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THE POWER-COMPOSITIONS DETERMINANT AND ITS APPLICATION TO GLOBAL OPTIMIZATION*

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Abstract. Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\alpha = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$. The main result of this paper is an explicit formula for the determinant of the matrix whose entries are $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_p^{\beta_p}$ where $\alpha, \beta \in C(n,p)$. The formula shows that the determinant is positive and has a nice factorization. As an application, it is shown that the polynomials $p_\alpha(x) = (\alpha_1 x_1 + \cdots + \alpha_p x_p)^n$ with $\alpha \in C(n,p)$ form a basis of the vector space $H_n[x_1, \dots, x_p]$ of homogeneous polynomials of degree n in p variables. The result is of interest in the context of global optimization because it allows an explicit representation of polynomials as a difference of convex functions.

Key words. composition, homogeneous polynomial, Vandermonde's determinant, difference of convex functions

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1. Introduction.

Let n, p be positive integers. A *p-composition* of n is a p-tuple $\alpha = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers α_i such that $\alpha_1 + \cdots + \alpha_p = n$. The set of p-compositions of n is denoted by $C(n,p)$. It is well known that its cardinality is

$$|C(n,p)| = \binom{n+p-1}{p-1}.$$

Let $\alpha, \beta \in C(n,p)$. We denote $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_p^{\beta_p}$ where, to be consistent, it is assumed that $0^0 = 1$. Let $M(n,p)$ be the matrix whose entries are α^β. The power-compositions determinant $\Delta(n,p)$ is the determinant of the matrix $M(n,p)$. The rows and columns of $M(n,p) = (\alpha^\beta)$ are labeled by the compositions given in the same ordering. In this way, the determinant $\Delta(n,p)$ does not depend on the ordering in $C(n,p)$.

The main theorem is the following.

THEOREM 1.1. *The determinant of the matrix $M(n,p) = (\alpha^\beta)$, $\alpha, \beta \in C(n,p)$, is*

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}}. \tag{1.1}$$

The determinant $\Delta(n,p)$ appears in the following context. Let $H_n[x_1, \dots, x_p]$ be the vector space of homogeneous polynomials of degree n in p variables with coefficients in a field of characteristic zero. The monomials $x^\beta = x_1^{\beta_1} \cdots x_p^{\beta_p}$, for all $\beta \in C(n,p)$, form a basis of $H_n[x_1, \dots, x_p]$. The dimension of $H_n[x_1, \dots, x_p]$ is

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$$\dim H_n[x_1,\dots,x_p] = |C(n,p)| = \binom{n+p-1}{p-1}.$$

Consider the polynomials

$$p_\alpha(x) = (\alpha_1 x_1 + \cdots + \alpha_p x_p)^n, \qquad \alpha = (\alpha_1,\dots,\alpha_p) \in C(n,p).$$

The multinomial theorem gives

$$(\alpha_1 x_1 + \cdots + \alpha_p x_p)^n = \sum_{\beta \in C(n,p)} \binom{n}{\beta} \alpha^\beta x^\beta.$$

Since the x^β form a basis, the polynomials $\binom{n}{\beta}x^\beta$ form a basis, too. By Theorem 1.1, $\Delta(n,p) \neq 0$ for all $n, p \geq 1$, therefore we have the following corollary.

Corollary 1.2. *The set of polynomials*

$$p_\alpha(x) = (\alpha_1 x_1 + \cdots + \alpha_p x_p)^n, \quad \alpha = (\alpha_1,\dots,\alpha_p) \in C(n,p)$$

is a basis of $H_n[x_1,\dots,x_p]$.

In section 8 we will discuss why this result can be of interest in the context of global optimization.

It will be useful to abbreviate the numbers on the right-hand side of (1.1). Define

$$\begin{aligned} D^*(n,k) &= n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}}; \\ D(n,p) &= \prod_{k=1}^{\min\{n,p\}} (D^*(n,k))^{\binom{p}{k}}. \end{aligned} \tag{1.2}$$

With this notation, the theorem to be proved is $\Delta(n,p) = D(n,p)$.

The paper is basically devoted to the proof of Theorem 1.1 and is organized as follows.

