Tutorial for the Singular grobcov.lib library

Canonical Gröbner Cover of parametric ideals

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   - The Wibmer Theorem and the Gröbner cover
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References

- Antonio Montes, Michael Wibmer. “Gröbner Bases for Polynomial Systems with Parameters".
  Journal of Symbolic Computation **45** (2010) 1391 - 1425.
- Slades of MAP-2010 Presentation, Logroño 2010, presenting the definition, existence and algorithm of the Gröbner cover:
  http://www-ma2.upc.edu/∼montes/
- Software download (beta version):
  http://www-ma2.upc.edu/∼montes/
- Standard software version will be distributed with the next Singular release.

The problem

Goal

Given: Parametric polynomial system of equations

$$\begin{cases} p_1(a_1,\dots,a_m,x_1,\dots,x_n) = 0 \\ \dots \\ p_r(a_1,\dots,a_m,x_1,\dots,x_n) = 0 \end{cases}$$

Goal: describe the different kind of solutions $(x_1,\dots,x_n)$ in dependence of the parameters $a_1,\cdots,a_m$.

The Gröbner Cover: grobcov(I)

Input: A generating set $\{p_1, \cdots, p_s\} \subset K[\overline{a}][\overline{x}]$ of the ideal I and a monomial order $\succ_{\overline{x}}$.

Output: The canonical Gröbner cover of $\overline{K}^m$ wrt I. It consist of a set of triplets $\{(lpp_1, B_1, S_1), \ldots, (lpp_r, B_r, S_r)\}$.

- the S_i are locally closed, disjoint segments such that $\bigcup_i (S_i) = \overline{K}^m$, given in canonical prime-representation (P-representation),
- the B_i are a set of monic I-regular functions having constant lpp on S_i, such that for every point $a \in S_i$ are the reduced Gröbner basis of I_a. They are provided in full representation (or optionally in generic representation).
- Moreover if the ideal is homogeneous, then different segments have different lpp's.
- For non-homogeneous ideals, homogenizing, computing the Gröbner cover and dehomogenizing leads also to a Gröbner cover. It is canonical if we homogenize the ideal and not only a basis. More than one segment can have the same lpp.

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S10. Inverse kinematic problem of a simple robot

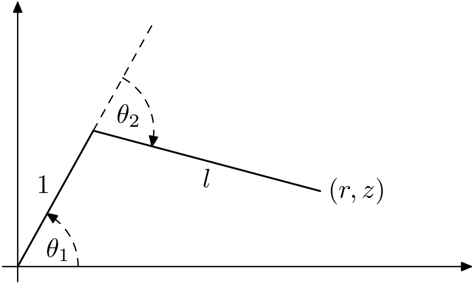


Figure: Simple two arms robot.

$$\begin{cases} c_1 = \cos(\theta_1); \\ s_1 = \sin(\theta_1); \\ c_2 = \cos(\theta_2); \\ s_2 = \sin(\theta_2); \end{cases} \qquad \begin{cases} r = c_1 + l(c_1c_2 - s_1s_2), \\ z = s_1 + l(s_1c_2 + c_1s_2) \\ c_1^2 + s_1^2 - 1, \\ c_2^2 + s_2^2 - 1, \end{cases}$$

Problem S10. Inverse kinematic problem of a simple robot

```
> LIB "grobcov.lib";
> ring R=(0,r,z),(s1,c1,s2,c2,l),lp;
> ideal S10=r-c1-l*c1*c2+l*s1*s2,
            z-s1-l*c1*s2-l*s1*c2,
            c1^2+s1^2-1,
            c2^2+s2^2-1;
> grobcov(S10);
```

```
[1]:
   [1]:
      _[1]=c2*l
      _[2]=s2^2
      _[3]=c1
      _[4]=s1
   [2]:
      _[1]=2*c2*l+l^2+(-r^2-z^2+1)
      _[2]=s2^2+c2^2-1
      _[3]=(2*r^2+2*z^2)*c1+(-2*z)*s2*l+(r)*l^2+(-r^3-r*z^2-r)
      _[4]=(2*r^2+2*z^2)*s1+(2*r)*s2*l+(z)*l^2+(-r^2*z-z^3-z)
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(r^2+z^2)
```

```
[2]:
   [1]:
      _[1]=c2*l
      _[2]=s2
      _[3]=c1*l^2
      _[4]=c1*c2
      _[5]=s1
   [2]:
      _[1]=2*c2*l+l^2+1
      _[2]=(z)*s2+(-r)*c2+(-r)*l
      _[3]=(4*r)*c1*l^2+(-4*r)*c1+l^4-2*l^2+(-4*z^2+1)
      _[4]=(8*r)*c1*c2+(8*r)*c1*l+(8*z^2-2)*c2+l^3+(4*z^2-3)*l
      _[5]=(2*z)*s1+(2*r)*c1+l^2-1
   [3]:
      [1]:
         [1]:
            _[1]=(r^2+z^2)
         [2]:
            [1]:
               _[1]=(z)
               _[2]=(r)
```

```
[3]:
   [1]:
      _[1]=l^2
      _[2]=c2
      _[3]=s2
      _[4]=s1^2
   [2]:
      _[1]=l^2-1
      _[2]=c2+l
      _[3]=s2
      _[4]=s1^2+c1^2-1
   [3]:
      [1]:
         [1]:
            _[1]=(z)
            _[2]=(r)
         [2]:
            [1]:
               _[1]=1
```

1. Generic segment::

lpp:	$c_2 l, s_2^2, c_1, s_1.$
Segment:	$\mathbb{C}^2 \setminus \mathbb{V}(r^2 + z^2).$
Basis:	$\begin{cases} 2c_2 l + l^2 + (-r^2 - z^2 + 1), \\ s_2^2 + c_2^2 - 1, \\ (2r^2 + 2z^2)c_1 + (-2z)s_2 l + (r)l^2 + (-r^3 - rz^2 - r), \\ (2r^2 + 2z^2)s_1 + (2r)s_2 l + (z)l^2 + (-r^2 z - z^3 - z). \end{cases}$

c_2 is free and $l = -c_2 \pm \sqrt{c_2^2 + r^2 + z^2 - 1}$.

