 1, 2b - y, 4a - 3\}$
6. Segment with \( \text{lpp} = \{p, n, m, b, a, r^2\} \)
Segment: \( V_8(\cup V_7) \)
Description: 2 real points \( P_{81}, P_{82} \) (and other complex points)
Basis:

\[
B_6 = \left\{ (7284y^6 + 88197y^4 - 15633y^2 - 3849)p + (8820y^6 + 97285y^4 - 5905y^2 - 265)r + (-11380y^6 - 103045y^4 + 1425y^2 - 1015), \\
(116y^6 + 1493y^4 + 2403y^2 + 179)n + (660y)r, \\
(116y^6 + 1493y^4 + 2403y^2 + 179)m + (-72y^6 - 866y^4 - 1006y^2 - 58)r, \\
(87932y^6 + 779351y^4 + 109221y^2 - 31747)b + (-35280y^7 - 389140y^5 + 23620y^3 + 1060y)r + (16384y^7 + 59392y^5 + 56832y^3 + 19456y), \\
(87932y^6 + 779351y^4 + 109221y^2 - 31747)a + (17640y^6 + 194570y^4 - 11810y^2 - 530)r + (-51068y^6 - 786519y^4 - 157349y^2 + 5123), \\
660r^2 - 1320r + (-116y^6 - 1493y^4 - 2403y^2 - 179) \right\}.
\]
7. Segment with 1pp = \{r, n, m, b, a, p^2\} 
Segment: V_9
Description: Points P_{91}, P_{92}
Basis:

\[ B_7 = \left\{ (23y^2 - 1)r + (-83y^2 + 6), (134y^2 - 13)n + (83y^3 - 6y), \\
(134y^2 - 13)m + (-268y^2 + 26), \\
(y^2 + 3)b + (-5y)p + (5y), (y^2 + 3)a + (-2y^2 - 1)p + (y^2 - 2), \\
5p^2 + (-y^2 - 8) \right\}. \]
The Gröbner cover of the Steiner-Lehmus system
Solutions at the special points

<table>
<thead>
<tr>
<th>Point</th>
<th>$(p, 1 - r)$</th>
<th>Bisectors</th>
</tr>
</thead>
</table>
| $P_{51}, P_{52}$ | $(0.5773502693, 0.5773502693)$  
               | $(0.5773502693, -0.5773502693)$  
               | $(-0.5773502693, 0.5773502693)$  
               | $(-0.5773502693, -0.5773502693)$ | $i_A = i_B$  
               |                     | $i_A = e_B$  
               |                     | $e_A = i_B$  
               |                     | $e_A = e_B$  
| $P_{61}, P_{62}$ | $(1,1)$                                                         | $i_A = i_B$                                    |
| $P_{81}, P_{82}$ | $(-0.3819659526, -1.272019650)$  
               | $(-0.3819659526, 1.272019650)$                        | $e_A = e_B$  
               |                     | $e_A = i_B$                                    |
| $P_{91}, P_{92}$ | $(-1.272019650, -0.381965976)$  
               | $(1.272019650, -0.381965976)$                    | $e_A = e_B$  
               |                     | $i_A = e_B$                                    |