In section 2 we define an order $\succ$ of the monomials for which the matrix $M(n,p)$ is block triangular, with zeros at the upper right part. Ordering the monomials x^β is equivalent to ordering the compositions in $C(n,p)$, which are the labels of rows and columns of the matrix $M(n,p)$. In the diagonal blocks we recognize submatrices $M^*(n,k)$ corresponding to proper k-compositions, which are k-compositions with all its entries different from zero. Thus, we can express the determinant $\Delta(n,p)$ in terms of a product of determinants $\Delta^*(n,k)$ of the matrices $M^*(n,k)$ corresponding to proper k-compositions with values of $k \in [\min\{n,p\}] = \{1,\dots,\min\{n,p\}\}$. Then the proof of $\Delta(n,p) = D(n,p)$ is reduced to the proof of $\Delta^*(n,p) = D^*(n,p)$.

Given m elements $a_1,\dots,a_m$ in a field, the Vandermonde matrix $V(a_1,\dots,a_m)$ is the $m \times m$ matrix whose ith column is $1, a_i, \dots, a_i^{m-1}$. Many generalizations of the Vandermonde matrix have been studied [3, 8, 10], including combinatorial ones [1, 7, 9]. Theorem 1.1 can be considered in this context in the sense that $\Delta^*(n,2)$ can be easily reduced to the Vandermonde's determinant $V(1, 1/(n-1), 1/(n-2), \dots, 1/2)$. In section 3 we study this determinant and some others special cases which are needed later.

In section 4 it is shown that the proposed ordering splits the matrix $M^*(n,p)$ into new block submatrices and that the diagonal blocks are matrices $M^*(n-j,p-1)$ except for constant factors, denoted $f_1(i,j)$. We proceed with a triangulation method which at each step reduces to zero the block matrices to the right of the diagonal block. This process modifies only the initial block matrices by constant factors $f_k(i,j)$. The method is recursive, but a compact form for the recurrence is not simple to obtain. Nevertheless, the recursion that modifies only the numerical factors $f_k(i,j)$ is independent of p. So comparing with the case $p = 2$ that is solved in section 3, we are able to establish an explicit formula for $f_i(i,i)$, which are the only values of f appearing in the recursive formula for the determinant $\Delta^*(n,p)$. The result is a concrete recursion formula for $\Delta^*(n,p)$ in terms of $\Delta^*(i,p-1)$ for i running from $n-1$ till $p-1$.

In section 5 we introduce a class of numbers that we call multiplicative binomial numbers because they have multiplicative properties similar to those of the additive ones for the binomial numbers.

In sections 6 and 7 a recursion rule for $D^*(n,p)$ involving the multiplicative binomial numbers is obtained. It is shown that this recursion is the same and has the same initial values as the recursion satisfied by $\Delta^*(n,p)$. So the proof becomes complete.

Finally, section 8 is concerned with the applications to global optimization theory, which was the initial motivation of the problem. As it is known, representing a function as a difference of convex functions is an essential tool in this context. We show how Corollary 1.2 allows us to do it explicitly for any polynomial function.

Throughout the paper, some elementary properties of binomial numbers are used. They can be found, for instance, in [4].

2. Reduction to proper compositions. A *proper* p-composition of n is a p-composition $\alpha = (\alpha_1, \dots, \alpha_p)$ such that $\alpha_i \geq 1$ for all $i \in [p]$. We denote by $C^*(n,p)$ the set of proper p-compositions of n. Its cardinality is

$$|C^*(n,p)| = \binom{n-1}{p-1}.$$

Let $M^*(n,p)$ be the matrix (α^β) with $\alpha, \beta \in C^*(n,p)$ and let $\Delta^*(n,p)$ be the determinant of $M^*(n,p)$. We first reduce the problem to the calculation of the determinant $\Delta^*(n,p)$. To this end, we need to define an order in $C(n,p)$. Denote by

- $w(\alpha)$ the number of zeros of α;
- $z(\alpha)$ the vector of the positions in increasing order of the zeros in α if $w(\alpha) > 0$, and the vector (0) if $w(\alpha) = 0$;
- $\ell(v)$ the last nonzero entry of v if v is a vector with nonzero entries, and 0 otherwise.