To have real solutions we must choose $c_2^2 > 1 - r^2 - z^2$, and

$$\begin{aligned} l &= -c_2 + \sqrt{c_2^2 + r^2 + z^2 - 1} \\ s_2^2 &= 1 - c_2^2, \\ c_1 &= \frac{(2z)s_2 l - (r)l^2 + (r^3 + rz^2 + r)}{2(r^2 + z^2)}, \\ s_1 &= \frac{-(2r)s_2 l - (z)l^2 + (r^2 z + z^3 + z)}{2(r^2 + z^2)}. \end{aligned}$$

2. Segment representing complex points:

lpp:	$c_2 l, s_2, c_1 l^2, c_1 c_2, s_1.$
Segment:	$\mathbb{V}(z^2+r^2) \setminus \mathbb{V}(z,r).$
Basis:	$\left\{ \begin{array}{l} 2c_2 l + l^2 + 1, \\ zs_2 - rc_2 - rl, \\ 4rc_1 l^2 - 4rc_1 + l^4 - 2l^2 + (-4z^2+1), \\ 8rc_1 c_2 + 8rc_1 l + (8z^2-2)c_2 + l^3 + (4z^2-3)l, \\ 2zs_1 + 2rc_1 + l^2 - 1. \end{array} \right.$

3. Segment representing the origin:

lpp:	$l^2, c_2, s_2, c_1^2.$	
Segment:	$\mathbb{V}(z,r).$	
Basis:	$\left\{ \begin{array}{l} l^2 - 1, \\ c_2 + 1, \\ s_2, \\ s_1^2 + c_1^2 - 1. \end{array} \right.$	$l = 1,$ $\theta_2 = \pi,$ θ_1 free

S42. Need of sheaves

- Consider the following simple system:

$$\begin{cases} u_1 x + u_2 = 0, \\ u_3 x + u_4 = 0; \end{cases}$$

- Compute the Gröbner cover:

Problem S42: Need of sheaves.

```
> ring R=(0,u1,u2,u3,u4),(x),dp;
> short=0;
> ideal S42=u1*x+u2,
            u3*x+u4;
> grobcov(S42);
[1]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(u1*u4-u2*u3)
```

```
[2]:
   [1]:
      _[1]=x
   [2]:
      [1]:
         _[1]=(u3)*x+(u4)
         _[2]=(u1)*x+(u2)
   [3]:
      [1]:
         [1]:
            _[1]=(u1*u4-u2*u3)
         [2]:
            [1]:
               _[1]=(u3)
               _[2]=(u1)
```

```
[3]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=(u3)
            _[2]=(u1)
         [2]:
            [1]:
               _[1]=(u4)
               _[2]=(u3)
               _[3]=(u2)
               _[4]=(u1)
```

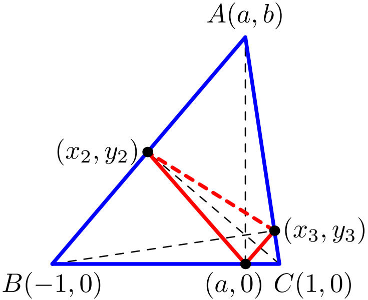
```
[4]:
   [1]:
      _[1]=0
   [2]:
      _[1]=0
   [3]:
      [1]:
         [1]:
            _[1]=(u4)
            _[2]=(u3)
            _[3]=(u2)
            _[4]=(u1)
         [2]:
            [1]:
               _[1]=1
```

S42. Need of sheaves

The result is:

lpp	Basis	Segment
1	1	$\mathbb{C}^4 \setminus \mathbb{V}(u_1u_4 - u_2u_3)$
x	$u_3x + u_4, u_1x + u_2$	$\mathbb{V}(u_1u_4 - u_2u_3) \setminus \mathbb{V}(u_3, u_1)$
1	1	$\mathbb{V}(u_3, u_1) \setminus \mathbb{V}(u_4, u_3, u_2, u_1)$
0	0	$\mathbb{V}(u_4, u_3, u_2, u_1)$

S53. Conditions for isosceles orthic triangle



Fix $B = (-1, 0)$, $C = (1, 0)$ and let $A = (a, b)$ be a parametric point. Construct the orthic triangle (i.e. the triangle trough the feets of the heights).

The question is:

1. for which points A the orthic triangle is isosceles at A'?

S53. Conditions for isosceles orthic triangle

1. The construction corresponds to the following equations:

$$\begin{aligned} G \quad = \quad & (a-1)y_2 - b(x_2-1), \\ & (a-1)(x_2+1) + by_2, \\ & (a+1)y_3 - b(x_3+1), \\ & (a+1)(x_3-1) + by_3. \end{aligned}$$

2. Add the condition for equal length of both sides $H_1 = (x_2 - a)^2 + y_2^2 - (x_3 - a)^2 - y_3^2$.
3. Compute the Gröbner cover:

S53. Automatic theorems discovering: Isosceles orthic triangle

```
> ring R=(0,a,b),(x2,x3,y2,y3),dp;
> ideal S53=(-b)*x2+(a-1)*y2+(b),
            (a-1)*x2+(b)*y2+(a-1),
            (b)*x3+(-a-1)*y3+(b),
            (a+1)*x3+(b)*y3+(-a-1),
            -x2^2+x3^2-y2^2+y3^2+(2*a)*x2+(-2*a)*x3;
> grobcov(S53);
```

```
[1]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=0
         [2]:
            [1]:
               _[1]=(a^2-b^2-1)
            [2]:
               _[1]=(a^2+b^2-1)
            [3]:
               _[1]=(a)
```

```
[2]:
   [1]:
      _[1]=y3
      _[2]=y2
      _[3]=x3
      _[4]=x2
   [2]:
      _[1]=(a^2+2*a+b^2+1)*y3+(-2*a*b-2*b)
      _[2]=(a^2-2*a+b^2+1)*y2+(2*a*b-2*b)
      _[3]=(a^2+2*a+b^2+1)*x3+(-a^2-2*a+b^2-1)
      _[4]=(a^2-2*a+b^2+1)*x2+(a^2-2*a-b^2+1)
   [3]:
      [1]:
         [1]:
            _[1]=(a^2-b^2-1)
         [2]:
            [1]:
               _[1]=(b)
               _[2]=(a-1)
            [2]:
               _[1]=(b)
               _[2]=(a+1)
            [3]:
```

```
            _[1]=(b^2+1)
            _[2]=(a)
[2]:
   [1]:
         _[1]=(a^2+b^2-1)
   [2]:
      [1]:
            _[1]=(b)
            _[2]=(a-1)
      [2]:
            _[1]=(b)
            _[2]=(a+1)
[3]:
   [1]:
         _[1]=(a)
   [2]:
      [1]:
            _[1]=(b^2+1)
            _[2]=(a)
```

```
[3]:
   [1]:
      _[1]=y3
      _[2]=x3
      _[3]=x2^2
   [2]:
      _[1]=y3
      _[2]=x3-1
      _[3]=x2^2+y2^2-2*x2+1
   [3]:
      [1]:
         [1]:
            _[1]=(b)
            _[2]=(a-1)
         [2]:
            [1]:
               _[1]=1
```

```
[4]:
   [1]:
      _[1]=1
   [2]:
      _[1]=1
   [3]:
      [1]:
         [1]:
            _[1]=(b^2+1)
            _[2]=(a)
         [2]:
            [1]:
               _[1]=1
```

```
[5]:
   [1]:
      _[1]=y2
      _[2]=x2
      _[3]=x3^2
   [2]:
      _[1]=y2
      _[2]=x2+1
      _[3]=x3^2+y3^2+2*x3+1
   [3]:
      [1]:
         [1]:
            _[1]=(b)
            _[2]=(a+1)
         [2]:
            [1]:
               _[1]=1
```

