Computing the canonical representation of constructible sets

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Abstract. Constructible sets are needed in many algorithms of Computer Algebra, particularly in the Gröbner Cover and other algorithms for parametric polynomial systems. In this paper we review the canonical form of constructible sets and give algorithms for computing it.

1. Introduction

In the basic paper defining the Gröbner Cover [Montes and Wibmer (2010)] for discussing parametric polynomial systems of equations, we introduced algorithms that have been improved since then. We used our own algorithm BUILDTREE for computing the initial Comprehensive Gröbner System (CGS), needed for the Gröbner Cover, now substituted in the Singular [Decker et al. (2015)] library "grobcov.lib" by the more efficient Kapur-Sun-Wang algorithm [Kapur et al. (2010)]. The algorithm GROBCOV used specially simple locally closed sets, whose union is certified to be also locally closed by Wibmer's theorem [Wibmer (2007)] (algorithm LCUNION).

The Gröbner Cover is used in [Montes and Recio (2014)] for the automatic deduction of geometric theorems. It is also essential for computing geometrical loci and defining a taxonomy of the components of loci in [Abanades et al. (2014)], as well as for envelopes. In general in these tasks, the representation of locally closed sets, i.e. difference of varieties, is sufficient. But for more general applications, where Wibmer's theorem [Wibmer (2007)] is not applicable, the union of locally closed sets is not always locally closed. This is the reason for reviewing here the canonical representation of constructible sets giving algorithms to compute it, as well as to use the new algorithms inside the library for computing higher dimensional geometrical loci's.

Canonical form of constructible sets were already introduced by [Allouche (1996)], in the context of general topology. More recently, [O'Halloran and Schilmoeller (2002)] have given a description of invariant sequences for constructible sets in Zariski topology. The object of this paper is, taken this last description as starting point, to give formulas and algorithms for computing effectively the canonical form of constructible sets.

In Section 2, we give the canonical representation of locally closed sets and an algorithm CREP for computing it, that is central for our purposes. In Section 3, we recall the canonical structure of constructible sets introduced by [O'Halloran and Schilmoeller (2002)], complementing it with dimension characteristics and an effective formula. This formula allows us to give an algorithm in Section 4 to build the canonical representation of constructible sets, using the CREP for locally

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closed sets. In Section 4 we also propose an acceleration method. Finally in Section 5 clarifying examples are given.

Some remarks about notation. All along the paper we shall use the notations $\subseteq$ and $\subset$ to represent inclusion and strict inclusion, respectively. If $r \geq 1$ is an integer the symbol $[r]$ means the set $[r] = \{i \in \mathbb{N} : 1 \leq i \leq r\}$. For a set $S \subseteq \mathbb{C}^n$, the complementary set $\mathbb{C}^n \setminus S$ of S is denoted S^c. Finally $A \uplus B$ means disjoint reunion, that is, $A \cup B$ with the additional information that $A \cap B = \emptyset$. Except in the examples, where the ring is given, all ideals considered are ideals in the ring $\mathbb{Q}[x_1, \ldots, x_n]$.

2. Canonical C-representation of locally closed sets

Consider the ring $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, \ldots, x_n]$ of polynomials in n indeterminates $x_1, \ldots, x_n$ with rational coefficients. If $N \subseteq \mathbb{Q}[\mathbf{x}]$, the *variety* of N is the set

$$\mathbf{V}(N) = \{\mathbf{u} \in \mathbb{C}^n : g(\mathbf{u}) = 0 \text{ for all } g \in N\}.$$

Let $\mathfrak{a} = \text{RAD}(\langle N \rangle)$. Then $\mathbf{V}(N) = \mathbf{V}(\langle N \rangle) = \mathbf{V}(\mathfrak{a})$. The ideal $\mathfrak{a}$ is called the *ideal of the variety* $\mathbf{V}(N)$, and is denoted $\mathfrak{a} = \mathbf{I}(\mathbf{V}(N))$. If $S \subseteq \mathbb{C}^n$, the *closure* of S is the smallest variety containing S, and is denoted $\overline{S}$. The ideal of S, denoted $\mathbf{I}(S)$, is defined by $\mathbf{I}(S) = \mathbf{I}(\overline{S})$. There is a one-to-one correspondence between varieties V and radical ideals $\mathfrak{a}$. For a radical ideal $\mathfrak{a}$ and a variety V, both $\mathbf{I}(\mathbf{V}(\mathfrak{a})) = \mathfrak{a}$ and $\mathbf{V}(\mathbf{I}(V)) = V$ hold.

By taking varieties as closed sets, we have a topology in $\mathbb{C}^n$ called the $\mathbb{Q}$-*Zariski topology* of $\mathbb{C}^n$. For concepts about varieties and the $\mathbb{Q}$-*Zariski topology* not defined here (such as irreducible varieties, irreducible components, dimension of a variety, etc.), we refer to [Cox et al. (1998)].

A set $S \subseteq \mathbb{C}^n$ is *locally closed* if it is the intersection of an open set and a closed set.

Remark 2.1. The concept of locally closed set admits different but equivalent definitions. Indeed, the following conditions are easily shown to be equivalent:

(a) The set S is locally closed;
(b) the set S is the difference of two closed sets;
(c) the set S is open in the closure $\overline{S}$ of S.
(d) the set $\overline{S} \setminus S$ is closed.

Let S be an open (resp. closed) set. As $\mathbb{C}^n$ is closed (resp. open), then $S = S \cap \mathbb{C}^n$ is a locally closed set. Thus, open sets and closed sets are locally closed.