The order $\succ$ is defined as follows: $\alpha \succ \beta$ if

(i) $w(\alpha) > w(\beta)$ or

(ii) $w(\alpha) = w(\beta)$ and $\ell(z(\alpha) - z(\beta)) > 0$ or

(iii) $w(\alpha) = w(\beta)$ and $\ell(z(\alpha) - z(\beta)) = 0$ and $\ell(\alpha - \beta) < 0$.

For instance, the compositions of $C(4,3)$ written in decreasing order $\succ$ are shown in Table 2.1.

A composition α with $w(\alpha) = 2$ can be identified in a natural way with a composition in $C^*(4,1)$; the compositions α with $w(\alpha) = 1$ and $z(\alpha) = (i)$, $i \in [3]$, can be put into one-to-one correspondence with the compositions of $C^*(4,2)$; and the com-

TABLE 2.1
$(C(4,3), \succ)$.

$w(\alpha)$	$z(\alpha)$	Compositions
2	$(2,3)$	(4,0,0),
2	$(1,3)$	(0,4,0),
2	$(1,2)$	(0,0,4),
1	(3)	(3,1,0),(2,2,0),(1,3,0),
1	(2)	(3,0,1),(2,0,2),(1,0,3),
1	(1)	(0,3,1),(0,2,2),(0,1,3),
0	(0)	(2,1,1),(1,2,1),(1,1,2)

positions with $w(\alpha) = 0$ form the set $C^*(4,3)$ of proper 3-compositions of 4. With this ordering, the matrix $M(5,3)$ becomes

$$\left(\begin{array}{c|c|c|ccc|ccc|ccc|ccc}
256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 256 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
81 & 1 & 0 & 27 & 9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 16 & 0 & 16 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 81 & 0 & 3 & 9 & 27 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
81 & 0 & 1 & 0 & 0 & 0 & 27 & 9 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 0 & 16 & 0 & 0 & 0 & 16 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 81 & 0 & 0 & 0 & 3 & 9 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 81 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 9 & 3 & 0 & 0 & 0 \\
0 & 16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 16 & 16 & 0 & 0 & 0 \\
0 & 1 & 81 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 9 & 27 & 0 & 0 & 0 \\ \hline
16 & 1 & 1 & 8 & 4 & 2 & 8 & 4 & 2 & 1 & 1 & 1 & 4 & 2 & 2 \\
1 & 16 & 1 & 2 & 4 & 8 & 1 & 1 & 1 & 8 & 4 & 2 & 2 & 4 & 2 \\
1 & 1 & 16 & 1 & 1 & 1 & 2 & 4 & 8 & 2 & 4 & 8 & 2 & 2 & 4
\end{array}\right).$$

A framed block corresponds to a set of entries such that the rows α of the block have constant values of $w(\alpha)$ and $z(\alpha)$, and the columns β of the block also have constant values of $w(\beta)$ and $z(\beta)$. Inside a block the ordering is given by condition (iii) of the definition, but at this point this is not important; nevertheless it will be significant later on.

From here onward, we assume that the matrix $M(n,p)$ is written with its rows and columns in decreasing $\succ$ order.

PROPOSITION 2.1.

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}} (\Delta^*(n,k))^{\binom{p}{k}}.$$

Proof. First assume $n \geq p$. For $p = 1$ the formulae are obvious. Let $p \geq 2$. Fix a $k \in [p-1]$ and a set of positions $1 \leq i_1 < i_2 < \cdots < i_k \leq p$. The set of rows of the matrix $M(n,p)$ labeled by compositions α with $w(\alpha) = k$ and $z(\alpha) = (i_1, \ldots, i_k)$ are consecutive by the ordering $\succ$, and they are in one-to-one correspondence with the compositions of $C^*(n, n-k)$. The diagonal submatrix of $M(n,p)$ formed by rows α and columns β such that $w(\alpha) = w(\beta)$ and $z(\alpha) = z(\beta)$ is the matrix $M^*(n, n-k)$ (recall the convention $0^0 = 1$).