S53. Orthic triangle is isosceles

The generic segment with $\mathrm{lpp} = \{1\}$ is:

$$S_1 = \mathbb{C}^2 \setminus \big(\mathbb{V}(a) \cup \mathbb{V}(a^2+b^2-1) \cup \mathbb{V}(a^2-b^2-1)\big)$$
$$B_1 = \{1\}$$

The segment with $\mathrm{lpp} = \{x_2, y_2, x_3, y_3\}$ is:

$$\begin{aligned} S_2 = {} & (\mathbb{V}(a) \setminus \mathbb{V}(b^2+1, a)) \cup \\ & (\mathbb{V}(a^2+b^2-1) \setminus (\mathbb{V}(b, a-1)) \cup (\mathbb{V}(b, a+1))) \cup \\ & (\mathbb{V}(a^2-b^2-1) \setminus (\mathbb{V}(b, a-1)) \cup (\mathbb{V}(b, a+1)) \cup (\mathbb{V}(b^2+1, a))). \end{aligned}$$

$$\begin{aligned} B_2 = {} & (a^2+2a+b^2+1)y_3 + (-2ab-2b), \\ & (a^2-2a+b^2+1)y_2 + (2ab-2b), \\ & (a^2+2a+b^2+1)x_3 + (-a^2-2a+b^2-1), \\ & (a^2-2a+b^2+1)x_2 + (a^2-2a-b^2+1) \end{aligned}$$

The Gröbner cover has 3 other segments corresponding to the points $B(-1,0)$, $C(1,0)$, and the pair of complex points $\mathbb{V}(b^2+1, a)$.

Conjecture

If a polynomial of degree n in x has a common root which each of its $n-1$ derivatives (not assumed to be the same), then it is of the form $P(x) = k(x+a)^n$, i.e. the common roots must be all the same.

Let

$$f(x) = x^n + \sum_{i=0}^{n-1} \binom{n}{i} a_i x^i.$$

We have

$$F_n(x,j) = \frac{j!}{n!} f^{(j)}(x) = x^{n-j} + \sum_{i=0}^{n-j-1} \binom{n-j}{i} a_{i+j} x^i$$

The system of the hypothesis becomes

$$\{F_n(x_1,0), F_n(x_1,1), \ldots, F_n(x_n,0), F_n(x_n,n-1)\}$$

S92. Casas Alberó Conjecture

```
> ring R=(0,a0,a1,a2,a3,a4),(x1,x2,x3,x4),dp;
> proc Fn(poly x,int n,int j)
  {
    int i; poly f=x^n;
    for(i=0;i<=n-1;i++)
    {
      f=f+binomial(n,i)*par(i+1+j)*x^i;
    }
    return(f);
  }
> int n=5;  ideal F;
> for (i=1;i<=n-1;i++)
  {
    F[size(F)+1]=Fn(var(i),n,0);
    F[size(F)+1]=Fn(var(i),n-i,i);
  }
```

```
> F;
F[1]=x1^5+(5*a4)*x1^4+(10*a3)*x1^3+(10*a2)*x1^2+(5*a1)*x1+(a0)
F[2]=x1^4+(4*a4)*x1^3+(6*a3)*x1^2+(4*a2)*x1+(a1)
F[3]=x2^5+(5*a4)*x2^4+(10*a3)*x2^3+(10*a2)*x2^2+(5*a1)*x2+(a0)
F[4]=x2^3+(3*a4)*x2^2+(3*a3)*x2+(a2)
F[5]=x3^5+(5*a4)*x3^4+(10*a3)*x3^3+(10*a2)*x3^2+(5*a1)*x3+(a0)
F[6]=x3^2+(2*a4)*x3+(a3)
F[7]=x4^5+(5*a4)*x4^4+(10*a3)*x4^3+(10*a2)*x4^2+(5*a1)*x4+(a0)
F[8]=x4+(a4)
```

```
> multigrobcov(F);
[1]:
   [1]:
      [1]:
         _[1]=1
      [2]:
         _[1]=1
      [3]:
         [1]:
            [1]:
               _[1]=0
            [2]:
               [1]:
                  _[1]=(a3-a4^2)
                  _[2]=(a2-a4^3)
                  _[3]=(a1-a4^4)
                  _[4]=(a0-a4^5)
```

```
[2]:
   [1]:
      [1]:
         _[1]=x4
         _[2]=x3^2
         _[3]=x2^3
         _[4]=x1^4
      [2]:
         _[1]=x4+(a4)
         _[2]=x3^2+(2*a4)*x3+(a4^2)
         _[3]=x2^3+(3*a4)*x2^2+(3*a4^2)*x2+(a4^3)
         _[4]=x1^4+(4*a4)*x1^3+(6*a4^2)*x1^2+(4*a4^3)*x1+(a4^4)
      [3]:
         [1]:
            [1]:
               _[1]=(a3-a4^2)
               _[2]=(a2-a4^3)
               _[3]=(a1-a4^4)
               _[4]=(a0-a4^5)
            [2]:
               [1]:
                  _[1]=1
```

If we can solve the system for every n we are done.
But for concrete values of n we can compute the Gröbner cover.
For $n = 5$ we obtain two segments:

Segment	Basis
$\mathbb{C}^5 \setminus \mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{1\}$
$\mathbb{V}(a_3 - a_4^2, a_2 - a_4^3, a_1 - a_4^4, a_0 - a_4^5)$	$\{x_4 + a_4, (x_3 + a_4)^2, (x_2 + a_4)^3, (x_1 + a_4)^3\}$

Thus the polynomial is $F_5(x, 0) = (x + a_4)^5$.