**Table:** Coincidences of bisectors of $A$ and $B$ at the special points
The colors of the curve

<table>
<thead>
<tr>
<th>Point</th>
<th>Branch</th>
<th>((p, 1 - r))</th>
<th>Bisectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, .7013671986))</td>
<td>(P_{62}-P_{82})</td>
<td>((- .7013, -1.2214))</td>
<td>(e_A = e_B)</td>
</tr>
<tr>
<td>((0, .4190287818))</td>
<td>(P_{52}-P_{82})</td>
<td>((- .4190, 1.0842))</td>
<td>(e_A = i_B)</td>
</tr>
<tr>
<td>((0, -.4190287818))</td>
<td>(P_{51}-P_{81})</td>
<td>((- .4190, 1.0842))</td>
<td>(e_A = i_B)</td>
</tr>
<tr>
<td>((0, -.7013671986))</td>
<td>(P_{61}-P_{81})</td>
<td>((- .7013, -1.2214))</td>
<td>(e_A = e_B)</td>
</tr>
<tr>
<td>((1, .7013671986))</td>
<td>(P_{62}-P_{92})</td>
<td>((-1.2215, -0.7013))</td>
<td>(e_A = e_B)</td>
</tr>
<tr>
<td>((1, .4190287818))</td>
<td>(P_{52}-P_{92})</td>
<td>((1.0842, -0.4190))</td>
<td>(i_A = e_B)</td>
</tr>
<tr>
<td>((1, -.4190287818))</td>
<td>(P_{51}-P_{91})</td>
<td>((1.0842, -0.4190))</td>
<td>(i_A = e_B)</td>
</tr>
<tr>
<td>((1, -.7013671986))</td>
<td>(P_{61}-P_{91})</td>
<td>((-1.2215, -0.7013))</td>
<td>(e_A = e_B)</td>
</tr>
</tbody>
</table>

**Table:** Coincidences of bisectors of \(A\) and \(B\) at some points of curve \(C_1\).
Generalized Steiner-Lehmus Theorem

**Theorem (Generalized Steiner-Lehmus)**

Let $ABC$ be a triangle and $i_A$, $e_A$, $i_B$, $e_B$ the lengths of the inner and outer bisectors of the angles $A$ and $B$. Then, considering the conditions for the equality of some bisector of $A$ and some bisector of $B$ the following excluding situations occur:

- the triangle $ABC$ is degenerate (i.e. $C$ is aligned with $A$ and $B$);
- $ABC$ is equilateral and then $i_A = i_B$ whereas $e_A$ and $e_B$ become infinite, $(P_{61}, P_{62})$;
- point $C$ is in the center of an equilateral triangle, and then $i_A = i_B = e_A = e_B$, $(P_{51}, P_{52})$;
- the triangle is isosceles but not of the special form of cases 2) or 3) and then $i_A = i_B \neq e_A = e_B$, (ordinary Theorem);

continues in the next slide ..
AC/AB = \frac{3-\sqrt{5}}{2}, \quad BC/AB = \sqrt{\frac{1+\sqrt{5}}{2}}, \quad \text{and then } e_A=e_B=i_B, (P_{81}, P_{82});

AC/AB = \sqrt{\frac{1+\sqrt{5}}{2}}, \quad BC/AB = \frac{3-\sqrt{5}}{2}, \quad \text{and then } e_A=e_B=i_A, (P_{91}, P_{92});

C \text{ lies in the curve of degree } 10 \text{ relative to points } A \text{ and } B \text{ (case of curve } C_1) \text{ passing through all the special points above but is none of these points, and then only one of the following things arrive: either } e_A=e_B \text{ or } i_A=e_B \text{ or } e_A=i_B \text{ depending on the branch of the curve (see Figure, the color representing which of the situations occur);}\n
\text{none of the above cases occur, and then no bisector of } A \text{ is equal to no bisector of } B.
S10. Inverse kinematic problem of a simple robot

\[ r = c_1 + l(c_1 c_2 - s_1 s_2), \]
\[ z = s_1 + l(s_1 c_2 + c_1 s_2), \]
\[ c_1^2 + s_1^2 - 1, \]
\[ c_2^2 + s_2^2 - 1, \]

\[ c_1 = \cos(\theta_1); \]
\[ s_1 = \sin(\theta_1); \]
\[ c_2 = \cos(\theta_2); \]
\[ s_2 = \sin(\theta_2); \]

Figure: Simple two arms robot.

\[ (r, z) \]

\[ \theta_1, \theta_2 \]
Problem S10. Inverse kinematic problem of a simple robot

