Generalizing the Steiner-Lehmus Theorem using the Gröbner Cover

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Abstract

In this note we present an application of a new method (the Gröbner Cover method, to algorithmically discuss parametric polynomial systems of equations) in the realm of automatic discovery of theorems in elementary geometry. Namely, we describe how the Gröbner Cover is particularly well suited to yield the missing hypothesis for a given geometric statement to hold true. This is achieved by addressing the following problem: find those triangles that have at least two bisectors of equal length. The case of two inner bisectors is the well known, XIXth century old, Steiner-Lehmus theorem, but the general case of inner and outer bisectors has been only recently addressed. We will show how the Gröbner Cover method provides automatically the conditions for a triangle to have two equal bisectors of whatever kind, yielding more insight than through any other automatic method.

Key words: automatic discovering, elementary geometry, comprehensive Gröbner system, Gröbner Cover.
MSC: 13P10, 68T15, 51M04.

Introduction

In (MoRe07) we have introduced and developed the foundations on the use of algorithmic methods for the discussion of parametric polynomial systems of equations in the field of automatic discovery of elementary geometry theorems. The merging of techniques from these two fields was exemplified through the application of an algorithm for the automatic case-analysis of polynomial systems with parameters (the algorithm *MCCGS*, standing for Minimal Canonical Comprehensive Gröbner System, cf. (MaMo09)) to a collection of geometric

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statements, of the kind: *Iff p, then q*, where p is missing. The automatic discovery protocol allowing such application stems from the work of (RV99) and has been further extended in (DaRe09) and, particularly, in (RV11). We refer the interested reader to the above mentioned papers for details and for references to previous and related work.

Now, since the Gröbner Cover algorithm, as described in (MoWi10), is a substantial improvement of the *MCCGS* concept and algorithm, it deserved being also tested in a challenging automatic theorem discovery situation, such as the Steiner-Lehmus theorem. This is the original goal of this paper.

The theorem of Steiner-Lehmus states that if a triangle has two (internal) angle-bisectors with the same length, then the triangle must be isosceles (the converse is, obviously, also true). This is an issue which has attracted along the years a considerable interest, and we refer to (StLe-web) for a large collection of references and comments on this classical statement and its proof. More recently, its generalization, regarding internal as well as external angle bisectors, has been approached through automatic tools, cf. (WL85), (Wang04) or (B07). The goal is to find a similar statement concerning triangles verifying the equality of two bisectors (of whatever kind) for different vertices. This generalization has been also achieved through the *FSDIC* automatic discovery protocol of (DaRe09), including the (perhaps new) case describing the simultaneous equality of three (either internal or external) bisectors, placed on each one of the vertices. We refer to (LoReVa09) (in Spanish) and to (LoReVa10) for further details.

All these results have been obtained through the use of ideal-theoretical elimination methods, which do not allow a fine grain analysis of the involved situation, in particular, concerning the behavior of some objects, indistinguishable from a complex-geometry point of view, such as the internal/external bisectors at a vertex. We think that the Gröbner Cover approach is particularly well suited in this context, bringing out, in its output, the possibility of a detailed case analysis that significantly extends our knowledge of the generalized Steiner-Lehums theorem.

Yet, we have to warn the reader that, in the current state of its implementation, the application of the Grobner Cover algorithm to the Steiner-Lehmus problem requires a non-trivial (and non automatic) analysis of the obtained output (see Section 3 for details).

1 On Gröbner Covers

There exist different methods to discuss parametric polynomial system of equations that can be used to find new geometrical theorems (some recent ones are (SuSa06; Na07; KaSuWa10; CDMXX11)). We have recently introduced the Gröbner Cover (in short: GC) algorithm (MoWi10), that gives precise and compact information about parametric polynomial systems of equations. What follows is a short digest of this method.

Let $\overline{a} = a_1, \ldots, a_m$ be a set of parameters, $\overline{x} = x_1, \ldots, x_n$ a set of variables and $I \subset K[\overline{a}][\overline{x}]$ an ideal (for example, generated by the set of equations describing a geometric construction, the parameters representing the coordinates of the free points), where K is a computable field (usually $\mathbb{Q}$). Denote $\overline{K}$ an algebraically closed extension of K (usually $\mathbb{C}$). Then $\overline{K}^m$ is the parameter space.

Selecting a monomial order $\succ$ for the variables, the Gröbner Cover of $\overline{K}^m$ with respect to I is a set of pairs $GC = \{(S_i, B_i) : 1 \leq i \leq s\}$, where the S_i, called segments, are locally closed subsets of the parameter space $\overline{K}^m$, and the B_i are sets of I-regular functions (c.f. (MoWi10); the reader can think of polynomials instead of regular functions in order to understand what follows) $g_{ij} : S_i \to \mathcal{O}_{S_i}[\overline{x}]$, that for every point $a \in S_i$ specialize to the reduced Gröbner basis of the specialized ideal I_a, i.e. the ideal obtained from I_a by evaluating the parameters $\overline{a}$ at point a.

Moreover, the segments are disjoint and cover the whole parameter space, the set of leading power products (lpp_i) of the bases B_i on each segment are constant (and characteristic of the segment if the ideal is homogeneous) and the whole description is arranged to be canonical in some sense. When the system is not homogeneous it can happen that more than one segment corresponds to the same lpp, but often in this case the corresponding solutions have to behave differently at infinity. It is known (see (CLS92)) that the set of lpp of the reduced Gröbner basis of a polynomial system characterizes the type of solutions (no solution, finite number of solutions, dimension of the solution set, etc.). Thus, it is natural to attach the information about the lpp_i as a third component of the label associated to the S_i-segments (even if it is apparent form the B_i's). The Gröbner Cover provides, as well, a very compact (i.e. minimal in some sense) discussion of all the involved cases.

There are many different ways of expressing a locally closed set, but for the GC-segments we have chosen a canonical description (the so called *P-representation*, too involved and irrelevant for our current goal in this paper to be described in detail here, but see (MoWi10)) providing the irreducible components of the Zariski closure of S_i and the irreducible components of the parts not included in S_i (holes). The I-regular functions g_{ij} in the basis B_i

$$g_{ij} : S_i \to \mathcal{O}_{S_i}[\overline{x}]$$

are described in terms of one or more polynomials in $\overline{K}[\overline{a}][\overline{x}]$ verifying that, for every point $a \in S_i$, if one of them does not specialize to 0, then it specializes (after normalizing) to the corresponding polynomial of the reduced Gröbner basis, and such that, always, at least one of these polynomials specializes to non-zero.