And the conjecture for the Gröbner cover for n becomes:

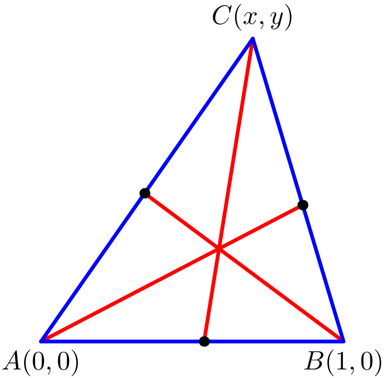
Segment	Basis
$\mathbb{C}^n \setminus \mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{1\}$
$\mathbb{V}\left(a_{n-2} - a_{n-1}^2, \ldots, a_0 - a_{n-1}^n\right)$	$\{x_{n-1} + a_{n-1}, \ldots, (x_1 + a_{n-1})^{n-1}\}$

Thus the polynomial is $F_n(x, 0) = (x + a_{n-1})^n$.

Theorem (Classical Steiner-Lehmus)

The inner bisectors of angles A and B of a triangle ABC are of equal length if and only if the triangle is isosceles with AC=BC.

Proved: 1848



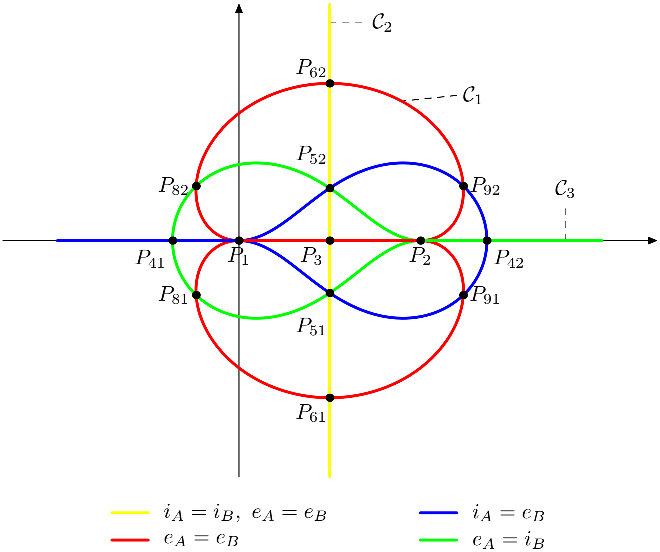
Recent studies

Generalization of the Steiner-Lehmus Theorem using automatic deduction of geometrical theorems.

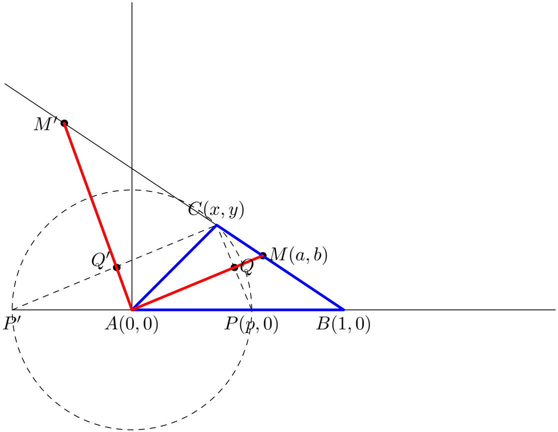
- [Wa04] D. Wang, Elimination practice: software tools and applications, Imperial College Press, London, (2004), p. 144-159.
- [LoReVa09] R. Losada, T. Recio, J.L. Valcarce, Sobre el descubrimiento automático de diversas generalizaciones del Teorema de Steiner-Lehmus, Boletín de la Sociedad Puig Adam, no. 82, pp. 53-76, (2009).
- http://www.mathematik.uni-bielefeld.de/∼sillke/PUZZLES/steiner-lehmus

Applying the Gröbner cover, we get rich information.

The Gröbner cover of the Steiner-Lehmus system

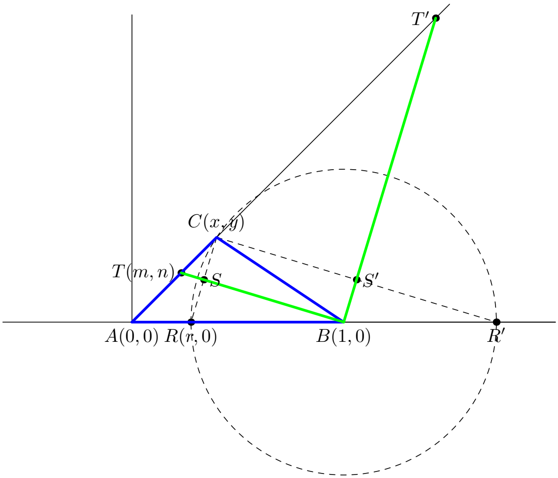


S93. Trying to prove it automatically



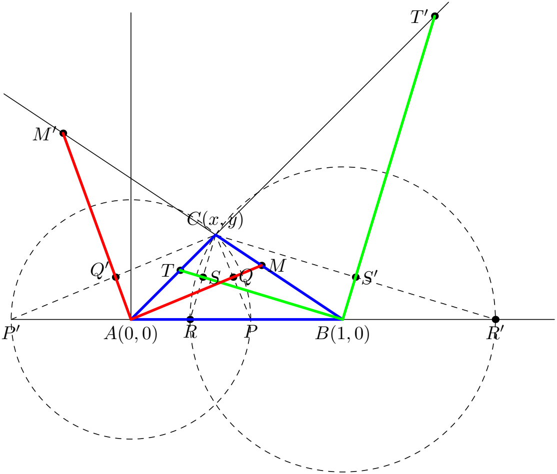
$$x^2 + y^2 = p^2, \begin{vmatrix} 0 & 0 & 1 \\ (x+p)/2 & y/2 & 1 \\ a & b & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 0 & 1 \\ a & b & 1 \\ x & y & 1 \end{vmatrix} = 0,$$

S93. Trying to prove it automatically



$$(1-x)^2+y^2=(1-r)^2, \begin{vmatrix} 1 & 0 & 1 \\ (x+r)/2 & y/2 & 1 \\ m & n & 1 \end{vmatrix}=0, \begin{vmatrix} 0 & 0 & 1 \\ m & n & 1 \\ x & y & 1 \end{vmatrix}=0,$$

S93. Trying to prove it automatically



$$a^2 + b^2 = (1 - m)^2 + n^2$$

One bisector of A equal to one bisector of B.

System of equations:

$$\begin{cases} x^2+y^2-p^2, \\ (a-1)y+b(1-x), \\ -ay+b(x+p), \\ (1-x)^2+y^2-(1-r)^2, \\ my-xn, \\ (1-m)y+(x+r-2)n, \\ a^2+b^2=(1-m)^2+n^2. \end{cases}$$

Parameters: x, y

Variables: a, b, m, n, p, r

Solutions:

	$+$	$-$
p	i_A	e_A
$1-r$	i_B	e_B

One bisector of A equal to one bisector of B.