Let S be a locally closed set. As $\overline{S}$ and $\overline{S} \setminus S$ are closed, there exist radical ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\overline{S} = \mathbf{V}(\mathfrak{a})$ and $\overline{S} \setminus S = \mathbf{V}(\mathfrak{b})$. These ideals satify

$$S = \overline{S} \setminus (\overline{S} \setminus S) = \mathbf{V}(\mathfrak{a}) \setminus \mathbf{V}(\mathfrak{b}). \tag{2.1}$$

Taking into account the one-to-one correspondence between radical ideals and varieties, the ideals $\mathfrak{a} = \mathbf{I}(\overline{S})$ and $\mathfrak{b} = \mathbf{I}\left(\overline{S} \setminus S\right)$ are uniquely determined by S. The pair $\text{CREP}(S) = [\mathfrak{a}, \mathfrak{b}]$ is called the *canonical representation* (CREP) *of the locally closed set* S. It is *canonical* in the sense that it does not depend on how the locally closed set S is given: it depends only on S.

Remark 2.2. If $[\mathfrak{a}, \mathfrak{b}] = \text{CREP}(S)$, then S is closed if and only if $\mathfrak{b} = \langle 1 \rangle$.

The following Proposition explains how to obtain $\text{CREP}(S) = [\mathfrak{a}, \mathfrak{b}]$ for a locally closed set S given in the form $S = \mathbf{V}(P) \setminus \mathbf{V}(Q)$ for two ideals P and Q. It uses the decomposition of $\mathbf{V}(P)$ into irreducible varieties, which can be done by [Gianni et al. (1988)] algorithm. Moreover, some additional properties of $[\mathfrak{a}, \mathfrak{b}]$ are given.

Proposition 2.3. *Let P and Q be two ideals, and let $S = \mathbf{V}(P) \setminus \mathbf{V}(Q)$ be a non empty locally closed set. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the prime decomposition of P, and let $J = \{i \in [s] : \mathbf{V}(\mathfrak{p}_i) \not\subseteq (\mathbf{V}(Q))\}$. Then*

(i) $\text{CREP}(S) = [\mathfrak{a}, \mathfrak{b}]$, *where* $\mathfrak{a} = \bigcap_{i\in J} \mathfrak{p}_i$ *and* $\mathfrak{b} = \text{RAD}(\mathfrak{a} + Q)$*;*
(ii) $\overline{S} = \mathbf{V}(\mathfrak{a})$*;*
(iii) $\mathfrak{a} \subset \mathfrak{b}$*;*
(iv) $\dim\ \mathbf{V}(\mathfrak{b}) < \dim\ \mathbf{V}(\mathfrak{a})$.

Proof. (i) For $i \in [s]$ let $V_i = \mathbf{V}(\mathfrak{p}_i)$. Then $\mathbf{V}(P) = V_1 \cup \cdots \cup V_s$ is the decomposition of $\mathbf{V}(P)$ into irreducible varieties. We have

$$S = \mathbf{V}(P) \setminus \mathbf{V}(Q) = \left(\bigcup_{i=1}^{s} V_i\right) \setminus \mathbf{V}(Q) = \bigcup_{i=1}^{s} \left(V_i \setminus (\mathbf{V}(Q) \cap V_i)\right).$$

If $i \in [s] \setminus J$, then $V_i \setminus (\mathbf{V}(Q) \cap V_i) = \emptyset$, and the set $V_i \setminus (\mathbf{V}(Q) \cap V_i)$ can be excluded from the union, obtaining

$$S = \bigcup_{i\in J} \left(V_i \setminus (\mathbf{V}(Q) \cap V_i)\right).$$

For $i \in J$, we have $\mathbf{V}(Q) \cap V_i \subset V_i$. As V_i is irreducible, the closure of $V_i \setminus (\mathbf{V}(Q) \cap V_i)$ is V_i. Therefore,

$$\overline{S} = \bigcup_{i\in J} V_i, \tag{2.2}$$

and

$$\mathfrak{a} = \mathbf{I}(\overline{S}) = \bigcap_{i\in J} \mathbf{I}(V_i) = \bigcap_{i\in J} \mathfrak{p}_i. \tag{2.3}$$

To obtain $\mathfrak{b} = \mathbf{I}(\overline{S} \setminus S)$ note that

$$\begin{aligned} S &= \bigcup_{i\in J} (V_i \setminus (\mathbf{V}(Q) \cap V_i)) = \bigcup_{i\in J} (V_i \setminus \mathbf{V}(Q)) = \left(\bigcup_{i\in J} V_i\right) \setminus \mathbf{V}(Q) = \overline{S} \setminus V(Q), \\ \overline{S} \setminus S &= \overline{S} \setminus (\overline{S} \setminus \mathbf{V}(Q)) = \overline{S} \cap \mathbf{V}(Q) = \mathbf{V}(\mathfrak{a}) \cap \mathbf{V}(Q) = \mathbf{V}(\text{RAD}(\mathfrak{a} + Q)), \end{aligned}$$

so that, $\mathfrak{b} = \mathbf{I}(\overline{S} \setminus S) = \text{RAD}(\mathfrak{a} + Q)$.

(ii) Is a direct consequence of (2.2) and (2.3).

(iii) From $\mathfrak{b} = \text{RAD}(\mathfrak{a} + Q)$, clearly $\mathfrak{b} \supseteq \mathfrak{a}$. Now $\mathfrak{b} = \mathfrak{a}$ implies $S = \emptyset$, a contradiction. Therefore, $\mathfrak{b} \supset \mathfrak{a}$.