Finally, the application of the GC in Section 3 involves a technical issue that deserves some comment in the following Remark, since it could be of more general interest. In fact, the GC algorithm is usually set for the homogenization of a given ideal I and then one has to consider its dehomogenization. Notice that, in general, the homogenization of the whole ideal is larger than the ideal generated by the homogenization of a basis, since the later can include some extra solutions at infinity generated by the above equations. But homogenizing the ideal I generated by some equations arising on a geometry problem can lead to a very large system of equations, and the current implementation of GC could fail providing an answer.

Yet, it is not too difficult to show that if, instead, one just homogenizes each polynomial in the basis of the given ideal, and then dehomogenizes, the same ideal I is obtained. If the GC procedure is applied to this homogeneous ideal generated by the homogenization of a basis of I and then the GC output is dehomogenized, the result is still relevant to discuss the parametric system given by I (see Proposition 10 in (MoWi10)).

In fact: let $\{f_1, \ldots, f_r\}$ be a collection of polynomials in $K[x_1, \ldots, x_n]$, and let I be the generated ideal. Denote by f^* the homogenization of f with respect to a new variable t, and by I^* the full homogenization of I, that is, the ideal generated by all $f^*, f \in I$. Notice that, in general, I^* contains, but could be larger than $J = (f_1^*, \ldots, f_r^*)$. For instance, take $I = (y - x^2, y + x^2)$, then $I^* = (y, x^2)$, but $J = (yt, x^2)$.

Now, given a homogeneous polynomial (or form) F, in the variables $\{x_1, \ldots, x_n, t\}$, its dehomogenization is defined as $F_* = F(x_1, \ldots, x_n, 1)$. Given a homogeneous ideal L, its dehomogenization is the ideal generated by the dehomogenization of all forms in L. Let us denote this dehomogenization ideal by L_*.

We want to prove that $(I^*)_* = (f_1^*, \ldots, f_r^*)_* = I$, i.e. that to homogenize and then to dehomogenize an ideal yields the same ideal as if we had homogenize just its generators and then to dehomogenize it.

Proof. Recall $J = (f_1^*, \ldots, f_r^*)$. In order to prove that $(I^*)_* = (J^*)_* = I$, first notice that $I \subseteq J_*$. In fact, every polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ trivially verifies that $(f^*)_* = f$. Thus $f_i \in J_*$, since $f_i^* \in J$, for all $i = 1 \ldots r$. Therefore $I = (f_1, \ldots, f_r) \subseteq J_*$.

Now, as $J \subseteq I^*$, we have $J_* \subseteq (I^*)_*$, and thus $I \subseteq J_* \subseteq (I^*)_*$.

Finally, let us prove that $(I^*)_* \subseteq I$. Let F be a form in I^*. By definition there is a polynomial $f \in I$ such that $f^* = F$. But the polynomials F_* generate $(I^*)_*$. And, therefore, the polynomial $(f^*)_*$ generate $(I^*)_*$. But these polynomials $(f^*)_*$ coincide with f, as observed above. And the polynomials f belong to I. Thus, $(I^*)_* \subseteq I$. □

Concerning the GC algorithm, one needs, in general, to homogenize the given ideal, then compute the Gröbner Cover and finally dehomogenize and reduce the bases. But if the homogenization is applied just to a basis of the ideal and the GC algorithm is then applied to J, the only difference is that the output can contain different segments with the same lpp, due to the extra points at infinity that J can include, if taken instead of I^*. In the above example the system $yt = 0, x^2 = 0$ has solutions $(0, 1, 0), (0, 0, 1)$, considering the last coordinate in the infinity hyperplane. If we consider the ideal (I^*), there is a single solution, namely $(0, 0, 1)$. But dehomogenizing, both solutions yield $(0, 0)$.

2 Automatic Discovery of Geometric Theorems

Our point of departure is a geometric statement of the kind $\{H \Rightarrow T\}$ (such as: *Given a triangle, if we construct the bisectors with respect to the vertices then... there are at least two bisector segments, from the vertex to the opposite side, of equal length*, where H stands for the equations describing the construction (bisector segments) and T describes the desired property (equality of lengths, etc.). By abuse of notation, we will denote also by H and T the ideals generated by the polynomials involved in the equations describing the construction associated to the given statement or the given thesis.

Now, since it is quite reasonable to assume that a given discovery statement is generally false (for instance, not all triangles have two bisectors with equal length), the automatic discovery goal is to search for complementary hypotheses (say, the given triangle should be not degenerate to a line and should be equilateral or isosceles, etc.) providing necessary and sufficient conditions for the thesis to hold.

Although this formulation could seem straightforward, things are quite subtle and involved (for instance, why not to consider the thesis itself as the only needed complementary hypothesis?). Therefore, as stated in the Introduction, there is a variety of protocols (precise formulation of goals and algorithmic procedures to achieve them) concerning the automatic discovery of geometric theorems. Among them, those of (Wang04), (RV99), (DaRe09), are -*grosso modo*– founded in ideal theoretic elimination theory, searching for a single conjunction of equations and negated equations as the complementary hypothesis.

On the other hand, the approaches of (MoRe07) and (RV11) rely –roughly speaking– on finding a finite union of collections of equations R'_i in the parameters, and inequalities R''_i (some of them in the parameters, to take care of the possible degenerate cases of the free variables for the given construction, and some in a subset of variables from these parameters, to consider the possible degenerate cases after including the new hypotheses R'_i), which would provide

- when added to H, sufficient conditions for T, so that

$$\{(H \wedge (\vee_i(R'_i \wedge \neg R''_i))) \Rightarrow T\},$$

- which are as well necessary, so that $\{(T \wedge H) \Rightarrow (H \wedge (\vee_i(R'_i \wedge \neg R''_i)))\}$

Therefore, as argued in detail in (MoRe07) and (RV11), a reasonable way to proceed in order to find a collection of polynomials R'_i, R''_i verifying the above conditions could consist in computing the projection over the parameter space of the solution set of all hypotheses and theses equations, $V(H) \cap V(T)$, and express it as $\bigcup_i (V(R'_i) \setminus V(R''_i))$. Yet, we should check if over each component of the union, the corresponding set of equations and inequations yield sufficient conditions for T.