If $w(\alpha) > w(\beta)$, then there exists a position i such that $\alpha_i = 0$ and $\beta_i \neq 0$. Hence $\alpha^\beta = 0$. If $w(\alpha) = w(\beta)$ and $\ell(z(\alpha) - z(\beta)) \neq 0$, then there also exists a position i such that $\alpha_i = 0$ and $\beta_i \neq 0$. Hence $\alpha^\beta = 0$. It follows that $M(n,p)$ has the following diagonal form:

$$M(n,p) = \begin{pmatrix} M^*(n,1) & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & & & \vdots \\ 0 & \cdots & M^*(n,1) & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ * & \cdots & * & M^*(n,2) & \cdots & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots & & & \vdots \\ * & \cdots & * & 0 & \cdots & M^*(n,2) & \cdots & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * & \cdots & \cdots & M^*(n,p) \end{pmatrix},$$

where the diagonal block $M^*(n,k)$ is repeated $\binom{p}{k}$ times, the $\binom{p}{k}$ possible positions of the k zeros. Since $\Delta(n,p)$ is the product of the determinants of the diagonal blocks, the formula of the proposition follows.

If $p > n$, the sets $C^*(n,k)$ are empty for $k > n$ and the last diagonal block in the matrix $M(n,p)$ is $M(n,n)$. Then the product is to be taken from $k = 1$ to $k = n$. □

According to (1.1) and (1.2), Proposition 2.1 reduces the problem to prove that

$$\Delta^*(n,p) = D^*(n,p) \tag{2.1}$$

for all $n \geq p \geq 1$. In the next section it is shown that (2.1) holds for $p = 1$, $p = 2$, and $p = n$, and in sections 4 and 6 it will be shown that both $\Delta^*(n,p)$ and $D^*(n,p)$ satisfy the same recurrence.

3. Special cases. There are two extreme cases, namely, $p = 1$ and $p = n$. In those cases we have

$$\begin{aligned} &\Delta^*(n,1) = n^n; \\ &D^*(n,1) = n^{\binom{n-1}{1}}\, 1^0\, 2^0 \,\cdots\, (n-1)^0\, n^{\binom{-1}{-1}} = n^{n-1}\, n = n^n; \\ &\Delta^*(n,n) = 1; \\ &D^*(n,n) = n^{\binom{n-1}{n}}\, 1^{n\binom{n-2}{n-2}} = n^0 = 1; \end{aligned}$$

and we notice the coincidence when binomials are extended in the usual form. We are interested in another special case: $p = 2$. In this case the determinant $\Delta^*(n,2)$ can be easily reduced to the well-known Vandermonde's determinant. We will need later the value not only of $\Delta^*(n,2)$ but also of its principal minor of order r, which we denote by $\Delta^*_r(n,2)$. We have

$$\begin{aligned} \Delta^*_r(n,2) &= \begin{vmatrix} (n-1)^{n-1}\cdot 1 & (n-1)^{n-2}\cdot 1^2 & \cdots (n-1)^{n-r}\cdot 1^r \\ (n-2)^{n-1}\cdot 2 & (n-2)^{n-2}\cdot 2^2 & \cdots (n-2)^{n-r}\cdot 2^r \\ & \cdots & \cdots \\ (n-r)^{n-1}\cdot r & (n-r)r^{n-2}\cdot r^2 & \cdots (n-r)^{n-r}\cdot r^r \end{vmatrix} \\ &= \left(\prod_{i=1}^{r}(n-i)^{n-1}\cdot i\right) \begin{vmatrix} 1 & 1/(n-1) & \cdots & (1/(n-1))^{r-1} \\ & \cdots & \cdots & \\ 1 & r/(n-r) & \cdots & (r/(n-r))^{r-1} \end{vmatrix} \\ &= \left(\prod_{i=1}^{r}(n-i)^{n-1}\cdot i\right) \prod_{1\leq i<j\leq r}\left(\frac{j}{n-j}-\frac{i}{n-i}\right) \end{aligned}$$

$$= \left(\prod_{i=1}^{r}(n-i)^{n-1}\cdot i\right)\prod_{1\le i<j\le r}\frac{n(j-i)}{(n-i)(n-j)}$$
$$= \left(\prod_{i=1}^{r}(n-i)^{n-1}\cdot i\right) n^{\binom{r}{2}}\frac{\prod_{i=1}^{r-1} i^{r-i}}{\prod_{i=1}^{r}(n-i)^{r-1}}$$
$$(3.1)\qquad = n^{\binom{r}{2}}\left(\prod_{i=1}^{r} i^{r-i+1}\right)\left(\prod_{i=1}^{r}(n-i)\right)^{(n-r)}.$$

In particular, for $r=n$, we have

$$\Delta^*(n,2) = n^{\binom{n}{2}}\prod_{i=1}^{n} i^{n-i+1},$$

which is the expression given by formula (1.2) for $D^*(n,2)$. Therefore, $\Delta^*(n,p) = D^*(n,p)$ for $p=1$, $p=2$, and $p=n$.