(iv) Eliminating the $\mathfrak{p}_j$ for $j \notin J$ and reindexing the ideals, the prime decomposition of $\mathfrak{a}$ can be written $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. For $i \in [r]$, let $\mathfrak{b}_i = \mathfrak{p}_i + \mathfrak{b} = \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_{r_i}}$ be the prime decomposition of $\mathfrak{b}_i$. By (i), $\mathbf{V}(\mathfrak{p}_i) \not\subseteq \mathbf{V}(\mathfrak{b})$ and so $\mathbf{V}(\mathfrak{b}_i) = \mathbf{V}(\mathfrak{b} + \mathfrak{p}_i) \subset \mathbf{V}(\mathfrak{p}_i)$, being $\mathbf{V}(\mathfrak{p}_{ij}) \subset \mathbf{V}(\mathfrak{p}_i)$. As $\mathfrak{p}_i$ and $\mathfrak{p}_{ij}$ are irreducibles we have $\dim\ \mathbf{V}(\mathfrak{p}_{ij}) < \dim\ V(\mathfrak{p}_i)$ for all j, and

$$\begin{aligned} S =& \mathbf{V}(\mathfrak{a}) \setminus \mathbf{V}(\mathfrak{b}) = \bigcup_{i=1}^{r} (\mathbf{V}(\mathfrak{p}_i) \setminus \mathbf{V}(\mathfrak{b} + \mathfrak{p}_i)) \\ =& \bigcup_{i=1}^{r} \left(\mathbf{V}(\mathfrak{p}_i) \setminus \mathbf{V}\left(\bigcap_{j=1}^{r_i} \mathfrak{p}_{ij}\right)\right) = \bigcup_{i=1}^{r} \left(\mathbf{V}(\mathfrak{p}_i) \setminus \bigcup_{j=1}^{r_i} \mathbf{V}(\mathfrak{p}_{ij})\right). \end{aligned}$$

Thus

$$\begin{aligned} \dim V(\mathfrak{b}) &= \max\{\dim \mathbf{V}(\mathfrak{p}_{ij}) : i \in [r],\ j \in [r_i]\} \\ &< \max\{\dim \mathbf{V}(\mathfrak{p}_i) : i \in [r]\} = \dim V(\mathfrak{a}). \end{aligned}$$

□

Proposition 2.3 (i) justifies Algorithm 1 CREP for obtaining $[\mathfrak{a}, \mathfrak{b}]$ from $[P, Q]$.

Corollary 2.4. *Let V and W be varieties and $S = V \setminus W$. If $W \subset V$ and $V = \overline{S}$, then* $\textsc{Crep}(S) = [\mathbf{I}(V), \mathbf{I}(W)]$ *and* $\dim W < \dim V$.

Proof. If $\overline{S} = V$ then $\mathfrak{a} = \mathbf{I}(\overline{S}) = \mathbf{I}(V)$. Moreover

$$\overline{S} \setminus S = (\overline{S} \cap (\overline{S}^c \cup W) = \overline{S} \cap W = W.$$

Thus $\mathfrak{b} = \mathbf{I}(W)$. The dimension relation is a consequence of Proposition 2.3. □

$T \leftarrow$ **Crep**(P, Q)
`Input:`
  $[P, Q]$: a pair of ideals representing the set $S = \mathbf{V}(P) \setminus \mathbf{V}(Q)$
`Output:`
  $[\mathfrak{a}, \mathfrak{b}]$ the C-representation of S

begin
  $Q = Q + P$
  $\mathfrak{a} = \langle 1 \rangle$
  $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} = \textsc{PrimeDecomp}(P)$
  for $i = 1$ to s **do**
    if $Q \not\subseteq \mathfrak{p}_i$ **then**
      $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{p}_i$
    end if
  end for
  $\mathfrak{b} = \textsc{Rad}(Q + \mathfrak{a})$
  return$([\mathfrak{a}, \mathfrak{b}])$
end

ALGORITHM 1. CREP algorithm

3. Canonical representation of constructible sets

A set $S \subseteq \mathbb{C}^n$ is *constructible* if it is a finite union of locally closed sets. In particular, locally closed sets are constructible. Constructible sets appear naturally in solving parametric polynomial systems of equations. Many authors give special representations for constructible sets [Leykin (2001), O'Halloran and Schilmoeller (2002), Kemper (2007), Chen et al. (2008a), Chen et al. (2008b)], [Chen et al. (2009)], adequate for its goals. Our goal is developing the invariant sequence of a constructible set described in [O'Halloran and Schilmoeller (2002)] setting the outlook on its effective computation, to generalize the CREP of a locally closed set.

Next lemma recalls the behaviour of locally closed sets and constructible sets respect to union, intersection and complementation. We omit the proofs which are straightforward.

Lemma 3.1. .

(i) *If S is locally closed, then S^c is constructible;*
(ii) *If S_1 and S_2 are locally closed, then $S_1 \cup S_2$ is constructible and $S_1 \cap S_2$ is locally closed;*
(iii) *If S_1 is locally closed and S_2 is constructible, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are constructible;*
(iv) *If S_1 and S_2 are constructible, then $S_1 \cup S_2$ and $S_1 \cap S_2$ are constructible.*
(v) *if S is constructible, then S^c is constructible.*
(vi) *if S_1 and S_2 are constructible, then $S_1 \setminus S_2$ is constructible.*

In the following $\mathcal{L}$ denotes be the family of locally closed sets and $\mathcal{C}$ the family of constructible sets.

Remark 3.2. According to Lemma 3.1, if S_1 and S_2 are constructible, then $S_1 \cup S_2$, $S_1 \cap S_2$ and S_1^c are constructible sets, too. Then $\mathcal{C}$ is a Boolean algebra of subsets of $\mathbb{C}^n$ containing $\mathcal{L}$. On the other hand, if a Boolean algebra $\mathcal{A}$ contains $\mathcal{L}$ then it must contain the finite union of locally closed sets, that is, $\mathcal{C} \subseteq \mathcal{A}$. We conclude that $\mathcal{C}$ is the Boolean algebra generated by $\mathcal{L}$. Let $\mathcal{T}$ the family union of the family of open sets and the family of closed sets. The boolean algebra generated by $\mathcal{T}$ contains $\mathcal{L}$, so $\mathcal{C}$ is also the boolean algebra generated by $\mathcal{T}$.