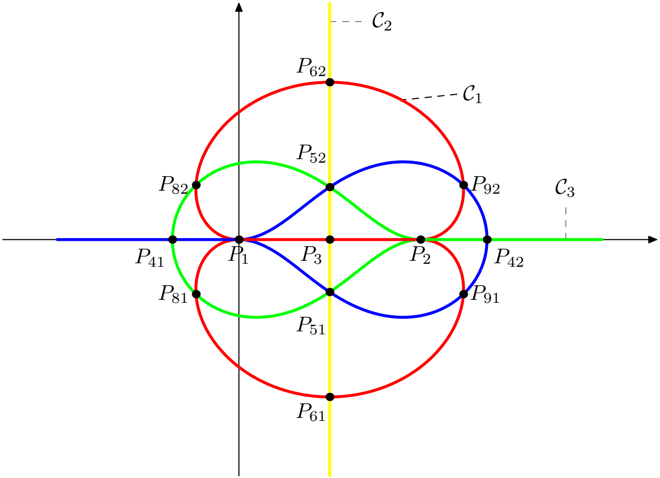
System of equations:

$$\begin{cases} x^2+y^2-p^2, \\ (a-1)y+b(1-x), \\ -ay+b(x+p), \\ (1-x)^2+y^2-(1-r)^2, \\ my-xn, \\ (1-m)y+(x+r-2)n, \\ a^2+b^2=(1-m)^2+n^2. \end{cases}$$

Parameters: x, y Variables: a, b, m, n, p, r

Solutions:

	$+$	$-$
p	i_A	e_A
$1-r$	i_B	e_B



$i_A = i_B,\ e_A = e_B$ $i_A = e_B$

$e_A = e_B$ $e_A = i_B$

The algorithm is used taking $\mathrm{grevlex}(a, b, m, n, p, r)$ order for the variables. The parameters are (x, y).
In the result they appear the following curves:

$$\begin{aligned}
\mathcal{C}_1 &= \mathbb{V}(8x^{10} + 41x^8y^2 + 84x^6y^4 + 86x^4y^6 + 44x^2y^8 + 9y^{10} - 40x^9 \\
&\quad -164x^7y^2 - 252x^5y^4 - 172x^3y^6 - 44xy^8 + 76x^8 + 246x^6y^2 \\
&\quad +278x^4y^4 + 122x^2y^6 + 14y^8 - 64x^7 - 164x^5y^2 - 136x^3y^4 \\
&\quad -36xy^6 + 16x^6 + 31x^4y^2 + 14x^2y^4 - y^6 + 8x^5 + 20x^3y^2 + 12xy^4 \\
&\quad -4x^4 - 10x^2y^2 - 6y^4 + y^2), \\
\mathcal{C}_2 &= \mathbb{V}(2x - 1). \\
\mathcal{C}_3 &= \mathbb{V}(y),
\end{aligned}$$

The Gröbner cover of the Steiner-Lehmus system

and the following varieties representing points (only the real points are represented in the table):

Varieties	Real points
$V_1 = \mathbb{V}(y, x)$	$P_1 = (0, 0)$
$V_2 = \mathbb{V}(y, x - 1)$	$P_2 = (1, 0)$
$V_3 = \mathbb{V}(y, 2x - 1)$	$P_3 = (\frac{1}{2}, 0)$
$V_4 = \mathbb{V}(y, 2x^2 - 2x - 1)$	$P_{4,12} = \left(\frac{1\pm\sqrt{3}}{2}, 0\right)$
$V_5 = \mathbb{V}(12y^2 - 1, 2x - 1)$	$P_{5,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right)$
$V_6 = \mathbb{V}(4y^2 - 3, 2x - 1)$	$P_{6,12} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
$V_7 = \mathbb{V}(4y^4 + 5y^2 + 2, 2x - 1)$	
$V_8 = \mathbb{V}(y^4 + 11y^2 - 1, 5x + 2y^2 + 1)$	$P_{8,12} = \left(2 - \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$
$V_9 = \mathbb{V}(y^4 + 11y^2 - 1, 5x - 2y^2 - 6)$	$P_{9,12} = \left(-1 + \sqrt{5}, \pm\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$

1. Segment with $\mathrm{lpp} = \{1\}$ — Generic segment

Segment: $\mathbb{C}^2 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$

Basis: $\{1\}$

3. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$

Segment: $\mathcal{C}_2 \setminus (V_3 \cup V_5 \cup V_6))$

Basis:

$$\{p + r - 1, (4y^2 - 3)n + (4y)r, (4y^2 - 3)m + 2r, (4y^2 - 3)b + (4y)r,\\ (4y^2 - 3)a - 2r + (-4y^2 + 3), 4r^2 - 8r + (-4y^2 + 3)\}\ .$$

2. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: $\mathcal{C}_1 \setminus (V_1 \cup V_2 \cup V_4 \cup V_5 \cup V_6 \cup V_7 \cup V_8 \cup V_9)$
Basis:

$$
\begin{aligned}
B_2 = \{&(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 + 3y^4 + 5y^2 - 1)r \\
&+(x^5 - 10x^4 + 2x^3y^2 + 17x^3 - 18x^2y^2 - 10x^2 + xy^4 + 17xy^2 - x - 8y^4 - 10y^2 + 2), \\
&(3x^4 - 6x^3 + 6x^2y^2 + 5x^2 - 6xy^2 - 4x + 3y^4 + 5y^2 + 1)p \\
&+(x^5 + 2x^4 + 2x^3y^2 - 7x^3 + 6x^2y^2 + 4x^2 + xy^4 - 7xy^2 - x + 4y^4 + 4y^2), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)n \\
&+(-3x^4y + 6x^3y - 6x^2y^3 - 5x^2y + 6xy^3 - 3y^5 - 5y^3 + y), \\
&(x^5 - 4x^4 + 2x^3y^2 + 5x^3 - 6x^2y^2 + xy^4 + 5xy^2 - x - 2y^4)m \\
&+(-3x^5 + 6x^4 - 6x^3y^2 - 5x^3 + 6x^2y^2 - 3xy^4 - 5xy^2 + x), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)b \\
&+(3x^4y - 6x^3y + 6x^2y^3 + 5x^2y - 6xy^3 - 4xy + 3y^5 + 5y^3 + y), \\
&(x^5 - x^4 + 2x^3y^2 - x^3 - x^2 + xy^4 - xy^2 + 3x + y^4 - y^2 - 1)a \\
&+(2x^5 - 8x^4 + 4x^3y^2 + 12x^3 - 12x^2y^2 - 8x^2 + 2xy^4 + 12xy^2 + 2x - 4y^4 - 4y^2)\}
\end{aligned}
$$