In practice, this could be achieved as follows. First, consider a geometrical construction depending on a set of points $\bar{A} = \{A_1, \ldots, A_s\}$, whose free coordinates are taken as parameters $\overline{a}$. The construction produces some new dependent points $\bar{P} = \{P_1, \ldots, P_r\}$, whose coordinates are taken as variables $\overline{x}$.

The problem is determining the configuration of the points $\bar{A}$, the parameters $\overline{a}$ varying in the parameter space $\mathbb{C}^m$, in order that the points $\bar{P}$ verify some property (for example, they are the end points of the bisectors with equal length). For this purpose, we write the equations reflecting the geometric construction and the theses, and we consider the corresponding parametric ideal $I \subset \mathbb{Q}[\overline{a}][\overline{x}]$.

Let $\{(S_i, B_i) : 1 \leq i \leq s\}$ be the Gröbner Cover of the parameter space wrt to I. Then we will have to carefully analyze its output, bearing in mind that

- As the locus of free points where the theorem holds should –when the given statement is not generally true, which is the usual case for discovery– have dimension less than the whole parameter space, the only open segment in the GC (also called the generic segment) must correspond to lpp = $\{1\}$. Thus, the generic segment will be of the form

$$S_1 = \overline{K}^m \setminus \bigcup_i \mathbb{V}(\mathfrak{p}_i)$$

- The remaining segments will be all inside $\bigcup_i V(\mathfrak{p}_i)$
- If the points P_i are uniquely determined by the points A_j, we will find a segment S_2 corresponding to a single solution in $\overline{x}$ with reduced Gröbner basis having the full set of coordinates as lpp.
- There can be segments lifting up to more than one solution, that we have then to analyze in detail.
- There can also exist segments corresponding to degenerate or lifting up to complex constructions in which we are in general less interested.

The important fact about the use of Gröbner Cover in this context is that it provides –in a compact and concise way– all the essential pieces (a finite number of them) on the parameter space, allowing to determine those that correspond to the validity of the given statement.

3 Steiner-Lehmus Theorem

To show the power of the outlined procedure, we will apply it to find a generalization of the Steiner-Lehmus Theorem. This theorem was proposed by the well known, XIXth century geometer, Steiner to Sturm and it was proved by Lehmus for the first time in 1848. It could be stated as follows:

Theorem 3.1 (Classical Steiner-Lehmus)**.** *The inner bisectors of angles A and B of a triangle ABC ($\alpha = \beta$ and $\gamma = \delta$) are of equal length ($\overline{AA'} = \overline{BB'}$) if and only if the triangle is isosceles with $\overline{AC} = \overline{BC}$ (see Figure 1).*

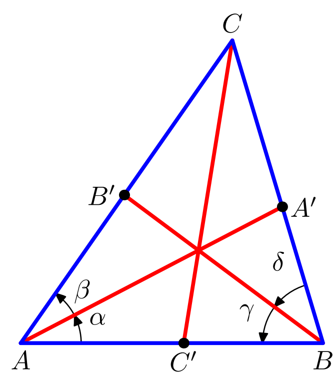

Figure 1: Triangle and inner bisectors

This is a statement which has attracted along the years a considerable interest for some intrinsic difficulties in its (traditional) proof, and we refer to (StLe-web) for a large collection of references and comments on this classical statement and its proof. More recently, its generalization, regarding internal as well as external angle bisectors, has been approached through automatic tools, cf. (WL85), (Wang04) or (B07). The goal is to find a similar statement concerning triangles verifying the equality of two bisectors (of whatever kind) for different vertices. This generalization has been also achieved through the automatic discovery protocol of (DaRe09), including the (perhaps new) case describing the simultaneous equality of three (either internal or external) bisectors, placed on each one of the vertices. We refer to (LoReVa09) (in Spanish) and to (LoReVa10) for further details.

Now, in order to automatically discover the Steiner-Lehmus theorem, we let ABC be the given triangle and consider the bisectors at angle A. To construct the bisectors (see Figure 2) we consider the circle with center A and radius $\overline{AC}$. There are two intersection points P and P' of the circle with side AB, and thus two middle points Q and Q' of $\overline{CP}$ and $\overline{CP'}$ determining the bisectors $\overline{AM}$ and $\overline{AM'}$ whose length we are interested in. So, if we only use the equations determining M and M' we will not distinguish between the inner and the outer bisector. This difficulty will allow to generalize the theorem.

Without loss of generality, we set coordinates $A(0,0)$, $B(1,0)$, $C(a,b)$. Then let $(p,0)$ be the intersection of the circle centered at A passing through C, (i.e. points P or P'), and let (x_1,y_1) stand for the feet of the bisectors, (i.e. points M or M'). The equation of the circle is $(a^2+b^2)-p^2$. Point (x_1,y_1) is on the line AQ. The middle point between $(0,p)$ and C is $Q=\left(\frac{a+p}{2},\frac{b}{2}\right)$ and so $bx_1-(a+p)y_1$ expresses that (x_1,y_1) is on the bisector line. Finally the equation staying that (x_1,y_1) lies on side BC, is $b(1-x_1)+(a-1)y_1$. Thus, the equations determining (x_1, y_1) in terms of (a, b) are:

$$(a^2 + b^2) - p^2, bx_1 - (a + p)y_1, b(1 - x_1) + (a - 1)y_1. \tag{1}$$

Notice the sign of p discriminates which bisector of A is being concerned with. If a solution of our problem has $p > 0$ it will correspond to the inner bisector of A, whereas a solution with $p < 0$ will correspond to the outer bisector of A. But the sign is not algebraically (from the complex point of view) relevant, so that both points M and M' are solutions of the same equations. The length of the bisector is $l_A^2 = x_1^2 + y_1^2$.