> LIB "grobcov.lib";
> ring R=(0,r,z),(s1,c1,s2,c2,l),lp;
> ideal S10=r-c1-l*c1*c2+l*s1*s2,
    z-s1-l*c1*s2-l*s1*c2,
    c1^2+s1^2-1,
    c2^2+s2^2-1;
> grobcov(S10);
[1]:
  _[1]=c2*l
  _[2]=s2^2
  _[3]=c1
  _[4]=s1

[2]:
  _[1]=2*c2*l+l^2+(-r^2-z^2+1)
  _[2]=s2^2+c2^2-1
  _[3]=(2*r^2+2*z^2)*c1+(-2*z)*s2*l+(r)*l^2+(-r^3-r*z^2-r)
  _[4]=(2*r^2+2*z^2)*s1+(2*r)*s2*l+(z)*l^2+(-r^2*z-z^3-z)

[3]:
  [1]:
    [1]:
      _[1]=0
  [2]:
    [1]:
      _[1]=(r^2+z^2)
\[\begin{array}{l}
\text{[2]}: \\
\quad \text{[1]}:\ \\
\quad \quad _{[1]} = c_2 \cdot l \\
\quad \quad _{[2]} = s_2 \\
\quad \quad _{[3]} = c_1 \cdot l^2 \\
\quad \quad _{[4]} = c_1 \cdot c_2 \\
\quad \quad _{[5]} = s_1 \\
\text{[2]}:\ \\
\quad \quad _{[1]} = 2 \cdot c_2 \cdot l + l^2 + 1 \\
\quad \quad _{[2]} = (z) \cdot s_2 + (-r) \cdot c_2 + (-r) \cdot l \\
\quad \quad _{[3]} = (4 \cdot r) \cdot c_1 \cdot l^2 + (-4 \cdot r) \cdot c_1 + l^4 - 2 \cdot l^2 + (-4 \cdot z^2 + 1) \\
\quad \quad _{[4]} = (8 \cdot r) \cdot c_1 \cdot c_2 + (8 \cdot r) \cdot c_1 \cdot l + (8 \cdot z^2 - 2) \cdot c_2 + l^3 + (4 \cdot z^2 - 3) \cdot l \\
\quad \quad _{[5]} = (2 \cdot z) \cdot s_1 + (2 \cdot r) \cdot c_1 + l^2 - 1 \\
\text{[3]}:\ \\
\quad \quad \text{[1]}:\ \\
\quad \quad \quad \quad _{[1]} = (r^2 + z^2) \\
\quad \quad \text{[2]}:\ \\
\quad \quad \quad \quad \text{[1]}:\ \\
\quad \quad \quad \quad \quad _{[1]} = (z) \\
\quad \quad \quad \quad \quad _{[2]} = (r)
\end{array}\]
\[ \begin{align*}
[3]: \\
[1]: \\
\quad [1] &= l^2 \\
\quad [2] &= c_2 \\
\quad [3] &= s_2 \\
\quad [4] &= s_1^2 \\
[2]: \\
\quad [1] &= l^2 - 1 \\
\quad [2] &= c_2 + 1 \\
\quad [3] &= s_2 \\
\quad [4] &= s_1^2 + c_1^2 - 1 \\
[3]: \\
[1]: \\
\quad [1]: \\
\quad [1] &= (z) \\
\quad [2] &= (r) \\
[2]: \\
\quad [1]: \\
\quad [1] &= 1
\end{align*} \]
1. Generic segment:

<table>
<thead>
<tr>
<th>lpp:</th>
<th>$c_2l, s_2^2, c_1, s_1$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment:</td>
<td>$\mathbb{C}^2 \setminus \mathbb{V}(r^2 + z^2)$.</td>
</tr>
</tbody>
</table>
| Basis:        | $\begin{cases} 
2c_2l + l^2 + (-r^2 - z^2 + 1), \\
2r^2 + 2s_2^2 - 1, \\
(2r^2 + 2s_2^2)c_1 + (-2z)s_2l + (r)l^2 + (-r^3 - rz^2 - r), \\
(2r^2 + 2s_2^2)s_1 + (2r)s_2l + (z)l^2 + (-r^2z - z^3 - z). 
\end{cases}$ |