The Gröbner cover of the Steiner-Lehmus system

4. Segment with $\mathrm{lpp} = \{n, b, r^2, p^2, a^2\}$
Segment: $\mathcal{C}_3 \setminus (V_1 \cup V_2)$
Includes the points $V_3 \cup V_4$
Basis: $\{n, b, r^2 - 2r - x^2 + 2x, p^2 - x^2, a^2 - m^2 + 2m - 1\}$

5. Segment with $\mathrm{lpp} = \{n, m, b, a, r^2, p^2\}$
Segment: V_5
Basis:

$$\{2n - 3yr, 4m - 3r, 2b + 3yp - 3y, 4a - 3p - 1, 3r^2 - 6r + 2, 3p^2 - 1\}$$

6. Segment with $\mathrm{lpp} = \{r, p, n, m, b, a\}$
Segment: V_6
Basis: $\{r, p - 1, 2n - y, 4m - 1, 2b - y, 4a - 3\}$

The Gröbner cover of the Steiner-Lehmus system

7. Segment with $\mathrm{lpp} = \{p, n, m, b, a, r^2\}$
Segment: $V_7 \cup V_8$
Basis:

$$
\begin{aligned}
B_7 = \big\{ & (7284y^6 + 88197y^4 - 15633y^2 - 3849)p + (8820y^6 + 97285y^4 \\
& -5905y^2 - 265)r + (-11380y^6 - 103045y^4 + 1425y^2 - 1015), \\
& (116y^6 + 1493y^4 + 2403y^2 + 179)n + (660y)r, \\
& (116y^6 + 1493y^4 + 2403y^2 + 179)m + (-72y^6 - 866y^4 - 1006y^2 - 58)r, \\
& (87932y^6 + 779351y^4 + 109221y^2 - 31747)b + (-35280y^7 - 389140y^5 \\
& +23620y^3 + 1060y)r + (16384y^7 + 59392y^5 + 56832y^3 + 19456y), \\
& (87932y^6 + 779351y^4 + 109221y^2 - 31747)a + (17640y^6 + 194570y^4 \\
& -11810y^2 - 530)r + (-51068y^6 - 786519y^4 - 157349y^2 + 5123), \\
& 660r^2 - 1320r + (-116y^6 - 1493y^4 - 2403y^2 - 179)\big\}.
\end{aligned}
$$

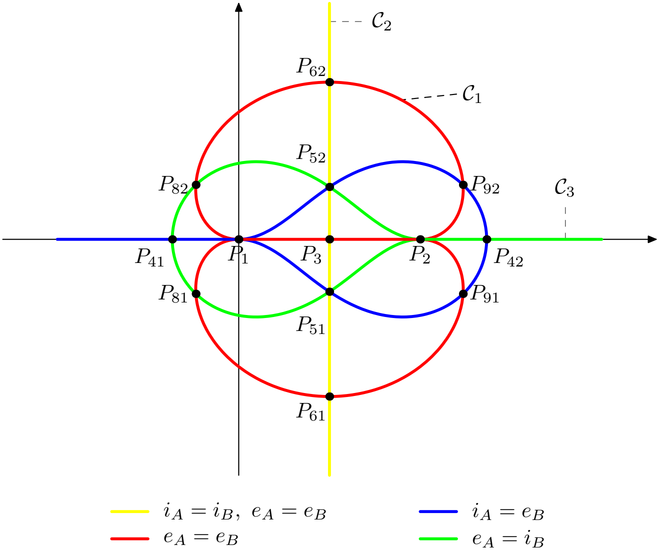
8. Segment with $\mathrm{lpp} = \{r, n, m, b, a, p^2\}$
Segment: V_9
Basis:

$$\{(23y^2-1)r+(-83y^2+6), (134y^2-13)n+(83y^3-6y), (134y^2-13)m+(-268y^2+26), (y^2+3)b+(-5y)p+(5y), (y^2+3)a+(-2y^2-1)p+(y^2-2), 5p^2+(-y^2-8)\}.$$

9. Segment with $\mathrm{lpp} = \{b, r^2, nr, p^2, a^2\}$
Segment: V_1
Basis: $\{b, r^2-2r, nr-2n, p^2, a^2-m^2-n^2+2m-1\}$

10. Segment with $\mathrm{lpp} = \{n, r^2, p^2, bp, a^2\}$
Segment: V_2
Basis: $\{n, r^2-2r+1, p^2-1, bp+b, a^2+b^2-m^2+2m-1\}$

The Gröbner cover of the Steiner-Lehmus system



The classical Steiner-Lehmus theorem enhanced

- Segment 3 corresponds to the mediatrix of side AB except the points $P_{51}, P_{52}, P_{61}, P_{62}, P_3$.
- On segment 2 there are two solutions corresponding to

$$\left.\begin{array}{l} p+r-1 \\ 4r^2-8r+(-4y^2+3) \end{array}\right\} \Rightarrow p = 1-r = \pm\sqrt{1+4y^2}$$

- Thus there are two solutions corresponding to

$$i_A = i_B, \qquad e_A = e_B.$$

Solutions at the special points

$$s_A = p, \quad s_B = 1 - r$$

Point	(s_A, s_B)	Bisectors
P_{51}, P_{52}	$(0.5773502693, 0.5773502693)$	$i_A = i_B$
	$(0.5773502693, -0.577350269)$	$i_A = e_B$
	$(-0.5773502693, 0.5773502693)$	$e_A = i_B,$
	$(-0.5773502693, -0.5773502693)$	$e_A = e_B$
P_{61}, P_{62}	(1,1)	$i_A = i_B$
P_{81}, P_{82}	$(-0.3819659526, -1.272019650)$	$e_A = e_B$
	$(-0.3819659526, 1.272019650)$	$e_A = i_B$
P_{91}, P_{92}	$(-1.272019650, -0.381965976)$	$e_A = e_B$
	$(1.272019650, -0.381965976)$	$i_A = e_B$

Table: Coincidences of bisectors of A and B at the special points

The colors of the curve

Point	Branch	(s_A, s_B)	Bisectors
$(0, .7013671986)$	P_{62}-P_{82}	$(-.7013, -1.2214)$	$e_A = e_B$
$(0, .4190287818)$	P_{52}-P_{82}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.4190287818)$	P_{51}-P_{81}	$(-.4190, 1.0842)$	$e_A = i_B$
$(0, -.7013671986)$	P_{61}-P_{81}	$(-.7013, -1.2214)$	$e_A = e_B$
$(1, .7013671986)$	P_{62}-P_{92}	$(-1.2215, -0.7013)$	$e_A = e_B$
$(1, .4190287818)$	P_{52}-P_{92}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.4190287818)$	P_{51}-P_{91}	$(1.0842, -0.4190)$	$i_A = e_B$
$(1, -.7013671986)$	P_{61}-P_{91}	$(-1.2215, -0.7013)$	$e_A = e_B$

Table: Coincidences of bisectors of A and B at some points of curve $\mathcal{C}_1$.