4. A recurrence for $\Delta^*(n,p)$. Unfortunately, the method for $p=2$ cannot be easily generalized for $p\ge 3$. Nevertheless, a triangulation method and the knowledge of the values $\Delta_r^*(p,2)$ lead to a recurrence for $\Delta^*(n,p)$.

Let $\alpha = (\alpha_1,\dots,\alpha_p)\in C^*(n,p)$. We denote $\overline{\alpha} = (\alpha_1,\dots,\alpha_{p-1})$. In the definition of $\succ$, condition (iii) orders the compositions in $C^*(n,p)$ in such a way that compositions α with $\alpha_p = i$ are consecutive and immediately succeeded by those with $\alpha_p = i+1$. The submatrix of $M^*(n,p)$ corresponding to rows α with $\alpha_p = i$ and columns β with $\beta_p = j$ can be written in the form

$$S_{ij} f_1(i,j),\quad i,j\in[n-1],$$

where S_{ij} is the matrix $(\overline{\alpha}^{\overline{\beta}})$ and $f_1(i,j) = i^j$. Observe that $S_{ii} = M^*(n-i,p-1)$. Now we begin a triangulation method that will only modify the factors $f_1(i,j)$. Fix a column β with $\beta_p = j\ge 2$. For each column γ with $\gamma_p = 1$ and $\gamma_k\ge\beta_k$ for $k\in[p-1]$, add to the column β the column γ multiplied by

$$-\frac{1}{(n-1)^{j-1}}\binom{j-1}{\overline{\gamma}-\overline{\beta}}\frac{f_1(1,j)}{f_1(1,1)}.$$

Since the differences $\overline{\gamma}-\overline{\beta}$ are all the compositions of $j-1$, the column β has been modified to the value

$$\begin{aligned}
&\overline{\alpha}^{\overline{\beta}} f_1(i,j) - \sum_{\gamma}\frac{1}{(n-1)^{j-1}}\binom{j-1}{\overline{\gamma}-\overline{\beta}}\frac{f_1(1,j)}{f_1(1,1)}\overline{\alpha}^{\overline{\gamma}} f_1(i,1)\\
= \;& \overline{\alpha}^{\overline{\beta}}\left\{f_1(i,j) - \left(\frac{n-i}{n-1}\right)^{j-1}\frac{f_1(1,j)}{f_1(1,1)} f_1(i,1)\right\}\\
= \;& \overline{\alpha}^{\overline{\beta}} f_2(i,j).
\end{aligned}$$

The transformed matrix, which has the same determinant as $M^*(n,p)$, is of the form

$$(S_{ij} f_2(i,j)),\quad i,j\in[n-1].$$

Observe that the blocks have been modified only by the numerical factor $f_2(i,j)$. Moreover, for $i=1$ and $j\ge 2$, we have $f_2(i,j)=0$ and the corresponding block is the zero matrix.

The procedure can be iterated. Assume that we have the $(r-1)$-iterated matrix in the form

$$\left(S_{ij} f_r(i,j)\right), \quad i,j \in [n-1],$$

with $f_k(i,j) = 0$ for $k < r$ and $j > i$. Select a column β with $\beta_p = j > r$. For each column γ with $\gamma_p = r$ and $\gamma_k \geq \beta_k$ for $k \in [p-1]$, add to the column β the column γ multiplied by

$$-\frac{1}{(n-r)^{j-r}} \binom{j-r}{\overline{\gamma}-\overline{\beta}} \frac{f_r(r,j)}{f_r(r,r)}.$$