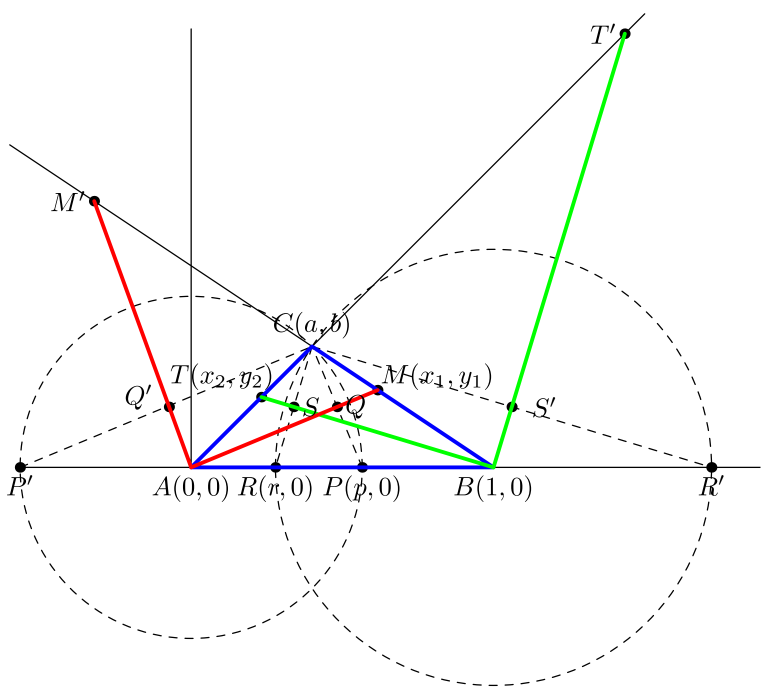

Figure 2: The bisectors of A and B are equal

Consider now the bisectors of B (see Figure 2). Denoting $(r, 0)$ the intersection point of the circle centered in B with radius $\overline{BC}$ (points R or R') and (x_2, y_2) the coordinates of the foot of the bisector of B (points T or T') the corresponding equations for them are:

$$(a - 1)^2 + b^2 - (r - 1)^2, (1 - x_2)b + (a + r - 2)y_2, ay_2 - bx_2. \tag{2}$$

In that case, a discriminator between inner and outer bisectors of B is $1 - r$. A solution with $1 - r > 0$ will correspond to the inner bisector whereas a solution with $1 - r < 0$ will correspond to the outer bisector. The length of the bisector of the angle B is $l_B^2 = (x_2 - 1)^2 + y_2^2$.

Now, using the set of all the above equations, we turn to searching the necessary and also sufficient conditions for assuring that the length of one bisector of the angle A is equal to that of one bisector of angle B, but we are not distinguishing between which inner or outer bisector is concerned. It can happen that the two equal bisectors are the two inner bisectors ($i_A = i_B$), or the two outer bisectors ($e_A = e_B$), or one inner and one outer bisector (cases $i_A = e_B$ and $e_A = i_B$). There are, thus, four possibilities.

In order to compute the Gröbner Cover, we include the set of equations (1), the set of equations (2), plus the condition that the length of one bisector of A is equal to that of one bisector of B, i.e. $x_1^2 + y_1^2 = (x_2 - 1)^2 + y_2^2$. Thus the complete set of equations is:

$$\begin{cases} a^2 + b^2 - p^2, \\ bx_1 - (a+p)y_1, \\ b(1 - x_1) + (a-1)y_1, \\ (a-1)^2 + b^2 - (r-1)^2, \\ b(1-x_2) + (a+r-2)y_2, \\ ay_2 - bx_2, \\ x_1^2 + y_1^2 = (x_2-1)^2 + y_2^2. \end{cases} \tag{3}$$

Now, we take the point $C(a, b)$ as the only parametric point, for which we want to obtain the conditions for the system (3) with variables x_1, y_1, x_2, y_2, p, r to have solutions. These solutions will correspond to one bisector of A being equal to one bisector of B, but the conditions over a, b will not distinguish between internal and external bisectors. When p is positive, the bisector of A will be internal and it will be external if p is negative. The same happens considering the sign of $1 - r$, for the bisector of B.

The GC algorithm is used here taking the grevlex$(x_1, y_1, x_2, y_2, p, r)$ order for the variables. The call in Singular (after charging the grobcov library) is:

```
> ring R=(0,a,b),(x1,y1,x2,y2,p,r),dp;
> ideal S93= a^2+b^2-p^2,
            b*x1-(a+p)*y1,
            b*(1-x1)+(a-1)*y1+,
            (a-1)^2+b^2-(r-1)^2,
            b*(1-x2)+(a+r-2)*y2,
            a*y2-b*x2,
            x1^2+y1^2-(x2-1)^2-y2^2;
> short=0;
> grobcov(S93);
```

Let us describe below and in the following tables the output of the Gröbner

Cover algorithm. The following irreducible curves and varieties (over $\mathbb{Q}$) appear:

$$\begin{aligned}
\mathcal{C}_1 &= \mathbb{V}(8a^{10} - 40a^9 + 41a^8b^2 + 76a^8 - 164a^7b^2 - 64a^7 \\
&\quad +84a^6b^4 + 246a^6b^2 + 16a^6 - 252a^5b^4 - 164a^5b^2 \\
&\quad +8a^5 + 86a^4b^6 + 278a^4b^4 + 31a^4b^2 - 4a^4 - 172a^3b^6 \\
&\quad -136a^3b^4 + 20a^3b^2 + 44a^2b^8 + 122a^2b^6 + 14a^2b^4 \\
&\quad -10a^2b^2 - 44ab^8 - 36ab^6 + 12ab^4 + 9b^{10} + 14b^8 \\
&\quad -b^6 - 6b^4 + b^2, \\
\mathcal{C}_2 &= \mathbb{V}(2a-1). \\
\mathcal{C}_3 &= \mathbb{V}(b),
\end{aligned}$$

We are interested only in the real points, so we separate the real from the complex points appearing in the segments.