$l$ is free. But in order to have real values with $-1 \leq c_2 \leq 1$, we must choose $|\sqrt{r^2 + z^2} - 1| \leq l \leq \sqrt{r^2 + z^2} + 1$, and then
2. Segment representing complex points:

<table>
<thead>
<tr>
<th>Ipp:</th>
<th>$c_2 l, s_2, c_1 l^2, c_1 c_2, s_1$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment:</td>
<td>$\mathbb{V}(z^2 + r^2) \setminus \mathbb{V}(z, r)$.</td>
</tr>
</tbody>
</table>
| Basis:        | \[
\begin{cases}
2c_2 l + l^2 + 1, \\
zs_2 - rc_2 - rl, \\
4rc_1 l^2 - 4rc_1 + l^4 - 2l^2 + (-4z^2 + 1), \\
8rc_1 c_2 + 8rc_1 l + (8z^2 - 2)c_2 + l^3 + (4z^2 - 3)l, \\
2zs_1 + 2rc_1 + l^2 - 1. 
\end{cases}
\] |

3. Segment representing the origin:

<table>
<thead>
<tr>
<th>Ipp:</th>
<th>$l^2, c_2, s_2, c_1^2$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment:</td>
<td>$\mathbb{V}(z, r)$</td>
</tr>
</tbody>
</table>
| Basis:        | \[
\begin{cases}
l^2 - 1, \\
c_2 + 1, \\
s_2, \\
s_1^2 + c_1^2 - 1. 
\end{cases} \quad l = 1, \\
\theta_2 = \pi, \\
\theta_1 \text{ free}
\] |
Consider the following simple system:

\[
\begin{align*}
    u_1 x + u_2 &= 0, \\
    u_3 x + u_4 &= 0;
\end{align*}
\]

Compute the Gröbner cover:
Problem S42: Need of sheaves.

```plaintext
> ring R=(0,u1,u2,u3,u4),(x),dp;
> short=0;
> ideal S42=u1*x+u2,
>          u3*x+u4;
> grobcov(S42);
[1]:
    [1]:
      _[1]=1
[2]:
    _[1]=1
[3]:
    [1]:
      [1]:
        _[1]=0
[2]:
    [1]:
      _[1]=(u1*u4-u2*u3)
```
\[2]:
\[1]:
\[1]=x
\[2]:
\[1]:
\[1]=(u3) \times x + (u4)
\[2]=(u1) \times x + (u2)
\[3]:
\[1]:
\[1]:
\[1]=(u1 \times u4 - u2 \times u3)
\[2]:
\[1]:
\[1]=(u3)
\[2]=(u1)
[3]:
  [1]:
    _[1]=1
  [2]:
    _[1]=1
[3]:
  [1]:
    [1]:
      _[1]=(u3)
      _[2]=(u1)
  [2]:
    [1]:
      _[1]=(u4)
      _[2]=(u3)
      _[3]=(u2)
      _[4]=(u1)
\[\begin{align*}
[4]: & \\
[1]: & \quad \_[1]=0 \\
[2]: & \quad \_[1]=0 \\
[3]: & \quad \_[1]: \\
[1]: & \quad \_[1]=(u4) \\
[2]: & \quad \_[2]=(u3) \\
[3]: & \quad \_[3]=(u2) \\
[4]: & \quad \_[4]=(u1) \\
[2]: & \quad \_[1]=1
\end{align*}\]
S42. Need of sheaves

The result is:

<table>
<thead>
<tr>
<th>Ipp</th>
<th>Basis</th>
<th>Segment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\mathbb{C}^4 \setminus \mathcal{V}(u_1u_4 - u_2u_3)$</td>
</tr>
<tr>
<td>$x$</td>
<td>$u_3x + u_4, u_1x + u_2$</td>
<td>$\mathcal{V}(u_1u_4 - u_2u_3) \setminus \mathcal{V}(u_3, u_1)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\mathcal{V}(u_3, u_1) \setminus \mathcal{V}(u_4, u_3, u_2, u_1)$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\mathcal{V}(u_4, u_3, u_2, u_1)$</td>
</tr>
</tbody>
</table>
Introduction