The differences $\overline{\beta} - \overline{\gamma}$ are all the compositions of $j-r$. Then, the column β has been modified to the value

$$\begin{aligned} & \overline{\alpha}^{\overline{\beta}} f_r(i,j) - \sum_{\gamma} \frac{1}{(n-r)^{j-r}} \binom{j-r}{\overline{\gamma}-\overline{\beta}} \frac{f_r(r,j)}{f_1(r,r)} \overline{\alpha}^{\overline{\gamma}} f_r(i,r) \\ = \ & \overline{\alpha}^{\overline{\beta}} \left\{ f_r(i,j) - \left(\frac{n-i}{n-r}\right)^{j-r} \frac{f_r(r,j)}{f_r(r,r)} f_r(i,r) \right\} \\ = \ & \overline{\alpha}^{\overline{\beta}} f_{r+1}(i,j), \end{aligned}$$

and $f_{r+1}(i,j) = 0$ for $i < r+1$ and $j > i$. We have the recurrence

$$f_{r+1}(i,j) = f_r(i,j) - \left(\frac{n-i}{n-r}\right)^{j-r} \frac{f_r(r,j)}{f_r(r,r)} f_r(i,r), \tag{4.1}$$

and, after $n-p$ iterations, the matrix becomes

$$\left(S_{ij} f_{n-p+1}(i,j)\right), \quad i,j \in [n-1],$$

with $f_{n-p+1}(i,j) = 0$ for $j > i$. Therefore $\Delta^*(n,p)$ is the product of the determinants of the diagonal blocks $S_{ii} f_i(i,i) = M^*(n-i, p-1) f_i(i,i)$. Denoting $f_i = f_i(i,i)$, we have

$$\Delta^*(n,p) = \left(\prod_{i=1}^{n-p+1} f_i^{\binom{n-i-1}{p-2}} \right) \left(\prod_{i=1}^{n-p+1} \Delta^*(n-i, p-1) \right).$$

The recurrence (4.1) does not seem easy to solve. Nevertheless, note that it does not depend on p. In the product

$$F(n,p) = \prod_{i=1}^{n-p+1} f_i^{\binom{n-i-1}{p-2}} \tag{4.2}$$

the exponents and the number of factors depend on p, but the f_i's themselves do not. Then, consider the case $p = 2$ and apply the method for calculating the minors of order r and $r-1$. We have

$$\frac{\Delta_r^*(n,2)}{\Delta_{r-1}^*(n,2)} = \frac{\left(\prod_{i=1}^{r} f_i \right) \left(\prod_{i=1}^{r} (n-i)^{n-i} \right)}{\left(\prod_{i=1}^{r-1} f_i \right) \left(\prod_{i=1}^{r-1} (n-i)^{n-i} \right)} = (n-r)^{n-r} f_r. \tag{4.3}$$

On the other hand, the quotient of the minors can be evaluated directly by (3.1):

$$\begin{aligned}\frac{\Delta^*_r(n,2)}{\Delta^*_{r-1}(n,2)} &= \frac{n^{\binom{r}{2}}\prod_{i=1}^{r} i^{r-i+1}\prod_{i=1}^{r}(n-i)^{(n-r)}}{n^{\binom{r-1}{2}}\prod_{i=1}^{r-1} i^{r-i}\prod_{i=1}^{r-1}(n-i)^{(n-r+1)}}\\ &= \frac{n^{r-1}\left(\prod_{i=1}^{r} i\right)(n-r)^{n-r}}{\prod_{i=1}^{r-1}(n-i)}\\ &= \frac{n^r}{\binom{n}{r}}(n-r)^{n-r}. \end{aligned} \tag{4.4}$$

Then, comparing (4.3) to (4.4), we obtain

$$f_r = \frac{n^r}{\binom{n}{r}} = r!\frac{n^{r-1}}{(n-1)\cdots(n-r+1)}, \tag{4.5}$$

which gives an explicit formula for $F(n,p)$.

We have proved that the determinants $\Delta^*(n,p)$ satisfy the recurrence

$$\Delta^*(n,p) = F(n,p)\prod_{i=1}^{n-p+1}\Delta^*(n-i,p-1), \tag{4.6}$$

with $F(n,p)$ given by (4.2) and (4.5). The last part of the proof consists of proving that the numbers $D^*(n,p)$ satisfy the same recurrence.

