Presentation of the book “The Gröbner Cover" Springer ACM Series (2018)

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Index of the Book

Software

All the algorithms described in the book are implemented in the

"*grobcov.lib*" library of SINGULAR.

The book can also be used as software Manual for the library.

The Gröbner Cover

The existence of the Gröbner Cover of a parametric polynomial $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal is a consequence of Wibmer's Theorem.
It was described in (2010) by A. Montes and M. Wibmer.

Wibmer's theorem establishes that for an **homogeneous parametric polynomial ideal** $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal, it exists

- a **canonical partition** of the parameter space into locally closed segments $S_i = \mathbb{V}(\mathfrak{p}_i) \setminus \mathbb{V}(\mathfrak{q}_i)$;
- each segment accepts a set of I-regular functions that specializes to the reduced Gröbner basis for every point $\mathbf{a} \in S_i$,
- **preserving** the set of lpp_i (leading power products),
- and each segment S_i has **different** lpp_i.

The set of triplets $(\mathrm{lpp}_i, B_i, S_i)$ constitutes its **Gröbner Cover**.

The Gröbner Cover

For a non-homogeneous parametric ideal

- homogenize the ideal,
- determine its Gröbner Cover,
- and dehomogenize.

The result is its canonical **Gröbner Cover**.

It can contain segments with the same set of lpp_i but coming from different sets of lpph_i of the homogenized ideal.

It is still canonic.

Example: Singular points of a conic

Equation of the conic:

$$x^2 + by^2 + 2cxy + 2dx + 2ey + f = 0$$

Input:

> LIB "grobcov.lib";
> ring R=(0,b,c,d,e,f),(x,y),dp;
> ideal $S = x^2 + by^2 + 2cxy + 2dx + 2ey + f$,
$2x + 2cy + 2d$,
$2by + 2cx + 2e$;
> grobcov(S,"showhom",1);

Output: Singular points of a conic

1.	lpp =	$\{1\}$	lpph= $\{1\}$
	Basis =	$\{1\}$	
	Segment =	$\mathbb{C}^5 \setminus \mathbb{V}(bd^2 - bf + c^2f - 2cde + e^2)$	
	Description:	Conic without singular points.	
2.	lpp =	$\{y, x\}$	lpph= $\{y, x\}$
	Basis =	$\{(cd - e)y + d^2 - f,\ (cd - e)x + cf - de\}$	
	Segment =	$\mathbb{V}(bd^2 - bf + c^2f - 2cde + e^2) \setminus \mathbb{V}(cd - e, b - c^2)$	
	Description:	Two intersecting lines.	
3.	lpp =	$\{1\}$	lpph= $\{@t, x\}$
	Basis =	$\{1\}$	
	Segment =	$\mathbb{V}(cd - e, b - c^2) \setminus \mathbb{V}(d^2 - f, cf - de, cd - e, b - c^2)$	
	Description:	Two parallel lines.	
4.	lpp =	$\{x\}$	lpph= $\{x\}$
	Basis =	$\{x + cy + d\}$	
	Segment =	$\mathbb{V}(d^2 - f, cf - de, cd - e, b - c^2)$	
	Description:	Double line.	

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Consider a geometric proposition of the form

$$(H \wedge \neg H_1) \Rightarrow (T \wedge \neg T_1).$$

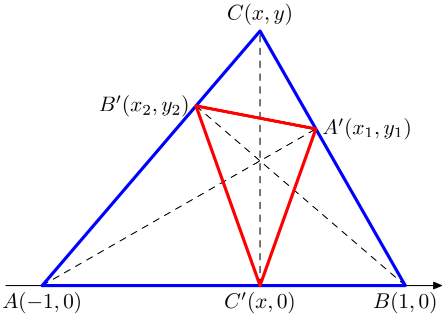
We provide an **automatic algorithm** (ADGT) for obtaining **supplementary conditions** for transforming the Proposition into a Theorem.

An example using the command ADGT

Let $A(-1,0), B(1,0), C(x,y)$ be the vertices of a triangle.

Consider the foots of the heights $A'(x_1,y_1), B'(x_2,y_2), C'(x,0)$.