Examples
- S53. Automatic discovery of theorems: isosceles orthic triangle
- S92. Casas Alberó conjecture
- S93. Generalization of the Steiner-Lehmus Theorem
- S10. Inverse kinematic problem of a simple robot
- S42. Need of sheaves

Description of the Gröbner cover
- Locally closed sets and I-regular functions
- The Wibmer Theorem and the Gröbner cover

Gröbner Cover algorithm

Representations
A subset $S \subset \overline{K}^m$ is **locally closed**, if it is difference of two varieties:

$$S = \mathbb{V}(M) \setminus \mathbb{V}(N).$$

**Definition (Open subset)**

A subset $U \subset S$ is said to be **open on $S$** if $S \setminus U \subsetneq S$. 

Antonio Montes (UPC)  
Canonical Gröbner Cover  
ISSAC-2011 San Jose  
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**Definition (\(I\)-Regular function)**

Let \(S\) be a locally closed subset of \(\overline{K}^m\). We call a function \(f : S \rightarrow \overline{K}[\bar{x}]\) \(I\)-regular, if \(\forall a \in S\) it exists an open \(U \subset S\) with \(a \in U\) and

\[
f(b) = \frac{P(b, \bar{x})}{Q(b)} \quad \text{for all } b \in U,
\]

where \(P \in I\) and \(Q \in K[\bar{a}]\) and \(Q(b) \neq 0\) for all \(b \in U\).

**Remark**

Let \(P\) and \(Q\) be a polynomials as above, (they are not unique), \(S = \nabla(a) \setminus \nabla(b)\) and \(p(b, \bar{x}) = P(b, \bar{x}) \mod a\). If \(f\) is monic and \(\text{lpp}(f)\) is constant on \(S\), then, for all \(b \in U\) is

- \(\text{lpp}_x(p(b, \bar{x})) = \text{lpp}_x(f), \text{ and}\)
- \(\text{lc}_x(p(b, \bar{x})) = Q(b) \mod a\).
Parametric subsets

Definition (Parametric subset of $\overline{K}^m$)

A locally closed subset $S \in \overline{K}^m$ is called parametric (wrt to $I$ and $\succ_{\bar{x}}$) if there exist monic $I$-regular functions $\{g_1, \ldots, g_s\}$ over $S$ so that $
abla g_1(a, \bar{x}), \ldots, g_s(a, \bar{x})$ is the reduced Gröbner basis of $I_a$ for all $a \in S$.

Note

Note that the definition immediately implies that if $a, b$ lie in a parametric set $S$, then $\text{lpp}_{\bar{x}}(I_a) = \text{lpp}_{\bar{x}}(I_b)$. The amazing thing is that the converse also holds if we additionally assume that $I \subset K[a][\bar{x}]$ is homogeneous (wrt to the variables).
Wibmer’s Theorem

**Theorem (M. Wibmer)**

Let $I \subset K[a][x]$ be a homogeneous ideal and $a \in \overline{K}^m$. Then the set

$$S_a = \{ b \in \overline{K}^m : \text{lpp}_x(I_b) = \text{lpp}_x(I_a) \}$$

is **parametric**.

In particular, $S_a$ is **locally closed**.
Definition (Gröbner cover)

By a **Gröbner cover** of $\overline{K}^m$ wrt to $I$ and $\succ_{\overline{x}}$ we mean a finite set of pairs \{(S_1, B_1), \ldots, (S_r, B_r)\} such that

1. the $S_i$’s are parametric and so, $B_i \subset O(S_i)[\overline{x}]$ is the reduced Gröbner basis of $I$ over $S_i$ for $i = 1, \ldots, r$, and
2. the union of all $S_i$’s equals $\overline{K}^m$.