The first step of the construction of the canonical structure of the constructible set S given as a union of locally closed sets is to separate S into two disjoint sets: $S = L \uplus C$ where L is largest locally closed set included in S and C its complement respect to $\overline{S}$. Having this in mind we define:

$$\mathbf{C}(S) = \overline{S} \setminus S, \quad \mathbf{L}(S) = \overline{S} \setminus \overline{\mathbf{C}(S)},$$

(If the set S is clear from the context, we often write C and L instead of $\mathbf{C}(S)$ and $\mathbf{L}(S)$ respectively).

If $S \in \mathcal{C}$, then, $\overline{S}$ and S^c are constructible and $\mathbf{C}(S) = \overline{S} \setminus S$ is a difference of constructibles, so it is a constructible set. Thus, the map

$$\begin{array}{rcl} \mathbf{C}\colon \mathcal{C} & \to & \mathcal{C} \\ S & \mapsto & \mathbf{C}(S) = \overline{S} \setminus S \end{array}$$

is well defined. Note:

(i) $\overline{S} = \mathbf{C}(S) \uplus S$;
(ii) S is closed if and only if $\mathbf{C}(S) = \emptyset$;
(iii) S is locally closed if and only if $\mathbf{C}(S)$ is closed.

The set $\mathbf{L}(S) = \overline{S} \setminus \overline{C}$ (where $C = \mathbf{C}(S)$) is a difference of closed sets, so it is a locally closed set. Then,

$$\begin{array}{rcl} \mathbf{L}\colon \mathcal{C} & \to & \mathcal{L} \\ S & \mapsto & \mathbf{L}(S) = \overline{S} \setminus \overline{C} \end{array}$$

is a well defined map. Clearly $\overline{S} = \mathbf{L}(S) \uplus \overline{C}$. Moreover, $\mathbf{L}(S) \subseteq S$. Indeed,

$$\mathbf{L}(S) = \overline{S} \setminus \overline{C} = \overline{S} \setminus \left(\overline{\overline{S} \setminus S}\right) \subseteq \overline{S} \setminus \left(\overline{S} \setminus S\right) = S.$$

For a constructible set S, the set $\mathbf{L}(S)$ can be characterized as the largest locally closed set included in S.

We give now a Proposition that determines an explicit expression of C as a union of locally closed sets in terms of the input expression of S.

Proposition 3.3. *Let $S = S_1 \cup \cdots \cup S_r$ be a constructible set with each S_i a locally closed set. For $i \in [r]$ let* $\textsc{CRep}(S_i) = [\mathfrak{a}_i, \mathfrak{b}_i]$, $V_i = \mathbf{V}(\mathfrak{a}_i)$ *and* $W_i = \mathbf{V}(\mathfrak{b}_i)$. *Then,*

$$\begin{aligned} C = \overline{S} \setminus S &= \bigcup_{T \subset [r]} \left(\left(\bigcap_{j \in T} V_j^c \right) \cap \left(\bigcap_{j \notin T} W_j \right) \right) \\ &= \bigcup_{T \subset [r]} \left(\left(\bigcap_{j \notin T} W_j \right) \setminus \left(\bigcup_{j \in T} V_j \right) \right). \end{aligned} \tag{3.1}$$

Proof. We have

$$\begin{aligned} S =&(V_1 \setminus W_1) \cup \cdots \cup (V_r \setminus W_r) = (V_1 \cap W_1^c) \cup \cdots \cup (V_r \cap W_r^c) \\ =& \bigcap_{T\subseteq[r]} \left(\left(\bigcup_{j\in T} V_j \right) \cup \left(\bigcup_{j\notin T} W_j^c \right) \right), \end{aligned}$$

and thus

$$S^c = \bigcup_{T\subseteq[r]} \left(\left(\bigcap_{j\in T} V_j^c \right) \cap \left(\bigcap_{j\notin T} W_j \right) \right).$$

For a subset $T \subseteq [r]$, let

$$Z_T = \left(\bigcap_{j\in T} V_j^c \right) \cap \left(\bigcap_{j\notin T} W_j \right),$$

so that $S^c = \bigcup_{T\subseteq[r]} Z_T$. With this notation, the equality to prove is $\overline{S} \setminus S = \bigcap_{T\subset[r]} Z_T$. For a set $T \subseteq [r]$ and an index $\ell \in T$ we have

$$V_\ell \cap Z_T \subseteq V_\ell \cap \bigcap_{j\in T} V_j^c \subseteq V_\ell \cap V_\ell^c = \emptyset,$$

(in particular, $V_\ell \cap Z_{[r]} = \emptyset$) and, if $\ell \notin T$, then $W_\ell \subset V_\ell$ and

$$V_\ell \cap \bigcap_{j\notin T} W_j = \bigcap_{j\notin T} W_j,$$

and we have $V_\ell \cap Z_T = Z_T$. Therefore, by using the distributive law,

$$\begin{aligned} \overline{S} \setminus S =&(V_1 \cup \cdots \cup V_r) \cap S^c = (V_1 \cup \cdots \cup V_r) \cap \bigcup_{T\subseteq[r]} Z_T \\ =&\bigcup_{\ell=1}^{r} \bigcup_{T\subseteq[r]} (V_\ell \cap Z_T) = \bigcup_{T\subset[r]} Z_T. \end{aligned}$$

□

Proposition 3.3 provides an explicit formula of $C = \overline{S} \setminus S$, as a union of locally closed sets. We can compute the CREP of each one of these subsets of C and obtain an expression that allows us to handle $C \subset \overline{S}$ in the same way as we have done with S. This provides an iterative method to build the canonical representations of S. Next Proposition summarizes the basic properties of the first step in the recursive construction.