The **ortic triangle** of ABC is $A'B'C'$.



Ortic triangle

Consider the following ideals of hypothesis and thesis:
Hypothesis H: The triangle $A'B'C'$ **is the ortic triangle** of ABC

$$H = -yx_1 + (x-1)y_1 + y,\ (x-1)(x_1+1) + yy_1,$$
$$-yx_2 + (x+1)y_2 - y,\ (x+1)(x_2-1) + yy_2$$

Hypothesis H_1: The triangle ABC is **degenerate** (we shall deny it):

$$H_1 = y$$

Thesis T: The ortic triangle $A'B'C'$ **is isosceles** ($\overline{A'C'} = \overline{B'C'}$):

$$T = (x_1 - x)^2 + y_1^2 - (x_2 - x)^2 - y_2^2$$

Thesis T_1: The ortic triangle $A'B'C'$ is **degenerate** (points aligned, we shall deny it):

$$T_1 = x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1$$

Ortic triangle

Proposition 1: $H \Rightarrow T$.

Calling sequence: $\texttt{ADGT}(H, T, 1, 1)$;

Result:

$$(x, y) \in S_1 \cup S_2 \cup S_3$$

$$S_1 = \mathbb{V}(x^2 - y^2 - 1) \setminus \mathbb{V}(y^2 + 1, x)$$

$$S_2 = \mathbb{V}(x^2 + y^2 - 1)$$

$$S_3 = \mathbb{V}(x) \setminus \mathbb{V}(y^2 + 1, x)$$

Proposition 2: $H \wedge \neg H_1 \Rightarrow T \wedge \neg T_1$.

Calling sequence: $\texttt{ADGT}(H, T, H_1, T_1)$;

Result:

$$(x, y) \in (S_1 \setminus (A \cup B)) \cup (S_3 \setminus O)$$

Ortic triangle

Proposition 1: $H \Rightarrow T$.

Calling sequence: $\texttt{ADGT}(H, T, 1, 1)$;

Result:

$$
\begin{aligned}
(x, y) \in & S_1 \cup S_2 \cup S_3 \\
& S_1 = \mathbb{V}(x^2 - y^2 - 1) \setminus \mathbb{V}(y^2 + 1, x) \\
& S_2 = \mathbb{V}(x^2 + y^2 - 1) \\
& S_3 = \mathbb{V}(x) \setminus \mathbb{V}(y^2 + 1, x)
\end{aligned}
$$

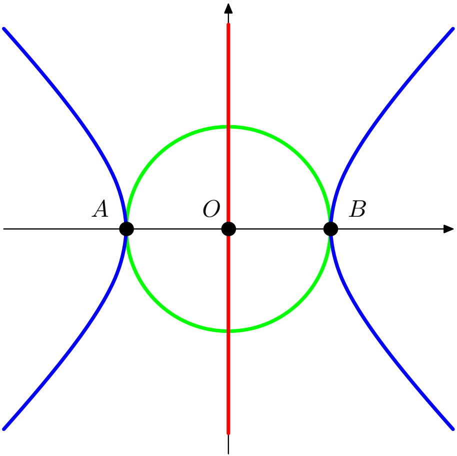
Proposition 2: $H \wedge \neg H_1 \Rightarrow T \wedge \neg T_1$.

Calling sequence: $\texttt{ADGT}(H, T, H_1, T_1)$;

Result:

$$
(x, y) \in (S_1 \setminus (A \cup B)) \cup (S_3 \setminus O)
$$

Ortic triangle



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Geometric Loci: generalized to n–dimensional space

Loci problems are usually considered in 2d or at most 3d, and are obtained by elimination techniques with unprecise definition.

Using the Gröbner Cover we are able to

- generalize them for n-dimensional space,
- precise exactly their irreducible components,
- and assign them a **taxonomy**.

A locus problem has

- a **tracer point** $T(\mathbf{x}) = T(x_1, \dots, x_n)$,
- a set of **auxiliary variables** $\mathbf{u} = (u_1, \dots, u_m)$ usually containing a **mover point** $M(w_1, \dots, w_n)$,
- an **ideal** $F[\mathbf{x}, \mathbf{u}]$ expected to have $n - 1$ degrees of freedom.