Theorem (Canonical Gröbner cover)

Let $I \subset K[a][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\text{lpp}_{\overline{x}}(I_a) = \text{lpp}_{\overline{x}}(I_b)$.
**Definition (Gröbner cover)**

By a *Gröbner cover* of $\overline{K}^m$ wrt to $I$ and $\succ_{\bar{x}}$ we mean a finite set of pairs \{$(S_1, B_1), \ldots, (S_r, B_r)$\} such that

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**Theorem (Canonical Gröbner cover)**

Let $I \subset K[\bar{a}][\bar{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\text{lpp}_x(I_a) = \text{lpp}_x(I_b)$. 
Note (Homogenization and dehomogenization)

For homogenization introduce a new variable $x_0$ and extend $\succ_{\bar{x}}$ to the monomials in $\bar{x}, x_0$ by setting

$$\bar{x}^\alpha x_0^i \succ_{\bar{x}, x_0} \bar{x}^\beta x_0^j \text{ iff } (\bar{x}^\alpha \succ_{\bar{x}} \bar{x}^\beta) \text{ or } (\bar{x}^\alpha = \bar{x}^\beta \text{ and } i > j)$$

Denote $\tau$ the dehomogenization consisting of substituting $x_0 = 1$. 
Affine canonical Gröbner cover

Definition (Affine canonical Gröbner cover)

Let \( I \subset K[\overline{a}][\overline{x}] \) be a non-homogeneous ideal and let \( J \subset K[\overline{a}][\overline{x}, x_0] \) denote its homogenization. The disjoint Gröbner cover of \( \overline{K}^m \) with respect to \( I \) and \( \succ_{\overline{x}} \) obtained by dehomogenization and reduction will be called the **canonical Gröbner cover of** \( \overline{K}^m \) **with respect to** \( I \) and \( \succ_{\overline{x}} \).

Remark

The affine canonical Gröbner cover does not necessarily summarize in a unique segment all the points corresponding to the same lpp. Nevertheless it is canonical, and when two segments occur with the same lpp they correspond to different kind of solutions at infinity.
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Algorithm (Homogeneous GröbnerCover)

\[ \text{GCover}(F, \succ x, \succ a) \]

\( T := \text{BuildTree}(F, \succ x, \succ a). \) (Initial disjoint and reduced CGS)

\( G := \emptyset \)

Group the segments of \( T \) by lpp’s: \( T = \{ T_i : 1 \leq i \leq s \} \).

where \( T_i = \{ (S_{ij}, B_{ij}) : 1 \leq j \leq s_i \} \) with \( \text{lpp}(B_{ij}) = \text{lpp}(B_{ik}) \)

For each lpp-segment \( T_i \)

\( S_i := \text{LCUnion}(S_{ij} : 1 \leq j \leq s_i). \) (Summarizing lpp-segments)

\( B_i := \text{Basis}(S_i, T_i). \) (Determining the generic basis for \( S_i \) using \( T_i \).

\( G := G \cup (S_i, B_i) \)

end for

Return \( G \)
Gröbner Cover algorithm

Algorithm (Affine GröbnerCover)

GröbnerCover\( (F, \succ x, \succ a) \)

If \( F \) is homogeneous then \( G := \text{GCover}(F, \succ x, \succ a) \)
else
\( F' := \text{Homogenize}(F, x_0), \ y := \bar{x}, x_0, \succ y = \succ \bar{x}, x_0 \)
\( G := \text{GCover}(F', \succ y, \succ a) \)
\( \bar{y} := \bar{x}, 1, (\text{Dehomogenize the bases in } G) \)
Reduce the bases in \( G \)
end if

Extend the bases in \( G \) (to obtain a full representation)

Return \( G \)
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5. Representations
Representation of locally closed subsets

Proposition (Canonical representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist uniquely determined radical ideals $\alpha \subset \beta$ of $K[\overline{a}]$, with $S = \overline{\text{V}(\alpha)} \setminus \overline{\text{V}(\beta)}$, such that

- $\overline{S} = \overline{\text{V}(\alpha)}$,
- $\overline{S} \setminus S = \overline{\text{V}(\beta)}$.