Proposition 3.4. *Let $S \neq \emptyset$ be a constructible set, $C = \mathbf{C}(S)$, $L = \mathbf{L}(S)$, $\mathfrak{a} = \mathbf{I}(S)$ and $\mathfrak{b} = \mathbf{I}(C)$. Then,*

(i) $C \subset \overline{S}$;
(ii) $\overline{C} \subset \overline{S}$;
(iii) $\overline{S} = \overline{L}$;
(iv) $[\mathfrak{a}, \mathfrak{b}] = [\mathbf{I}(S), \mathbf{I}(C)] = [\mathbf{I}(\overline{S})], \mathbf{I}(\overline{C})]$ *is the C-representation of L.*
(v) $\dim C < \dim S$.

Proof. (i) Let $S = S_1 \cup \cdots \cup S_r$ with S_i locally closed. For $i \in [r]$, let $\textsc{Crep}(S_i) = [\mathfrak{a}_i, \mathfrak{b}_i]$, $V_i = \mathbf{V}(\mathfrak{a}_i)$ and $W_i = \mathbf{V}(\mathfrak{b}_i)$. Then, $S = \bigcup_{i=1}^{r}(V_i \setminus W_i)$ with $W_i \subset V_i$. By taking closures it results $\overline{S} = \bigcup_{i=1}^{r} V_i$. Now, from formula (3.1) of Proposition 3.3 it results

$$C \subseteq \bigcup_{i=1}^{r} W_i \subset \bigcup_{i=1}^{r} V_i = \overline{S}.$$

(ii) Taking closures in the preceding expression, it results

$$\overline{C} \subseteq \bigcup_{i=1}^{r} W_i \subset \bigcup_{i=1}^{r} V_i = \overline{S}.$$

(iii) From $\overline{C} \subseteq \bigcup_{j=1}^{r} W_j$ we have

$$L = \overline{S} \setminus \overline{C} \supseteq \overline{S} \setminus \left(\bigcup_{j=1}^{r} W_j\right) = \left(\bigcup_{i=1}^{r} V_i\right) \setminus \left(\bigcup_{j=1}^{r} W_j\right) = \bigcup_{i=1}^{r}\bigcup_{k=1}^{r_i}\left(V_{ik} \setminus \bigcup_{j=1}^{r} W_j\right),$$

where $V_i = \bigcup_{k=1}^{r_i} V_{ik}$ is the decomposition of V_i into irreducible varieties. If some irreducible variety V_{ik} of V_i of the segment i is cancelled by some W_j of a segment j, i.e. $W_j \supseteq V_{ik}$, then $V_j \supset W_j \supseteq V_{ik}$, and in this case the variety V_{ik} is included in V_j. So, V_{ik} does not cancel in the closure of L nor of S. Thus $\overline{L} \supseteq \bigcup_{i=1}^{r} V_i = \overline{S}$. As $L \subseteq S$ we also have $\overline{L} \subseteq \overline{S}$, and the inclusion is proved.

(iv), (v) From (ii) and (iii) the expression $L = \overline{S} \setminus \overline{C}$ satisfies the conditions of Corollary 2.4, and thus (iv) and (v) follow. □

We proceed now to describe the method for obtaining the canonical representation. Let S be a constructible set. Define the sequence (A_i) by

$$A_1 = S, \quad A_{i+1} = \mathbf{C}(A_i).$$

By Proposition 3.4 (ii) and (v), if $A_i \neq \emptyset$, we have $\overline{A_i} \supset \overline{A_{i+1}}$ and $\dim \overline{A_i} > \dim \overline{A_{i+1}}$. Therefore, there exists an integer $r \geq 1$ such that $A_{r+1} = \emptyset$ and A_r is closed. Consider the finite sequences

$$\begin{aligned} &S = A_1, A_2, \ldots, A_r, \\ &\overline{S} = \overline{A_1} \supset \overline{A_2} \supset \cdots \supset \overline{A_r}, \\ &\dim(S) = \dim(A_1) > \dim(A_2) > \ \ldots \ > \dim(A_r). \end{aligned} \tag{3.2}$$

By construction $A_2 = \mathbf{C}(A_1) = \overline{S} \setminus S$ is disjoint with $S = A_1$. But $A_3 = \overline{A_2} \setminus A_2$ is disjoint with A_2 and a subset of S. Thus, have two decreasing and disjoint subsequences

$$\begin{aligned} S =& A_1 \supset A_3 \supset \cdots \supset A_{2s\pm1}, \\ C =& A_2 \supset A_4 \supset \cdots \supset A_{2s}. \end{aligned}$$

Aplying $\mathbf{L}$ to sequence (3.2), i.e. $L_i = \mathbf{L}(A_i)$, we get a new sequence of disjoint locally closed sets that can be divided into two subsequences, considering the odd and even elements as follows:

$$S = L_1 \uplus L_3 \uplus \cdots \uplus L_{2s\pm1}, \tag{3.3}$$

$$C = L_2 \uplus L_4 \uplus \cdots \uplus L_{2s}. \tag{3.4}$$

The odd disjoint locally closed subsets $L_1, L_3 \ldots L_{2s-1}$ in which S is decomposed by the above procedure form the *canonical structure of the constructible set* S and is independent of the initially given locally closed sets defining S. We also obtain the canonical structure of the complement $C = \overline{S} \setminus S$ as the union of the even locally closed subsets $L_2 \cup L_4 \uplus \cdots \uplus L_{2s}$. From them it is obvious how to obtain the *canonical representation* of S and C whose levels are already given by their $\textsc{Crep}$'s.

```
L ← FistLevel(A)
Input:
A = {[a_1, b_1], ..., [a_r, b_r]}
   a set of CREP's of the segments defining a constructible set S
Output:
   C: a set of CREP's the segments defining C(S) and
   L = L(S): the CREP of the first level of S

begin
  Ā = ⋂_{i=1}^r a_i
  P = ⟨1⟩, Q = ⟨0⟩, C̄ = ⟨1⟩
  for all T ⊂ [r] do
    for j ∈ [r] do
      if j ∈ T then Q = Q + b_j
      else P = P ∩ a_j
      end if
    end for
    K = CREP(Q, P)
    C = APPEND(K to C)
    C̄ = C̄ ∩ K_1
  end do
  C = SIMPLIFYUNION(C)     # for reducing terms
  L = [Ā, C̄]               # it is unnecessary to compute the CREP
  return ([C, L])
end
```