Varieties	Real points
$V_1 = \mathbb{V}(b, a)$	$P_1 = (0, 0)$
$V_2 = \mathbb{V}(b, a-1)$	$P_2 = (1, 0)$
$V_3 = \mathbb{V}(b, 2a^2 - 2a - 1)$	$P_{31} = \left(\frac{1-\sqrt{3}}{2}, 0\right)$ $= (-.3660254038, 0.)$ $P_{32} = \left(\frac{1+\sqrt{3}}{2}, 0\right)$ $= (1.366025404, 0.)$
$V_4 = \mathbb{V}(b, 2a-1)$	$P_4 = (\frac{1}{2}, 0)$
$V_5 = \mathbb{V}(12b^2 - 1, 2a - 1)$	$P_{51} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{6}\right)$ $= (0.5, -0.2886751347)$ $P_{52} = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ $= (0.5, 0.2886751347)$
$V_6 = \mathbb{V}(4b^2 - 3, 2a - 1)$	$P_{61} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $= (.5000000000, -.8660254040)$ $P_{62} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $= (0.5, .8660254040)$
$V_7 = \mathbb{V}(b^4 + 11b^2 - 1, 5a - 2b^2 - 6)$	$P_{71} = \left(-1 + \sqrt{5}, -\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$ $= (1.236067977, -.3002831039)$ $P_{72} = \left(-1 + \sqrt{5}, \frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$ $= (1.236067977, .3002831039)$
$V_8 = \mathbb{V}(b^4 + 11b^2 - 1, 5a + 2b^2 + 1)$	$P_{81} = \left(2 - \sqrt{5}, -\frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$ $(-.236067977, -.3002831039)$ P_{82} $\left(2 - \sqrt{5}, \frac{\sqrt{-22+10\sqrt{5}}}{2}\right)$ $(-.236067977, .3002831039)$
$V_9 = \mathbb{V}(4b^4 + 5b^2 + 2, 2a - 1)$	

Vars.	Complex points	
V_7	$P_{73} = \left(-1-\sqrt{5}, -I\frac{\sqrt{22+10\sqrt{5}}}{2}\right)$	$= (-3.236067977, -3.330190676I)$
	$P_{74} = \left(-1+\sqrt{5}, I\frac{\sqrt{22+10\sqrt{5}}}{2}\right)$	$= (-3.236067977, 3.330190676I)$
V_8	$P_{83} = \left(2+\sqrt{5}, -I\frac{\sqrt{22+10\sqrt{5}}}{2}\right)$	$= (4.236067977, -3.330190676I)$
	$P_{84} = \left(2+\sqrt{5}, I\frac{\sqrt{22+10\sqrt{5}}}{2}\right)$	$= (4.236067977, 3.330190676I)$
V_9	$P_{91} = \left(\frac{1}{2}, -\frac{\sqrt{-10+2I\sqrt{7}}}{4}\right)$	$= (0.5, -.2026163631 - .8161209412I)$
	$P_{92} = \left(\frac{1}{2}, \frac{\sqrt{-10+2I\sqrt{7}}}{4}\right)$	$= (0.5, 0.2026163631 + .8161209412I)$
	$P_{93} = \left(\frac{1}{2}, -\frac{\sqrt{-10-2I\sqrt{7}}}{4}\right)$	$= (0.5, -0.2026163631 + .8161209412I)$
	$P_{94} = \left(\frac{1}{2}, \frac{\sqrt{-10-2I\sqrt{7}}}{4}\right)$	$=(0.5, 0.2026163631 - .8161209412I)$

These curves are represented in Figure 3. Special points are either singular points of $\mathcal{C}_1$ or intersection points between the three curves:

a) V_1, V_2, V_5, V_7, V_8 are singular points of $\mathcal{C}_1$. They contain the real points $P_1, P_2, P_{51}, P_{52}, P_{71}, P_{72}, P_{81}, P_{82}$ and some other complex points.

b) V_5, V_6, V_9 are intersection points between $\mathcal{C}_1$ and $\mathcal{C}_2$. They contain the real points $P_{51}, P_{52}, P_{61}, P_{62}$ plus other complex points.

c) V_1, V_2 are intersection points between $\mathcal{C}_1$ and $\mathcal{C}_3$. They contain the real points $P_1 = A$ and $P_2 = B$.

d) V_3 is the intersection between $\mathcal{C}_2$ and $\mathcal{C}_3$.

Variety V_9 contains only complex points, whereas V_7 and V_8 contain real and complex points. We distinguish both cases because of the particular behavior of complex points concerning Euclidean distance issues and because we are not interested in the complex points.

Let us give now the output of the Gröbner Cover. We obtain the following description with 9 segments:

1. Segment with lpp = $\{1\}$ Generic segment
   Segment: $\mathbb{C}^2 \setminus (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$
   Description: The whole parameter space except the curves $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$.
   Basis: $B_1 = \{1\}$
   There are no solution over this segment.

2. Segment with lpp $= \{p, y_2, x_2, y_1, x_1, r^2\}$

Segment: $(\mathcal{C}_2 \setminus (V_4 \cup V_5 \cup V_6)) \cup V_8$

Description: ($\mathcal{C}_2$ minus intersecting points with $\mathcal{C}_1$ and $\mathcal{C}_2$) plus V_8

Basis:

$$\begin{aligned} B_2 \quad = \quad & \{(35a-45)p + (-4ab^2-37a+2b^2-9)r + (65a-5) \\ & (a-2b^2+1)y_2 + (-4ab)r, \quad (7a+2b^2-5)x_2 + (-2a+2)r, \\ & (100ab^3-75ab+60b^3-45b)y_1 + (-28ab^2+16a+124b^2-8)r \\ & +(-940ab^2+80a+470b^2-40), \quad (220b^2-165)x_1 \\ & +(-16ab^2-148a+8b^2-36)r + (160ab^2+380a-300b^2-25), \\ & (4a)r^2 + (-8a)r + (a-2b^2+1)\} \end{aligned}$$

There are 2 solutions on each point of this segment.

3. Segment with lpp $= \{r, p, y_2, x_2, y_1, x_1\}$

Segment: $\mathcal{C}_1 \setminus (V_1 \cup V_2 \cup V_3 \cup V_5 \cup V_6 \cup V_7 \cup V_8 \cup V_9)$

Description of the real points: The curve $\mathcal{C}_1$ except the points $P_1, P_2, P_{31}, P_{32},\ P_{51}, P_{52},\ P_{61}, P_{62}, P_{71}, P_{72}, P_{81}, P_{82}$