Geometric Loci: Taxonomy

`locus` allows to define and determine the **taxonomy** of its components.

Consider the solution of the system:

$$\mathbb{V}(F) = \{(\mathbf{x}, \mathbf{u}) \in \mathbb{C}^{n+m} : \forall f \in F, f(\mathbf{x}, \mathbf{u}) = 0\}.$$

Denote π_1, π_2 the **projections** onto the $\mathbf{x}$ and the $\mathbf{u}$ spaces respectively:

$$\begin{array}{rccc} \pi_1\colon & \mathbb{C}^{n+m} & \to & \mathbb{C}^n \\ & (\mathbf{x}, \mathbf{u}) & \mapsto & \mathbf{x} \end{array} \qquad \begin{array}{rccc} \pi_2\colon & \mathbb{C}^{n+m} & \to & \mathbb{C}^m \\ & (\mathbf{x}, \mathbf{u}) & \mapsto & \mathbf{u} \end{array}$$

and the **anti-image** $\mathcal{A}(\mathbf{x})$ of a locus point $\mathbf{x}$ on the $\mathbf{u}$ space:

$$\begin{array}{rccc} \mathcal{A}\colon & \mathbb{C}^n & \to & \mathbb{C}^m \\ & \mathbf{x} & \mapsto & \pi_2\left(\mathbb{V}(F) \cap \pi_1^{-1}(\mathbf{x})\right) \end{array}$$

Definitions

We can give a formal generic definition of locus in algebraic terms.

Definition (Algebraic locus)

The locus L associated to the parametric polynomial system $F(\mathbf{x}, \mathbf{u})$, is the set

$$L = \pi_1(\mathbb{V}(F)) \subset \mathbb{C}^n.$$

Definition (Normal and non-normal locus)

A **point** $\mathbf{x} \in \mathbb{C}^n$ *of the locus is* **normal** *if* $\dim(\mathcal{A}(\mathbf{x})) = 0$. *The points* $\mathbf{x} \in \mathbb{C}^n$ *of the locus for which* $\dim(\mathcal{A}(\mathbf{x})) > 0$ *are called* **non-normal**. *The set of all normal points is called the* **normal locus** *and the set of all non-normal points is called the* **non-normal locus**.

Proposition

Proposition

The normal *locus* L_{nor} *and the* non-normal *locus* L_{nonor} *are* **disjoint constructible sets**. *We have* $L = L_{\mathrm{nor}} \oplus L_{\mathrm{nonor}}$, *and each of both subsets can be decomposed into* **irreducible components**

$$C_i = \mathbb{V}(\mathfrak{p}_i) \setminus \left(\bigcup_j \mathbb{V}(\mathfrak{p}_{ij}) \right) \tag{1}$$

where all the $\mathfrak{p}_i$ *and all the* $\mathfrak{p}_{ij}$ *are prime ideals, being*

$$L_{\mathrm{nor}} = \bigcup_j C_{\mathrm{nor},j}, \quad L_{\mathrm{nonor}} = \bigcup_j C_{\mathrm{nonor},j}$$

Taxonomy

Definition ("Normal" and "Special" components)

Let C be a component of the **normal locus** *and let $\mathbf{w}$ be the subset of the auxiliary variables $\mathbf{u}$ representing the mover coordinates (if they exist), or the last n' auxiliary variables $\mathbf{u}$ conveniently chosen. A component C of the normal locus is* **"Normal"** *if $\dim(C) = \dim(\mathcal{A}_m(C))$. But it can happen that $\dim(C) > \dim(\mathcal{A}_m(C))$, and then the component is* **"Special"**.

Definition ("Degenerate" and "Accumulation" components)

A component of the **non-normal locus** *is* **"Degenerate"** *if its dimension is $n-1$. If its dimension is smaller than $n-1$, then it is an* **"Accumulation"** *component.*

Surfaces generated by two parallel curves in 3-dimensional space.

Consider a parabola $y_1 = x_1^2$ on the plane $z_1 = -1$ (**floor**) and a parabola $x_2 = y_2^2$ on the plane $z_2 = 1$ (**ceiling**).