ALGORITHM 2. FIRSTLEVEL algorithm

For $i \in [r]$, define the ideals $\mathfrak{a}_i = \mathbf{I}(\overline{A_i})$. By using Proposition 3.4 (iv) and (v) it results

$$
\begin{aligned}
&L_i = \mathbf{V}(\mathfrak{a}_i) \setminus \mathbf{V}(\mathfrak{a}_{i+1}),\\
&\dim \mathbf{V}(\mathfrak{a}_i) > \dim \mathbf{V}(\mathfrak{a}_{i+1}),\\
&\mathbf{I}(S) = \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_r \subset \mathfrak{a}_{r+1} = \langle 1 \rangle,\\
&\overline{S} = \mathbf{V}(\mathfrak{a}_1) \supset \mathbf{V}(\mathfrak{a}_2) \supset \mathbf{V}(\mathfrak{a}_3) \supset \cdots \supset \mathbf{V}(\mathfrak{a}_r) \supset \mathbf{V}(\mathfrak{a}_{r+1}) = \emptyset.
\end{aligned}
$$

Remark 3.5. In $\mathbb{Q}[x_1, \ldots, x_n]$, Taking into account the decreasing dimensions of the levels of a constructible set we have

(i) The maximum number of levels of S and C is $n + 1$, that will occur when

$$\dim(L_1) = n,\ \dim(L_2) = n - 1,\ \dim(L_3) = n - 2,\ \ldots,\ \dim(L_{n+1}) = 0.$$

(ii) The maximum number of levels of S is $\lceil \frac{n+1}{2} \rceil$.

(iii) $\dim(L_{2i-1}) \geq \dim(L_{2i+1}) + 2$.

4. Algorithms for obtaining the canonical representation of a constructible set

The algorithms work with ideals, whereas the definitions of $\mathbf{C}$ and $\mathbf{L}$ as well as the formulas given in the previous sections are given in varieties. To set down the algorithms we must consider the one-to-one correspondence between ideals of varieties and varieties.

```
[L, C] ← ConsLevels(A)
Input:
A = {[a_1, b_1], ..., [a_r, b_r]}
   a set of CREP's of the segments of a constructible set S
Output: [L, C]
   L: the set of CREP's of the canonical locally closed levels of S
   C: the set of CREP's of the canonical locally closed levels of C = S̄ \ S

begin
 ℓ = 0                                  # ℓ = level
 B = A, L = ∅, C = ∅
 while B ≠ ∅ do
   ℓ = ℓ + 1
   K = FIRSTLEVEL(B)                     # K = [C_ℓ, [a_ℓ, a_{ℓ+1}]]
   if ℓ mod 2 = 1 then
     L = APPEND([ℓ, K_2] to L)
   else
     C = APPEND([ℓ, K_2] to C)
   end if
 end do
 return([L, C])
end
```

ALGORITHM 3. CONSLEVELS algorithm

To flexibilize language, if $S = S_1 \cup \cdots \cup S_r$ is a constructible set with each S_i locally closed, we call the sets S_i the *segments* of S in the expression $S = S_1 \cup \cdots \cup S_r$.

Algorithm 2 FIRSTLEVEL corresponds to Proposition 3.3. Given a constructible set S, we apply the algorithm CREP to its segments; the resulting set of pairs of ideals is the input of FIRSTLEVEL.

FIRSTLEVEL applied to A_i returns in fact $[A_{i+1}, \mathbf{L}(A_i)]$, following Proposition 3.3, being A_{i+1} given by the set of CREP's of its segments and $\mathbf{L}(A_i)]$ being already CREP$(\mathbf{L}(A_i))$.

The procedure CONSLEVELS applies FIRSTLEVEL to $S = A_1$ in its first step, obtaining $[A_2, \mathbf{L}(A_1)]$. Then it takes iteratively the first argument A_i as input for the next call to FIRSTLEVEL and separates alternatively the second element (the level $\mathbf{L}(A_i)$) into the odd and even levels to form the unions (3.3) and (3.4) of S and C respectively.

Moreover, the algorithms can be accelerated. Formula (3.1) of Proposition 3.3 for computing the complement $C = \mathbf{C}(S) = \overline{S} \setminus S$ can contain many terms as CREP's of locally closed sets, as it considers all the subsets of $[r]$. Observe that if there are two different segments of C such that CREP$(S_i) = [\mathfrak{a}_i, \mathfrak{b}_i]$ and CREP$(S_j) = [\mathfrak{a}_j, \mathfrak{b}_j]$ are such that $\mathfrak{b}_i = \mathfrak{a}_j$, then

$$S_i \cup S_j = (\mathbf{V}(\mathfrak{a}_i) \setminus \mathbf{V}(\mathfrak{b}_i)) \cup (\mathbf{V}(\mathfrak{b}_i) \setminus \mathbf{V}(\mathfrak{b}_j)) = \mathbf{V}(\mathfrak{a}_i) \setminus \mathbf{V}(\mathfrak{b}_j)$$

so that CREP$(S_i \cup S_j) = [\mathfrak{a}_i, \mathfrak{b}_j]$. This can be tested for every (i, j). After this process it can appear more than one segment that has become closed. All them can be summarized into a single one taking the intersection of the corresponding ideals of varieties. Doing so we can reduce the number of segments in C which will results in an acceleration of the algorithm CONSLEVELS. The acceleration algorithm 4 SIMPLIFYUNION is to be used inside FIRSTLEVEL after obtaining C.