Basis:

$$\begin{aligned} B_3 \quad = \quad & \{(3a^4-6a^3+6a^2b^2+5a^2-6ab^2+3b^4+5b^2-1)r + (a^5-10a^4 \\ & +2a^3b^2+17a^3-18a^2b^2-10a^2+ab^4+17ab^2-a-8b^4-10b^2+2), \\ & (3a^4-6a^3+6a^2b^2+5a^2-6ab^2-4a+3b^4+5b^2+1)p \\ & +(a^5+2a^4+2a^3b^2-7a^3+6a^2b^2+4a^2+ab^4-7ab^2-a+4b^4+4b^2), \\ & (a^5-4a^4+2a^3b^2+5a^3-6a^2b^2+ab^4+5ab^2-a-2b^4)y_2 \\ & +(-3a^4b+6a^3b-6a^2b^3-5a^2b+6ab^3-3b^5-5b^3+b), \\ & (a^5-4a^4+2a^3b^2+5a^3-6a^2b^2+ab^4+5ab^2-a-2b^4)x_2 \\ & +(-3a^5+6a^4-6a^3b^2-5a^3+6a^2b^2-3ab^4-5ab^2+a), \\ & (a^5-a^4+2a^3b^2-a^3-a^2+ab^4-ab^2+3a+b^4-b^2-1)y_1 \\ & +(3a^4b-6a^3b+6a^2b^3+5a^2b-6ab^3-4ab+3b^5+5b^3+b), \\ & (a^5-a^4+2a^3b^2-a^3-a^2+ab^4-ab^2+3a+b^4-b^2-1)x_1 \\ & +(2a^5-8a^4+4a^3b^2+12a^3-12a^2b^2-8a^2+2ab^4+12ab^2+2a \\ & -4b^4-4b^2)\} \end{aligned}$$

There is a single solution on each point of this segment.

4. Segment with lpp $= \{y_2, y_1, r^2, p^2, x_1^2\}$

Segment: $\mathcal{C}_3 \setminus (V_1 \cup V_2)$

Description: The line $\mathcal{C}_3$ except the points P_1, P_2

Basis:

$$\begin{aligned} B_4 \quad = \quad & \{y_2, y_1, r^2-2r+(-a^2+2a), p^2+(-a^2) \\ & x_1^2 - x_2^2 + 2x_2 - 1\} \end{aligned}$$

There are infinite solutions, but correspond to degenerate triangles.

5. Segment with lpp $= \{y_2, x_2, y_1, x_1, r^2, p^2\}$

   Segment: V_5

   Description: Points P_{51}, P_{52}

   Basis:

$$\begin{aligned} B_5 &= \{2y_2 - 3br, 4x_2 - 3r, 2y_1 + 3bp - 3b, \\ &\quad 4x_1 - 3p - 1, 3r^2 - 6r + 2, 3p^2 - 1\} \end{aligned}$$

   There are 4 solutions on each point of this segment.

6. Segment with lpp $= \{r, p, y_2, x_2, y_1, x_1\}$

   Segment: V_6

   Description: Points P_{61}, P_{62}

   Basis:

$$B_6 = \{r, p - 1, 2y_2 - b, 4x_2 - 1, 2x_1 - b, 4x_1 - 3\}$$

   There is a single solution on the points of this segment.

7. Segment with lpp $= \{r, y_2, x_2, y_1, x_1, p^2\}$

   Segment: V_7

   Description: Points P_{71}, P_{72}

   Basis:

$$\begin{aligned} B_7 &= \{5r + (b^2 - 7), (5b)y_2 + (3b^2 - 1), x_2 - 2, \\ &\quad (5b)y_1 + (3b^2 - 1)p + (-3b^2 + 1), \\ &\quad 5x_1 + (b^2 - 2)p + (-b^2 - 3), 5p^2 + (-b^2 - 8)\}. \end{aligned}$$

   There are 2 solutions on each point of this segment.

8. Segment with lpp $= \{y_1, r^2, y_2 r, p^2, x_1^2\}$

   Segment: V_1

   Description: Point P_1

   Basis:

$$B_8 = \{y_1,\ r^2 - 2r,\ y_2 r - 2y_2,\ p^2,\ x_1^2 - x_2^2 - y_2^2 + 2x_2 - 1\}$$

   There are infinite solutions, but correspond to degenerate triangles.

9. Segment with lpp $= \{y_2, r^2, p^2, y_1 p, x_1^2\}$

   Segment: V_2

   Description: Point P_2

   Basis:

$$B_9 = \{y_2,\ r^2 - 2r + 1,\ p^2 - 1,\ y_1 p + y_1,\ x_1^2 + y_1^2 - x_2^2 + 2x_2 - 1\}$$

   There are infinite solutions, but correspond to degenerate triangles.

3.1 Discussion and formulation of the generalized theorem

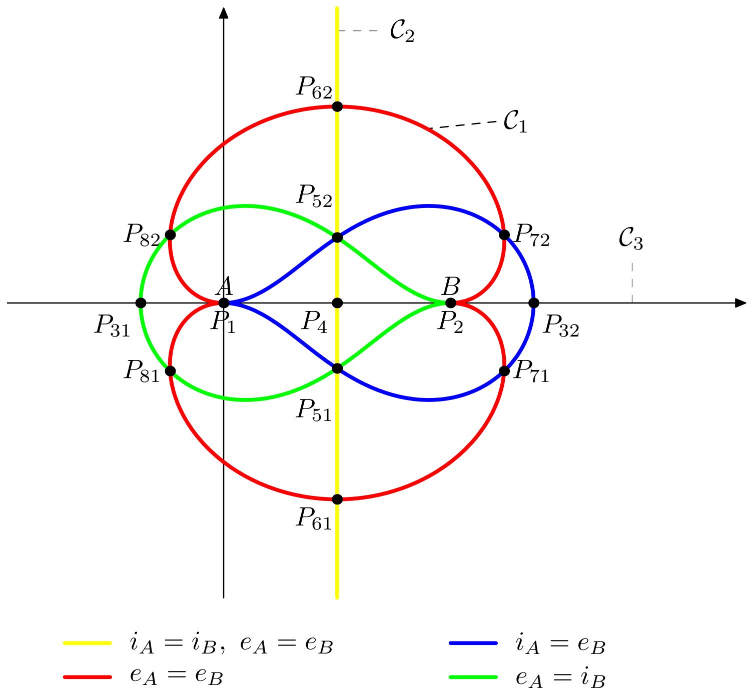

Figure 3: Problem 1: Curves $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and special points

– **Segment 1**: of the Gröbner Cover proves that the thesis does not hold in general, except for triangles with vertex C placed on the three curves $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ (see Figure 3). For the points inside these curves, system (3) has always some solution. Let us discuss which kind of solutions exist on these curves.