The pair $(\alpha, \beta)$ -top, hole- is called the canonical representation of $S$. 
Proposition (Canonical prime representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist a unique canonical prime representation of $S$ given the prime components of $\alpha$, say $p_i$, and associated to each, a set of prime ideals $p_{ij}$ (holes) in the form $((p_1, (p_{11}, \ldots, p_{1j_1})), \ldots, (p_k, (p_{k1}, \ldots, p_{kj_k})))$ so that

$$S = \bigcup_{i=1}^{k} \left( \bigcap_{j=1}^{j_i} \mathbb{V}(p_i) \setminus \bigcup_{j=1}^{j_i} \mathbb{V}(p_{ij}) \right).$$

and $p_i \subset p_{ij}$ for all $i, j$, such that

- $\overline{S} = \mathbb{V}(p_1) \cup \ldots \cup \mathbb{V}(p_r)$ and
- $(\overline{S} \setminus S) \cap \mathbb{V}(p_i) = \mathbb{V}(p_{i1}) \cup \ldots \cup \mathbb{V}(p_{iri})$

are the minimal decompositions into irreducible closed sets.
Definition (Generic representation)

Let \( S \subset \overline{K}^m \) be a locally closed set and \( f : S \rightarrow \overline{K}[\overline{x}] \) a monic \( I \)-regular function. We say that \( p \in K[\overline{a}][\overline{x}] \) generically represents \( f \) if

- \( \text{lpp}(f) = \text{lpp}(p) \),
- \( \text{lc}(p)(a) \neq 0 \) on an open and dense set of points in \( S \),
- if \( \text{lc}(p)(a) \neq 0 \) then \( f(a, \overline{x}) = p(a, \overline{x}) / \text{lc}(p)(a) \), otherwise is \( p(a, \overline{x}) = 0 \).

Proposition

Every monic \( I \)-regular function \( f : S \rightarrow \overline{K}[\overline{x}] \) admits a generic representation.
**Definition (Full representation)**

Let \( S \subset \overline{K}^m \) be a locally closed set and \( f : S \to \overline{K}[\overline{x}] \) a monic \( I \)-regular function. We say that a the set of polynomials \( \{p_1, \cdots, p_r\} \subset K[\overline{a}][\overline{x}] \) fully represents \( f \) if

- \( \text{lpp}(f) = \text{lpp}(p_i) \), for \( 1 \leq i \leq r \),
- for \( a \in S \) and \( 1 \leq i \leq r \) either \( \text{lc}(p_i)(a) \neq 0 \) or \( p_i(a, \overline{x}) = 0 \),
- for all \( a \in S \) it exist at least one \( i \) and an open \( U \subset S \) such that for every \( b \in U \) is \( \text{lc}(p_i)(a) \neq 0 \) and \( f(a, \overline{x}) = p(a, \overline{x}) / \text{lc}(p)(a) \).

**Proposition**

Given a generic representation of a monic \( I \)-regular function \( f : S \to \overline{K}[\overline{x}] \), the algorithm \text{EXTEND} computes a full representation.
Representation of $I$-regular functions

Example

Let $I = \langle ax + by, cx + dy \rangle$ and $F$ be the monic $I$-regular function

$$F : \quad S = \mathbb{V}(ad - bc) \setminus \mathbb{V}(a, c) \subset \mathbb{C}^4 \quad \rightarrow \quad \mathbb{C}[x, y]$$

$$(a, b, c, d) \quad \mapsto \quad \begin{cases} 
  x + \frac{b}{a}y & \text{if } a \neq 0 \\
  x + \frac{d}{c}y & \text{if } c \neq 0
\end{cases}$$

Then

Generic representation of $F$: \quad $p = ax + by$

Full representation of $F$: \quad $\{p_1 = ax + by, p_2 = cx + dy\}$