Generalized Steiner-Lehmus Theorem

Theorem (Generalized Steiner-Lehmus)

Let ABC be a triangle and i_A, e_A, i_B, e_B the lengths of the inner and outer bisectors of the angles A and B. Then, considering the conditions for the equality of some bisector of A and some bisector of B the following excluding situations occur:

- *the triangle ABC is degenerate (i.e. C is aligned with A and B);*
- *ABC is equilateral and then $i_A = i_B$ whereas e_A and e_B become infinite, (P_{61}, P_{62});*
- *point C is in the center of an equilateral triangle, and then $i_A = i_B = e_A = e_B$, (P_{51}, P_{52});*
- *the triangle is isosceles but not of the special form of cases 2) or 3) and then $i_A = i_B \neq e_A = e_B$, (ordinary Theorem);*

continues in the next slice ..

Generalized Steiner-Lehmus Theorem

Theorem (continues)

- $\frac{\overline{AC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, $\frac{\overline{BC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, *and then* $e_A = e_B = i_B$, (P_{81}, P_{82});
- $\frac{\overline{AC}}{\overline{AB}} = \sqrt{\frac{1+\sqrt{5}}{2}}$, $\frac{\overline{BC}}{\overline{AB}} = \frac{3-\sqrt{5}}{2}$, *and then* $e_A = e_B = i_A$, (P_{91}, P_{92});
- *C lies in the curve of degree 10 relative to points* A *and* B *(case of curve* $\mathcal{C}_1$*) passing through all the special points above but is none of these points, and then only one of the following things arrive: either* $e_A = e_B$ *or* $i_A = e_B$ *or* $e_A = i_B$ *depending on the branch of the curve (see Figure, the color representing which of the situations occur);*
- *none of the above cases occur, and then no bisector of* A *is equal to no bisector of* B.

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Topological concepts on the parameter space

Definition

A subset $S \subset \overline{K}^m$ is locally closed, if it is difference of two varieties: $S = \mathbb{V}(M) \setminus \mathbb{V}(N)$.

Definition (Open subset)

A subset $U \subset S$ is said to be open on S if $\overline{S \setminus U} \subsetneq S$.

I-regular functions

Definition (I-Regular function)

Let S be a locally closed subset of $\overline{K}^m$. We call a function $f : S \longrightarrow \overline{K}[\overline{x}]$ I-regular, if $\forall a \in S$ it exists an open $U \subset S$ with $a \in U$ and

$$f(b) = \frac{P(b,\overline{x})}{Q(b)} \text{ for all } b \in U,$$

where $P \in I$ and $Q \in K[\overline{a}]$ and $Q(b) \neq 0$ for all $b \in U$.

Remark

Let P and Q be a polynomials as above, (they are not unique), $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$ and $p(b,\overline{x}) = P(b,\overline{x}) \mod \mathfrak{a}$. If f is monic and $\mathrm{lpp}(f)$ is constant on S, then, for all $b \in U$ is

- $\mathrm{lpp}_{\overline{x}}(p(b,\overline{x})) = \mathrm{lpp}_{\overline{x}}(f)$, *and*
- $\mathrm{lc}_{\overline{x}}(p(b,\overline{x})) = Q(b) \mod \mathfrak{a}$.

Parametric subsets

Definition (Parametric subset of $\overline{K}^m$)

A locally closed subset $S \in \overline{K}^m$ is called parametric (wrt to I and $\succ_{\overline{x}}$) if there exist monic I-regular functions $\{g_1, \ldots, g_s\}$ over S so that $\{g_1(a, \overline{x}), \ldots, g_s(a, \overline{x})\}$ is the reduced Gröbner basis of I_a for all $a \in S$.

Note

Note that the definition immediately implies that if a, b lie in a parametric set S, then $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.
The amazing thing is that the converse also holds if we additionally assume that $I \subset K[\overline{a}][\overline{x}]$ is homogeneous (wrt to the variables).

Wibmer's Theorem

Theorem (M. Wibmer)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal and $a \in \overline{K}^m$. Then the set

$$S_a = \{b \in \overline{K}^m \ : \ \mathrm{lpp}_{\overline{x}}(I_b) = \mathrm{lpp}_{\overline{x}}(I_a)\}$$

is parametric.
In particular, S_a is locally closed.

Definition (Gröbner cover)

By a Gröbner cover of $\overline{K}^m$ wrt to I and $\succ_{\overline{x}}$ we mean a finite set of pairs $\{(S_1,B_1),\dots,(S_r,B_r)\}$ such that

1. *the S_i's are parametric and so, $B_i \subset \mathcal{O}(S_i)[\overline{x}]$ is the reduced Gröbner basis of I over S_i for $i=1,\dots,r$, and*
2. *the union of all S_i's equals $\overline{K}^m$.*

Theorem (Canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a,b \in \overline{K}^m$ lie in the same segment if and only if $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.

Definition (Gröbner cover)

By a Gröbner cover of $\overline{K}^m$ wrt to I and $\succ_{\overline{x}}$ we mean a finite set of pairs $\{(S_1, B_1), \ldots, (S_r, B_r)\}$ such that

1. *the S_i's are parametric and so, $B_i \subset \mathcal{O}(S_i)[\overline{x}]$ is the reduced Gröbner basis of I over S_i for $i = 1, \ldots, r$, and*
2. *the union of all S_i's equals $\overline{K}^m$.*

Theorem (Canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a homogeneous ideal. Then there exists a unique Gröbner cover of $\overline{K}^m$ with minimal cardinality which we call the canonical Gröbner cover. It is disjoint and two points $a, b \in \overline{K}^m$ lie in the same segment if and only if $\mathrm{lpp}_{\overline{x}}(I_a) = \mathrm{lpp}_{\overline{x}}(I_b)$.

Non-homogeneous ideals

Note (Homogenization and dehomogenization)

For homogenization introduce a new variable x_0 and extend $\succ_{\overline{x}}$ to the monomials in $\overline{x}, x_0$ by setting

$$\overline{x}^{\alpha} x_0^i \succ_{\overline{x},x_0} \overline{x}^{\beta} x_0^j \ \text{ iff } \ (\overline{x}^{\alpha} \succ_{\overline{x}} \overline{x}^{\beta}) \text{ or } \ (\overline{x}^{\alpha} = \overline{x}^{\beta} \text{ and } \ i > j)$$

Denote τ the dehomogenization consisting of substituting $x_0 = 1$.