$A' \leftarrow$ **SimplifyUnion**(A)
`Input:`
$A = \{[\mathfrak{p}_1, \mathfrak{q}_1], \ldots, [\mathfrak{p}_r, \mathfrak{q}_r]\}$
a set of CREP's of the locally closed sets defining C
`Output:` A'
A': a simpler set of CREP's of the C

begin
$A' = A$
for $i \in [r]$ **do**
for $j \in [r],\ j \neq i$ **do**
if $A'_{i,2} = A'_{j,1}$ **do** $A'_i = [A'_{i,1}, A'_{j,2}]$; DELETE$(A'_j)$ **end if**
end for
end for
$J = \{j \in A' : A'_{j2} = 1\}$
$\mathfrak{p} = \bigcap_{j \in J} A'_{j,2}$
DELETE$(A'_j$ for all $j \in J)$
$A' =$ APPEND$([\mathfrak{p}, \langle 1 \rangle]$ to A')
return(A')
end

ALGORITHM 4. SIMPLIFYUNION algorithm

5. Examples

We have implemented algorithms FIRSTLEVEL and CONSLEVELS (as well as the acceleration routine SIMPLIFYUNION) in Singular. They will be next included in the reformed GROBCOV library. We show here some examples of adding locally closed sets to obtain the canonical representation of the constructible.

Example 1. The first example is a simple geometric problem in 3-dimensional space with a nice geometrical interpretation.

Consider the constructible set $S = S_1 \cup S_2 \cup S_3$, where

$$S_1 = \mathbf{V}(x^2 + y^2 + z^2 - 1) \setminus \mathbf{V}(z, x^2 + y^2 - 1),$$
$$S_2 = \mathbf{V}(y, x^2 + z^2 - 1) \setminus \mathbf{V}(z(z+1), y, x + z + 1),$$
$$S_3 = \mathbf{V}(x) \setminus \mathbf{V}(5z - 4, 5y - 3, x).$$

The set S_1 is a sphere minus a maximum circle, S_2 is a maximum circle minus two points and S_3 is a plane minus one point. Applying CONSLEVELS to them the result is:

$$L_1 = \mathbf{V}(x(x^2 + y^2 + z^2 - 1)) \setminus \mathbf{V}(z, x^2 + y^2 - 1),$$
$$C_2 = \mathbf{V}(z, x^2 + y^2 - 1) \setminus \mathbf{V}(z, y(y^2 - 1), x + y^2 - 1),$$
$$L_3 = \mathbf{V}(z, y, x - 1).$$

The canonical representations of S and C are

$$S = L_1 \uplus L_3, \;\; C = \overline{S} \setminus S = C_2.$$

As expected from the geometrical interpretation. S_2 is completely included in S_1 except for point $P = \mathbf{V}(z, y, x - 1) = L_3$. Point P is not in S_1 because it is in the circle retrieved from the sphere, and cannot be included in L_1 because it does not form a locally closed set with L_1. This is the reason for its appearance in next level. Moreover, S_3 is completely included in L_1, and its hole is included in S_1 and in L_1.

Example 2. We consider now the system of equations of the Romin robot [Gonzalez and Recio (1993)] in the context of the computation of its Gröbner Cover [Montes and Wibmer (2010)].

Consider the ring $R = \mathbb{Q}(e,f,a,b,c,d)[c_3,s_3,c_2,s_2,c_1,s_1]$. The parametric polynomial system of equations from the Romin robot is determined by the ideal

$$\begin{aligned} S = \langle & a + ds_1, \\ & b - dc_1, \\ & ec_2 + fc_3 - d, \\ & es_2 + fs_3 - c, \\ & s_1^2 + c_1^2 - 1, \\ & s_2^2 + c_2^2 - 1, \\ & s_3^2 + c_3^2 - 1\rangle. \end{aligned}$$

The first step is to compute a CGS (Comprehensive Gröbner System). Using Kapur-Sun-Wang algorithm [Kapur et al. (2010)], the parameter space is divided into 22 disjoint segments, and for each segment a basis specializing to the reduced Gröbner basis on the whole segment is given. Here we are interested only in the segments. We give them in CREP and order them in decreasing dimension:

Num.	$\mathfrak{a}$	$\mathfrak{b}$	dim.
1.	0	$\langle acd(a^2+b^2)-d^2\rangle$	6
2.	$\langle a^2+b^2-d^2\rangle$	$\langle a^2+b^2-d^2, efd(c^2+d^2)\rangle$	5
3.	$\langle d, a^2+b^2\rangle$	$\langle d, a^2+b^2, efbc, efac\rangle$	4
4.	$\langle a^2+b^2-d^2, e\rangle$	$\langle a^2+b^2-d^2, fd(f^2-c^2-d^2), e\rangle$	4
5.	$\langle a^2+b^2-d^2, f\rangle$	$\langle a^2+b^2-d^2, f, ed(e^2-c^2-d^2)\rangle$	4
6.	$\langle c^2+d^2, a^2+b^2-d^2\rangle$	$\langle c^2+d^2, b, a^2+b^2-d^2,$ $efd(e^2-f^2), efd(e^2-f^2)\rangle$	4
7.	$\langle d, b, a\rangle$	$\langle d, b, a, efc\rangle$	3
8.	$\langle d, a^2+b^2, ec\rangle$	$\langle d, a^2+b^2, fb, fa, ec\rangle$	3
9.	$\langle d, a^2+b^2, f\rangle$	$\langle d, a^2+b^2, f, eb, ea\rangle$	3
10.	$\langle a^2+b^2-d^2,$ $f^2-c^2-d^2, e\rangle$	$\langle d(c^2+d^2), a^2+b^2-d^2,$ $fd, f^2-c^2-d^2, e\rangle$	3
11.	$\langle a^2+b^2-d^2, f, e\rangle$	$\langle d(c^2+d^2), a^2+b^2-d^2, f, e\rangle$	3
12.	$\langle a^2+b^2-d^2, f,$ $e^2-c^2-d^2\rangle$	$\langle d(c^2+d^2), a^2+b^2-d^2,$ $f, ed, e^2-c^2-d^2\rangle$	3
13.	$\langle c^2+d^2, a^2+b^2-d^2,$ $e^2-f^2\rangle$	$\langle c^2+d^2, a^2+b^2-d^2, fd,$ $fc, ed, ec, e^2-f^2\rangle$	3
14.	$\langle d, b, a, ec\rangle$	$\langle d, b, a, fc(f^2+c^2), ec,$ $f(e^2-f^2+c^2)\rangle$	2
15.	$\langle d, b, a, f\rangle$	$\langle d, b, a, f, e(e^2-c^2)\rangle$	2
16.	$\langle d, a^2+b^2, f, e\rangle$	$\langle d, bc, ac, a^2+b^2, f, e\rangle$	2
17.	$\langle c^2+d^2, a^2+b^2-d^2, f, e\rangle$	$\langle d, c, a^2+b^2, f, e\rangle$	2
18.	$\langle d, b, a, c(f^2-c^2), ec,$	$\langle d, c, b, a, f, e, e^2-f^2+c^2\rangle$	1
19.	$\langle d, b, a, f, e\rangle$	$\langle d, c, b, a, f, e\rangle$	1
20.	$\langle d, c, a, f, e^2-c^2\rangle$	$\langle d, c, b, a, f, e\rangle$	1
21.	$\langle d, c, a^2+b^2, f, e\rangle$	$\langle d, c, b, a, f, e\rangle$	1
22.	$\langle d, c, b, a, f, e\rangle$	$\langle 1\rangle$	0

Using ConsLevels to add the segments with equal dimension we also find locally closed sets. In the computation, the following varieties appear:

$$
\begin{aligned}
V_1 =&\mathbf{V}(0),\\
V_2 =&\mathbf{V}(a^2+b^2-d^2),\\
V_3 =&\mathbf{V}(a^2+b^2-d^2, efd(c^2+d^2),\\
V_4 =&\mathbf{V}(a^2+b^2-d^2, fd(f2c^2+f^2d^2-c^4-d^4), efd(c^2+d^2), efb(c^2+d^2),\\
&\quad efa(c^2+d^2), fd(e^2-f^2+c^2+d^2), ed(e^2-f^2-c^2-d^2),\\
&\quad efbc(e^2-f^2), efac(e^2-f^2)),\\
V_5 =&\mathbf{V}(d(c^2+d^2), a^2+b^2-d^2, fd, fb, fa, ed, eb, ea, efc),\\
V_6 =&\mathbf{V}(d, bc, ac, a^2+b^2, fb, fa, fc(f^2-c^2), eb, ea, efc, f(e^2-f^2+c^2)),\\
V_7 =&\mathbf{V}(d, c, b, a, f, e),\\
V_8 =&\mathbf{V}(1).
\end{aligned}
$$

We have:

$$
\begin{aligned}
&\biguplus\{S : \dim S = 6\} = V_1 \setminus V_2, && \biguplus\{C : \dim C = 6\} = V_2 \setminus V_8,\\
&\biguplus\{S : \dim S = 5\} = V_2 \setminus V_3, && \biguplus\{C : \dim C = 5\} = V_3 \setminus V_8,\\
&\biguplus\{S : \dim S = 4\} = V_3 \setminus V_4, && \biguplus\{C : \dim C = 4\} = V_4 \setminus V_8,\\
&\biguplus\{S : \dim S = 3\} = V_4 \setminus V_5, && \biguplus\{C : \dim C = 3\} = V_5 \setminus V_8,\\
&\biguplus\{S : \dim S = 2\} = V_5 \setminus V_6, && \biguplus\{C : \dim C = 2\} = V_6 \setminus V_8,\\
&\biguplus\{S : \dim S = 1\} = V_6 \setminus V_7, && \biguplus\{C : \dim C = 1\} = V_7 \setminus V_8,\\
&\biguplus\{S : \dim S = 0\} = V_7 \setminus V_8, && \biguplus\{C : \dim C = 0\} = \emptyset.
\end{aligned}
$$

But if we use ConsLevels to add separately the even-dimension and the odd-dimension ones then the results are

$$
\begin{aligned}
&\biguplus\{S : \dim S = 0 \pmod 2\} = (V_1 \setminus V_2) \uplus (V_3 \setminus V_4) \uplus (V_5 \setminus V_6) \uplus (V_7 \setminus V_8)\,,\\
&\biguplus\{C : \dim C = 0 \pmod 2\} = (V_2 \setminus V_3) \uplus (V_4 \setminus V_5) \uplus (V_6 \setminus V_7)\,,\\
&\biguplus\{S : \dim S = 1 \pmod 2\} = (V_2 \setminus V_3) \uplus (V_4 \setminus V_5) \uplus (V_6 \setminus V_7)\,,\\
&\biguplus\{C : \dim C = 1 \pmod 2\} = (V_3 \setminus V_4) \uplus (V_5 \setminus V_6) \uplus (V_7 \setminus V_8)\,,
\end{aligned}
$$

which are not locally closed and have respectively 4 and 3 proper levels.

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