– **Segment 2**: It has two components:

1) For $a = 1/2$, vertex C is on the bisector of side AB (so that the triangle ABC is isosceles), leaving out the points $P_{51}, P_{52}, P_{61}, P_{62}, P_4$. Specializing the basis on this branch (setting $a = 1/2$) yields to

$$\begin{aligned} B_{21} \quad = \quad & \{-p - r + 1, (4b^2 - 3)y_2 + 4(b)r, (4b^2 - 3)x_2 + 2r, \\ & (4b^2 - 3)y_1 + (4b)r, (4b^2 - 3)x_1 - 2r + (-4b^2 + 3), \\ & 4r^2 - 8r + 3 + (-4b^2)\} \end{aligned}$$

and considering the first and the last equations we have:

$$p = 1 - r = (1/2) \pm \sqrt{1 + 4b^2}.$$

Thus there are two solutions: In one case $p = 1 - r > 0$, so that it holds the equality of both internal bisectors (i.e. $i_A = i_B$), and in the other solution it holds that $p = 1 - r < 0$, corresponding to the case $e_A = e_B$, which is also obvious from the first equality, by symmetry.

Thus, on this part of the segment the two inner bisectors are equal, as well as the two outer ones. This corresponds to the classical Steiner-Lehmus Theorem, enlarging it with the coincidence of the outer bisectors too.

2) V_8 containing the pair of points P_{81} and P_{82}. For them we have

Point	$(p, 1-r)$	Bisectors
P_{81}, P_{82}	$(-0.3819659526, -1.272019650)$	$e_A = e_B$
	$(-0.3819659526, 1.272019650)$	$e_A = i_B$

– **Segment 5**: It contains the two real points P_{51}, P_{52}, and for each one there are four solutions, as it is clear by observing the values of p and $1 - r$ at each of the solutions.

Point	$(p, 1-r)$	Bisectors
P_{51}, P_{52}	$(0.5773502693, 0.5773502693)$	$i_A = i_B$
	$(0.5773502693, -0.577350269)$	$i_A = e_B$
	$(-0.5773502693, 0.5773502693)$	$e_A = i_B,$
	$(-0.5773502693, -0.5773502693)$	$e_A = e_B$

– **Segment 6**: contains the two real points P_{61}, P_{62}, and for each one there is a unique solution corresponding to $i_A = i_B$ (as it can be checked by actually solving the system associated to segment 6). We observe –see the Remark at the end of section 1– that, although at these points there should be –by symmetry– another solution, related to the equality $e_A = e_B$, it is actually missing, because both external angle bisectors become infinite. This is the reason why, even if the lpp on this segment 6 is equal to the lpp on segment 3, this common lpp appears in different segments. We will see below that, in the curve described in segment 3, in the neighborhood of P_{61} and P_{62}, we have the equality $e_A = e_B$, instead of $i_A = i_B$.

Point	$(p, 1-r)$	Bisectors
P_{61}, P_{62}	(1,1)	$i_A = i_B$

– **Segment 7**: V_7 containing the pair of points P_{71} and P_{72}. For them we have

Point	$(p, 1-r)$	Bisectors
P_{71}, P_{72}	$(-1.272019650, -0.381965976)$	$e_A = e_B$
	$(1.272019650, -0.381965976)$	$i_A = e_B$

– **Segment 3**: This segment contains all the points of the curve $\mathcal{C}_1$ except the special points. There is a unique solution on each point of the curve, and so only one equality between one bisector of A and one bisector of B can happen. The kind of solution cannot change, by continuity, on the curve except when the curve reaches a special point. The reason is that in the changing points one needs to have equality of more bisectors and this can only occur in some special segment. So we only need to determine the color (i.e. the kind of solution) in a single point of the curve between special points.

We can proceed, then, by choosing some simple vertical lines, determining its intersection with the curve and computing in each case the correspondent bisectors. For instance, for the lines $x = 0$ and $x = 1$ (which determined quite a few of the pieces of the curve; a similar procedure should be performed on the remaining parts) we obtain the following systems of equations for the intersections:

$$\begin{cases} a = 0 \\ b^2(3b^4 - 4b^3 + 5b^2 - 4b + 1)(3b^4 + 4b^3 + 5b^2 + 4b + 1) \end{cases}$$

$$\begin{cases} a = 1 \\ b^2(3b^4 - 4b^3 + 5b^2 - 4b + 1)(3b^4 + 4b^3 + 5b^2 + 4b + 1) \end{cases}$$

We do not consider the solutions $(a, b) = (0, 0)$ and $(a, b) = (1, 0)$ as they correspond to degenerate triangles. Substituting the solutions of these systems into the basis B_3 one can determine the pair $(p, 1 - r)$ for each of the points, thus determining which bisectors are equal at the point. We set a red color if $e_A = e_B$, blue color if $i_A = e_B$ and green if $e_A = i_B$. The possibility $i_A = i_B$ never occurs on $\mathcal{C}_1^*$. The following table gives the color of some points of the curve

Point	Branch	$(p, 1 - r)$	Bisectors
$(0, .7013671986)$	P_{62}-P_{82}	$(-.7013671074, -1.221439949)$	$e_A = e_B$
$(0, .4190287818)$	P_{52}-P_{82}	$(-.4190287676, 1.08424403111)$	$e_A = i_B$
$(0, -.4190287818)$	P_{51}-P_{81}	$(-.4190287676, 1.08424403111)$	$e_A = i_B$
$(0, -.7013671986)$	P_{61}-P_{81}	$(-.7013671074, -1.221439949)$	$e_A = e_B$
$(1, .7013671986)$	P_{62}-P_{92}	$(-1.221530232, -0.701371729)$	$e_A = e_B$
$(1, .4190287818)$	P_{52}-P_{92}	$(1.084234608, -0.419025294)$	$i_A = e_B$
$(1, -.4190287818)$	P_{51}-P_{91}	$(1.084234608, -0.419025294)$	$i_A = e_B$
$(1, -.7013671986)$	P_{61}-P_{91}	$(-1.221530232, -0.701371729)$	$e_A = e_B$

– **Segments 4, 8, 9**: These three segments correspond to degenerate triangles. Here there are infinite solutions, as the lengths of the bisectors are not defined. We need not analyze what happens exactly over them.

Obviously all the properties of the form of the curve and special points and colors obtained are easily transformed under scaling the distance $\overline{AB}$.