Determine the locus formed by the lines relying the points of both parabolas with parallel tangents. The tangent vectors are $(-2x_1, 1, 0)$ and $(1, -2y_2, 0)$. The condition of being parallels is $4x_1y_2 - 1 = 0$. The system is:

$$\begin{aligned} F = & x_1^2 - y_1, z_1 + 1, \\ & y_2^2 - x_2, z_2 - 1, \\ & 4x_1y_2 - 1, \\ & x - x_1 - \lambda(x_2 - x_1), y - y_1 - \lambda(y_2 - y_1), z - z_1 - \lambda(z_2 - z_1) \end{aligned}$$

Applying locus algorithm

Calling sequence:

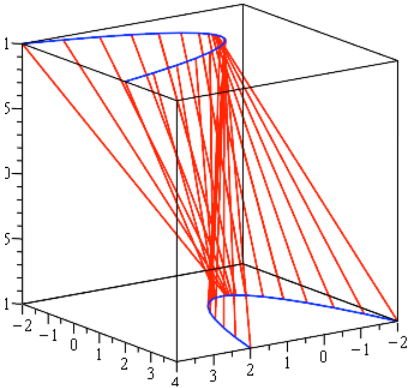
$$
\begin{aligned}
&> \text{LIB "grobcov.lib"};\\
&> \text{ring R} = (0, x, y, z), (\lambda, x_2, y_2, z_2, x_1, y_1, z_1), \text{lp};\\
&> \text{ideal F} = x_1^2 - y_1, y_2^2 - x_2, 4x_1y_2 - 1, z_1 + 1, z_2 - 1,\\
&\qquad x - x_1 - \lambda(x_2 - x_1), y - y_1 - \lambda(y_2 - y_1),\\
&\qquad z - z_1 - \lambda(z_2 - z_1);\\
&> \text{locus(F)};
\end{aligned}
$$

Result of `locus` command

We obtain the result:

$$
\begin{aligned}
S = {} & \mathbb{V}(2048x^3z + 2048x^3 - 4096x^2y^2 + 1152xyz^2 - 1152xy \\
& - 2048y^3z + 2048y^3 + 27z^4 - 54z^2 + 27) \\
& \setminus (\mathbb{V}(z+1, y) \cup \mathbb{V}(z-1, x)) \\
T = {} & \text{Normal} \\
A = {} & \mathbb{V}(z_1 + 1, x_1^2 - y_1, z_2 - 1, x_2 - y_2^2, 4y_2y_1 - x_1, 4y_2x_1 - 1)
\end{aligned}
$$

Richard Serra surface



Artistic perspective of the Richard Serra surface using SURFER-imaginary software

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Geometric Envelopes

Usually the definition of **Envelope** concerns a family of curves or a family of surfaces with $n-1$ degrees of freedom.

We **generalize** the problem to n dimensional space and higher degrees of freedom.

Geometric Envelopes: Definition

Definition (Family of hyper-surfaces)

We say that

$$F(\mathbf{x},\mathbf{u}) = F(x_1,\dots,x_n;u_1,\dots,u_m) = 0$$

and the independent restrictions $C = \{g_1,\dots,g_s\}$ *(with $s < m$)*

$$\begin{cases} g_1(u_1,\dots,u_m) = 0 \\ \dots \\ g_s(u_1,\dots,u_m) = 0 \end{cases}$$

represent a family of hyper-surfaces *of $\mathbb{C}^n$ if F depends at least on 1* parameter u. *Thus, if the G_i are independent*

$$d = \dim(C) = m - s \geq 1.$$

Algebraic Definition

Definition (Algebraic Envelope)

Given a family of hyper-surfaces $F(x_1, \ldots, x_n; u_1, \ldots, u_m) = 0$ *with* m *parameters* $\mathbf{u}$, *constrained by the* $s < m$ *independent equations* $C = \langle g_1(\mathbf{u}), \ldots, g_s(\mathbf{u}) \rangle$, *let*

$$J = \frac{\partial(F, g_1, \ldots, g_s)}{\partial(u_1, \ldots, u_m)}; \quad J_c = \{\mathit{minors}(J) \text{ of order } (s+1) \times (s+1)\}.$$