5. The multiplicative binomial numbers. To obtain a recurrence for $D^*(n,p)$ we introduce a class of numbers which have similar properties with respect to the product to those of the binomial numbers with respect to the sum.

Let $n \geq p \geq 1$ be integers. The *multiplicative binomial number* $a(n,p)$ is the number

$$a(n,p) = \prod_{i=1}^{n-p+1} i^{\binom{n-i}{p-1}} = 1^{\binom{n-1}{p-1}}\cdot 2^{\binom{n-2}{p-1}}\ \cdots\ (n-p+1)^{\binom{p-1}{p-1}}.$$

From the definition, we have

$$\begin{aligned} a(n,1) &= n!,\\ a(n,2) &= 1^{n-1}\,2^{n-2}\dots(n-1),\\ a(n,n-1) &= 2,\\ a(n,n) &= 1. \end{aligned} \tag{5.1}$$

A straightforward calculation and the binomial recurrence $\binom{n-k}{p-i}+\binom{n-k}{p-i-1}=\binom{n-k+1}{p-i}$ shows that

$$a(n,p) = a(n-1,p-1)\,a(n-1,p). \tag{5.2}$$

TABLE 5.1

$n \setminus p$	0	1	2	3	4	5	
1	2	1					
2	3	2	1				
3	4	6	2	1			
4	5	24	12	2	1		
5	6	120	288	24	2	1	
6	7	720	34560	6912	48	2	1

Moreover,

$$\begin{aligned} a(n,p) &= a(n-1,p-1)\,a(n-1,p) \\ &= a(n-1,p-1)\,a(n-2,p-1)\,a(n-2,p) \\ &\cdots \\ &= a(n-1,p-1)\,a(n-2,p-1)\,\cdots\,a(p-1,p-1). \end{aligned} \tag{5.3}$$

The properties (5.2) and (5.3) allow us to translate every additive property of the binomial numbers into a multiplicative one of the numbers $a(n,p)$.

If we want to define $a(n,p)$ also for $p=0$, then (5.1) and (5.2) give

$$(n+1)! = a(n+1,1) = a(n,0)\,a(n,1) = a(n,0)n!$$

and $a(n,0)$ must be defined as

$$a(n,0) = n+1.$$

Table 5.1 gives some values of $a(n,p)$. It is constructed as the Pascal triangle of binomial numbers but uses multiplication instead of addition.

6. A recurrence for $D^*(n,p)$. The number $D^*(n,p)$ has been defined in (1.2) by

$$D^*(n,p) = n^{\binom{n-1}{p}} \prod_{i=1}^{n-p+1} i^{(n-i+1)\binom{n-i-1}{p-2}}. \tag{6.1}$$

Then we have

$$\begin{aligned} &D^*(n-1,p-1)D^*(n-1,p) \\ &= \left((n-1)^{\binom{n-2}{p-1}} \prod_{i=1}^{n-p+1} i^{(n-i)\binom{n-i-2}{p-3}} \right) \cdot \left((n-1)^{\binom{n-2}{p}} \prod_{i=1}^{n-p} i^{(n-i)\binom{n-i-2}{p-2}} \right) \\ &= (n-1)^{\binom{n-1}{p}} \prod_{i=1}^{n-p+1} i^{(n-i)\binom{n-i-1}{p-2}}. \end{aligned}$$

Comparing this with (6.1) leads to

$$\begin{aligned} D^*(n,p) &= \left(\frac{n}{n-1}\right)^{\binom{n-1}{p}} \left(\prod_{i=1}^{n-p+1} i^{\binom{n-i-1}{p-2}} \right) D^*(n-1,p-1)D^*(n-1,p) \\ &= \left(\frac{n}{n-1}\right)^{\binom{n-1}{p}} a(n-1,p-1)D^*(n-1,p-1)D^*(n-1,p). \end{aligned}$$