Definition (Affine canonical Gröbner cover)

Let $I \subset K[\overline{a}][\overline{x}]$ be a non-homogeneous ideal and let $J \subset K[\overline{a}][\overline{x}, x_0]$ denote its homogenization. The disjoint Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$ obtained by dehomogenization and reduction will be called the canonical Gröbner cover of $\overline{K}^m$ with respect to I and $\succ_{\overline{x}}$.

Remark

The affine canonical Gröbner cover does not necessarily summarize in a unique segment all the points corresponding to the same lpp. Nevertheless it is canonical, and when two segments occur with the same lpp they correspond to different kind of solutions at infinity.

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Gröbner Cover algorithm

Algorithm (Homogeneous GröbnerCover)

GCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$

  $T :=$**BuildTree**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$. *(Initial disjoint and reduced CGS)*

  $G := \emptyset$

  Group the segments of T *by* lpp*'s:* $T = \{T_i : 1 \leq i \leq s\}$.

    where $T_i = \{(S_{i_j}, B_{i_j}) : 1 \leq j \leq s_i\}$ *with* $\mathrm{lpp}(B_{ij}) = \mathrm{lpp}(B_{ik})$

  For *each* lpp*-segment* T_i

    $S_i :=$**LCUnion**$(S_{i_j} : 1 \leq j \leq s_i)$. *(Summarizing* lpp*-segments)*

    $B_i :=$**Basis**(S_i, T_i). *(Determining the generic basis for* S_i *using* T_i*.)*

    $G := G \cup (S_i, B_i)$

  end for

Return G

Gröbner Cover algorithm

Algorithm (Affine GröbnerCover)

GröbnerCover$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
If F *is homogeneous* **then** $G :=$ **GCover**$(F, \succ_{\overline{x}}, \succ_{\overline{a}})$
else
$F' :=$ **Homogenize**(F, x_0), $\overline{y} := \overline{x}, x_0$, $\succ_{\overline{y}} = \succ_{\overline{x}, x_0}$
$G :=$ **GCover**$(F', \succ_{\overline{y}}, \succ_{\overline{a}})$
$\overline{y} := \overline{x}, 1$, (**Dehomogenize** *the bases in* G)
Reduce *the bases in* G
end if
Extend *the bases in* G *(to obtain a full representation)*
Return G

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Representation of locally closed subsets

Proposition (Canonical representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist uniquely determined radical ideals $\mathfrak{a} \subset \mathfrak{b}$ of $K[\overline{a}]$, with $S = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$, such that

- $\overline{S} = \mathbb{V}(\mathfrak{a})$,
- $\overline{S} \setminus S = \mathbb{V}(\mathfrak{b})$.

The pair $(\mathfrak{a}, \mathfrak{b})$ -top,hole- is called the canonical representation of S.

Representation of locally closed sets

Proposition (Canonical prime representation)

Let $S \subset \overline{K}^m$ be a locally closed set. Then, there exist a unique canonical prime representation of S given the prime components of $\mathfrak{a}$, say $\mathfrak{p}_i$, and associated to each, a set of prime ideals $\mathfrak{p}_{ij}$ (holes) in the form $((\mathfrak{p}_1, (\mathfrak{p}_{11}, \dots, \mathfrak{p}_{1j_1})), \dots, (\mathfrak{p}_k, (\mathfrak{p}_{k1}, \dots, \mathfrak{p}_{kj_k})))$ so that

$$S = \bigcup_{i=1}^{k} \left(\mathbb{V}(\mathfrak{p}_i) \setminus (\bigcup_{j=1}^{j_i} \mathbb{V}(\mathfrak{p}_{ij})) \right).$$

and $\mathfrak{p}_i \subset \mathfrak{p}_{ij}$ for all i, j, such that

- $\overline{S} = \mathbb{V}(\mathfrak{p}_1) \cup \dots \cup \mathbb{V}(\mathfrak{p}_r)$ *and*
- $(\overline{S} \setminus S) \cap \mathbb{V}(\mathfrak{p}_i) = \mathbb{V}(\mathfrak{p}_{i1}) \cup \cdots \cup \mathbb{V}(\mathfrak{p}_{ir_i})$

are the minimal decompositions into irreducible closed sets.

Representation of I-regular functions

Definition (Generic representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \to \overline{K}[\overline{x}]$ a monic I-regular function. We say that $p \in K[\overline{a}][\overline{x}]$ generically represents f if

- $\mathrm{lpp}(f) = \mathrm{lpp}(p)$,
- $\mathrm{lc}(p)(a) \neq 0$ *on an open and dense set of points in* S,
- *if* $\mathrm{lc}(p)(a) \neq 0$ *then* $f(a, \overline{x}) = p(a, \overline{x}) / \mathrm{lc}(p)(a)$, *otherwise is* $p(a, \overline{x}) = 0$.

Proposition

Every monic I-regular function $f : S \to \overline{K}[\overline{x}]$ admits a generic representation.

Representation of I-regular functions

Definition (Full representation)

Let $S \subset \overline{K}^m$ be a locally closed set and $f : S \to \overline{K}[\overline{x}]$ a monic I-regular function. We say that a the set of polynomials $\{p_1, \cdots, p_r\} \subset K[\overline{a}][\overline{x}]$ fully represents f if

- $\mathrm{lpp}(f) = \mathrm{lpp}(p_i)$, *for* $1 \le i \le r$,
- *for* $a \in S$ *and* $1 \le i \le r$ *either* $\mathrm{lc}(p_i)(a) \ne 0$ *or* $p_i(a, \overline{x}) = 0$,
- *for all* $a \in S$ *it exist al least one* i *and an open* $U \subset S$ *such that for every* $b \in U$ *is* $\mathrm{lc}(p_i)(a) \ne 0$ *and* $f(a, \overline{x}) = p(a, \overline{x}) / \mathrm{lc}(p)(a)$.

Proposition

Given a generic representation of a monic I-regular function $f : S \to \overline{K}[\overline{x}]$, the algorithm Extend *computes a full representation.*

Representation of I-regular functions

Example

Let $I = \langle ax+by, cx+dy \rangle$ and F be the monic I-regular function

$$F: \quad S = \mathbb{V}(ad-bc) \setminus \mathbb{V}(a,c) \subset \mathbb{C}^4 \quad \to \quad \mathbb{C}[x,y]$$
$$(a,b,c,d) \quad \mapsto \quad \begin{cases} x + \frac{b}{a}y & \text{if } a \neq 0 \\ x + \frac{d}{c}y & \text{if } c \neq 0 \end{cases}$$

Then

Generic representation of F: $p = ax + by$
Full representation of F: $\{p_1 = ax+by, p_2 = cx+dy\}$