In summary, now we have thus proved the following

Theorem 3.2 (Generalized Steiner-Lehmus). *Let ABC be a triangle and i_A, e_A, i_B, e_B the lengths of the inner and outer bisectors of the angles A and B. Then, considering the conditions for the* equality of some bisector of A and some bisector of B *the following excluding situations occur:*

1. *The triangle ABC is* degenerate *(i.e. C is aligned with A and B);*
2. *ABC is* equilateral *and then $i_A = i_B$ whereas e_A and e_B become infinite, (P_{61}, P_{62});*
3. *Point C is in the* center of an equilateral triangle*, and then $i_A = i_B = e_A = e_B$, (P_{51}, P_{52});*
4. *The triangle is* isosceles but *not of the special form of cases 2. or 3. and then $i_A = i_B \neq e_A = e_B$, (ordinary Theorem);*
5. $\dfrac{\overline{AC}}{\overline{AB}} = \sqrt{\dfrac{1+\sqrt{5}}{2}}$, $\dfrac{\overline{BC}}{\overline{AB}} = \dfrac{3-\sqrt{5}}{2}$, *and then $e_A = e_B = i_A$, (P_{71}, P_{72});*
6. $\dfrac{\overline{AC}}{\overline{AB}} = \dfrac{3-\sqrt{5}}{2}$, $\dfrac{\overline{BC}}{\overline{AB}} = \sqrt{\dfrac{1+\sqrt{5}}{2}}$, *and then $e_A = e_B = i_B$, (P_{81}, P_{82});*
7. C lies in the curve of degree 10 *relative to points A and B (case of curve $\mathcal{C}_1$) passing through all the special points above but is none of these points, and then only one of the following possibilities happen: either $e_A = e_B$ or $i_A = e_B$ or $e_A = i_B$, depending on the piece of the curve (see figure 3, the color representing which of those situations occur);*
8. None of the above cases *occur, and then no bisector of A is equal to no bisector of B.*

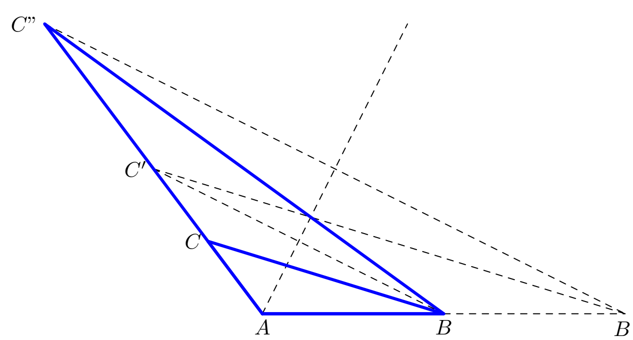

Figure 4: Transformation of coordinates

3.2 Bisectors at vertex C

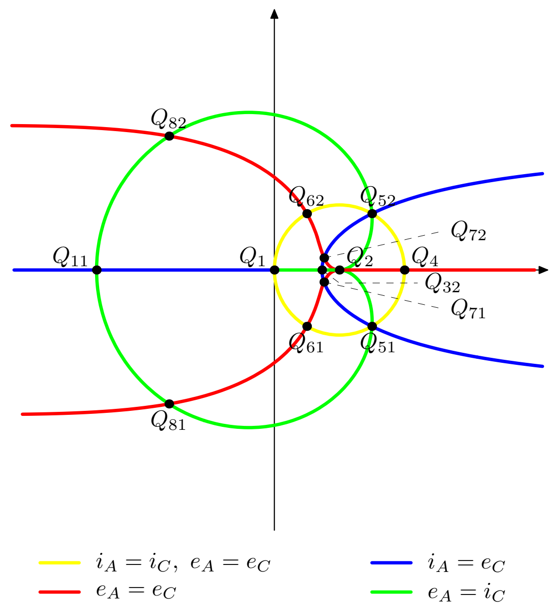

Figure 5: Problem 2: Curves $\mathcal{CC}_1, \mathcal{CC}_2$

It could be of some interest to analyze the conditions for one bisector of the fixed point A to be equal to one bisector of the moving point C.

This problem can be solved by a transformation of the previous solution for the case of equal length of the bisectors at the fixed points A, B. Each point C of the solution to the precedent problem corresponds to a triangle ABC (see Figure 4), where one bisector of A is equal to one bisector of B, with $\overline{AB} = 1$. Considering a parallel to the line BC one can form a similar triangle $AB'C'$ with $\overline{AC'} = 1$. Making a symmetry over the inner bisector of A will led to a new triangle ABC'' with one bisector of A equal to one bisector of C'' and $\overline{AB} = 1$ that corresponds to the requirements of the new problem. This yields a transformation of C into C'' that will conserve the direction: $\overrightarrow{AC}$ parallel to $\overrightarrow{AC''}$ but having inverse lengths. Thus, setting $C = (a, b)$ and $C'' = (a', b')$ the transformations is

$$\begin{cases} a' = \dfrac{a}{a^2 + b^2} \\ b' = \dfrac{b}{a^2 + b^2} \end{cases} \qquad \begin{cases} a = \dfrac{a'}{a'^2 + b'^2} \\ b = \dfrac{b'}{a'^2 + b'^2} \end{cases}$$

Substituting the transformation into the curves obtained in the precedent section and eliminating the denominator $(a^2+b^2)^s$ (where s depends on the curve), leads to the transformed curves

$$\begin{aligned}
\mathcal{CC}_1 &= \mathbb{V}(a^8b^2 + 4a^6b^4 + 6a^4b^6 + 4a^2b^8 + b^10 - 4a^8 - 18a^6b^2 - 30a^4b^4 - 22a^2b^6 - 6b^8 \\
&\quad +8a^7 + 28a^5b^2 + 32a^3b^4 + 12ab^6 + 16a^6 + 31a^4b^2 + 14a^2b^4 - b^6 - 64a^5 - 100a^3b^2 \\
&\quad -36ab^4 + 76a^4 + 94a^2b^2 + 14b^4 - 40a^3 - 44ab^2 + 8a^2 + 9b^2) \\
\mathcal{CC}_2 &= \mathbb{V}(a^2 - 2a + b^2), \qquad \mathcal{CC}_3 = \mathbb{V}(b)
\end{aligned}$$

where the curves $\mathcal{CC}_i$ correspond to $\mathcal{C}_i$ and the points Q_{ij} to P_{ij}. All of them are represented in Figure 5.

We can also consider the problem as a new one and compute the solutions of the new system using the Gröbner Cover. The computations are completely similar to those of the precedent section and we do not give the details.

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