The set of $\binom{m}{s+1}$ *equations* J_c, *imply that the* **rank of the Jacobian** *is less than or equal than* s. *Consider the ideal* S

$$S = \langle F,\ C,\ J_c \rangle,$$

The algebraic envelope of F, C, *if it exists, is*

$$L = \mathtt{locus}(S).$$

Theorem

Theorem (Associated Tangent element)

Let $F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$ be a family of hyper-surfaces with m parameters $\mathbf{u}$, constrained by the $s < m$ equations $C = \langle g_1(\mathbf{u}), \dots, g_s(\mathbf{u}) \rangle$.

Let E be the envelope, and $E_i = \mathbb{V}(p_i(\mathbf{x})) \setminus \mathbb{V}(\mathfrak{q}_i(\mathbf{x}))$ the C-representation of a **"Normal"** *standard $(n-1)$-dimensional component of E corresponding to a hyper-surface, and $\mathbf{x}^{(0)} \in E_i$ be a* **regular point** *of $p_i(\mathbf{x}) = 0$.*

Then there **exists one Associated Tangent hyper-surface** *$F(\mathbf{x}, \mathbf{u}^{(0)})$ of the family F (or at most a finite set) that passes at point $\mathbf{x}^{(0)}$ (i.e. $F(\mathbf{x}^{(0)}, \mathbf{u}^{0)}) = 0$) and is tangent to $\mathbb{V}(p_i(\mathbf{x}))$ at point $\mathbf{x}^{(0)}$.*

Algorithms

The SINGULAR `grobcov.lib` library has incorporated algorithms related to envelopes:

1. **envelop**: Determines the envelope components and their taxonomies.
2. **AssocTanToEnv**: Determines the associated tangent element of the family passing at a regular point of a “Normal" component.
3. **FamElemsAtEnvComPoints** Determines all the elements of the family passing at a point of the envelope.
4. **discrim** Determines the discriminant wrt to a variable, when an equation is of degree 2.

Example 2: more degrees of freedom

Example 2: In this example we can see the difficulties for determining the **real** points of the envelope, and how can we solve them in certain cases.

Consider the family of spheres of radius 1, centered at point (x_1, y_1, z_1) of a sphere of center $(0, 0, t)$ and radius $\sqrt{t}$. We have the following family and restrictions:

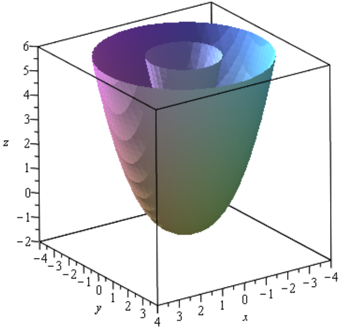
$$\begin{aligned} F =& (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - 1, \\ C =& \langle x_1^2 + y_1^2 + (z_1 - t)^2 - t \rangle. \end{aligned}$$

Result of applying `envelop` command

Applying `envelop`(F, C) we obtain 2 components:

$$\begin{aligned} E_1 = & \mathbb{V}((16x^6 + 48x^4y^2 + 16x^4z^2 - 32x^4z - 56x^4 + 48x^2y^4 + 32x^2y^2z^2 \\ & - 64x^2y^2z - 112x^2y^2 - 32x^2z^3 - 24x^2z^2 + 29x^2 + 16y^6 \\ & + 16y^4z^2 - 32y^4z - 56y^4 - 32y^2z^3 - 24y^2z^2 + 29y^2 \\ & + 16z^4 - 24z^3 - 15z^2 + 38z - 15)) \setminus \mathbb{V}(z - 1, x^2 + y^2) \\ & \text{Normal} \\ E_2 = & \mathbb{V}(z - 1, y, x) \\ & \text{Accumulation} \end{aligned}$$

Image of the envelope



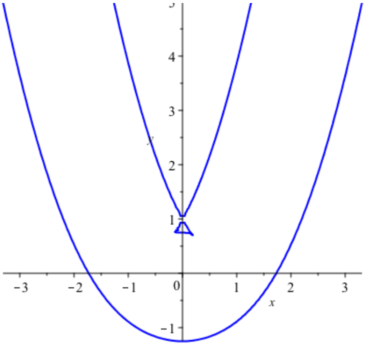


Figure: 3d Section in 2d

We observe two disjoint **real** parts of the envelop component, and the "Accumulation" point.

Result of applying `discrim` command

In order to determine which spheres of the family are real we compute the discriminant of C with respect to t (which is of degree 2 in t).

$$\texttt{discrim}(C,t) = 4z_1 - 1 - x_1^2 - y_1^2$$

The paraboloid $4z - 1 - x^2 - y^2 = 0$ separates two regions. In the interior region there are centers of spheres of the family, whereas there are not in the external part.

We can also consider the subfamily of spheres with centers for which the discriminant is 0.

Section for $y = 0$. Family elements, envelope and separating discriminant

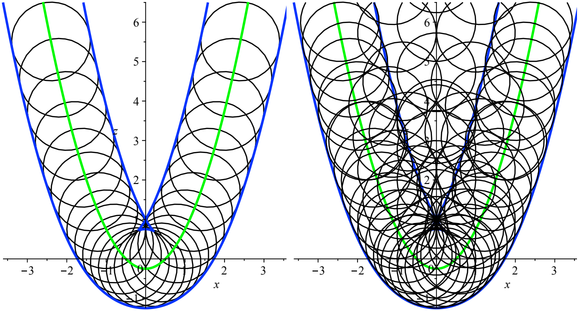


Figure: Section for $y = 0$

Result of applying `AssocTanToEnv` command

Considering a point $P = (x, y, z)$ of the envelop component E_1,

`AssocTanToEnv`(F, C, E_1) gives:

$$t = -\frac{12x^4 + 24x^2y^2 - 4x^2z^2 - 28x^2z - 31x^2 + 12y^4 - 4y^2z^2 - 28y^2z - 31y^2 - 16z^4 - 28z^3 - 3z^2 + 10z + 10}{16z^3 + 24z^2 + 12z - 25}$$

$$z_1 = -\frac{24x^4 + 48x^2y^2 - 8x^2z^2 - 56x^2z - 62x^2 + 24y^4 - 8y^2z^2 - 56y^2z - 62y^2 - 32z^4 - 40z^3 + 18z^2 + 32z - 5}{32z^3 + 48z^2 + 24z - 50}$$

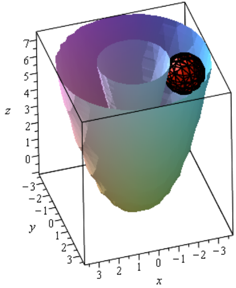
$$y_1 = -\frac{6x^4yz + 32x^4y + 32x^2y^3z + 64x^2y^3 + 16x^2yz^3 - 16x^2yz^2 - 100x^2yz - 116x^2y + 16y^5z + 32y^5}{32z^4 + 16z^3 - 24z^2 - 74z + 50} - \frac{16y^3z^3 - 16y^3z^2 - 100y^3z - 116y^3 - 48yz^4 - 68yz^3}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

$$x_1 = -\frac{16x^5z + 32x^5 + 32x^3y^2z + 64x^3y^2 + 16x^3z^3 - 16x^3z^2 - 100x^3z - 116x^3 + 16xy^4z + 32xy^4}{32z^4 + 16z^3 - 24z^2 - 74z + 50} - \frac{16xy^2z^3 - 16xy^2z^2 - 100xy^2z - 116xy^2 - 48xz^4 - 68xz^3 - 36xz^2 + 36xz + 35x}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

Tangent sphere at a particular point

determining the values of (t, x_1, y_1, z_1) of the parameters of the associated tangent sphere to the component at $P(x, y, z)$.

For example, at point $(-3.519505319, 0, 6)$, the sphere has center $(-2.538358523, 0., 6.193264030)$. It is represented in red in the picture.



This sphere is also the Associated Tangent Sphere of the family at point $(-1.557188274, 0, 6.386408905)$.

The book will be published in January 2019 in ACM Springer Series

Thank you for attending the talk!!