Let us define $D^{**}(n,p)$ by

$$D^*(n,p) = n^{\binom{n-1}{p}} D^{**}(n,p). \tag{6.2}$$

We have

$$D^{**}(n,p) = a(n-1,p-1)D^{**}(n-1,p-1)D^{**}(n-1,p).$$

Iterating and taking into account (5.3) and that $D^*(p,p) = 1$ for any p, we obtain

$$D^{**}(n,p) = a(n,p)\prod_{i=1}^{n-p} D^{**}(n-i,p-1).$$

Substituting (6.2), we obtain

$$D^*(n,p) = b(n,p)a(n,p)\prod_{i=1}^{n-p} D^*(n-i,p-1), \tag{6.3}$$

where

$$b(n,p) = \frac{n^{\binom{n-1}{p}}}{\prod_{i=1}^{n-p}(n-i)^{\binom{n-i-1}{p-1}}}. \tag{6.4}$$

7. The last step.

LEMMA 7.1. *$F(n,p) = b(n,p)a(n,p)$.*

Proof. From the definition (4.2) of $F(n,p)$ and the values (4.5) of f_i, we have

$$\begin{aligned} F(n,p) &= \prod_{i=1}^{n-p+1} f_i^{\binom{n-i-1}{p-2}} \\ &= \prod_{i=1}^{n-p+1}\left(i!\frac{n^{i-1}}{(n-1)\cdots(n-i+1)}\right)^{\binom{n-i-1}{p-2}} \\ &= \left(\prod_{i=1}^{n-p+1} i!^{\binom{n-i-1}{p-2}}\right)\prod_{i=1}^{n-p+1}\left(\frac{n^{i-1}}{(n-1)\cdots(n-i+1)}\right)^{\binom{n-i-1}{p-2}}. \end{aligned}$$

In the product of powers of factorials, an $i \in [n-p+1]$ has the exponent

$$\binom{n-i-1}{p-2} + \binom{n-i-2}{p-2} + \cdots + \binom{p-2}{p-2} = \binom{n-i}{p-1}.$$

So the first product is

$$\prod_{i=1}^{n-p+1} i!^{\binom{n-i-1}{p-2}} = \prod_{i=1}^{n-p+1} i^{\binom{n-i}{p-1}} = a(n,p).$$

In the second product, the exponent of n is

$$\binom{n-3}{p-2} + 2\binom{n-4}{p-2} + \cdots + (n-p)\binom{p-2}{p-2} = \binom{n-1}{p}$$

and the exponent of $n-i$ is

$$-\binom{n-i-2}{p-2}-\binom{n-i-3,}{p-2}-\cdots-\binom{p-2}{p-2}=-\binom{n-i-1}{p-1}.$$

So we have

$$F(n,p)=\frac{a(n,p)\,n^{\binom{n-1}{p}}}{\prod_{i=1}^{n-p}(n-i)^{\binom{n-i-1}{p-1}}}=a(n,p)\,b(n,p). \qquad \square$$

THEOREM 7.2. *The determinant of the matrix $M^*(n,p)$ is*

$$\Delta^*(n,p)=D^*(n,p)=n^{\binom{n-1}{k}}\prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}}.$$

Proof. By induction on p. For $p=1,2$ the equality has been proved in section 2. Assume that $p\geq 3$ and that the theorem holds for $p-1$. Then (4.6) gives

$$\Delta^*(n,p)=F(n,p)\prod_{i=1}^{n-p+1}\Delta^*(n-i,p-1).$$

By Lemma 7.1 and the induction hypothesis, we have

$$\Delta^*(n,p)=b(n,p)a(n,p)\prod_{i=1}^{n-p+1}D^*(n-i,p-1),$$

and by (6.3)

$$\Delta^*(n,p)=D^*(n,p). \qquad \square$$

COROLLARY 7.3. *Let $n\geq p\geq 1$ be integers. The set of polynomials*

$$p_\alpha(x)=(\alpha_1x_1+\cdots+\alpha_px_p)^n, \qquad \alpha=(\alpha_1,\ldots,\alpha_p)\in C^*(n,p)$$

is a basis of the subspace $\langle x^\alpha:\alpha\in C^*(n,p)\rangle$ of $H_n[x_1,\ldots,x_p]$ generated by the proper monomials.

Corollaries 1.2 and 7.3 can be interpreted in the context of Gröbner bases and convex polytopes [11]. Corollary 1.2 states that the ideal I generated by the convex polynomials $p_\alpha(x)$ is the monomial ideal $\langle x^\beta:\beta\in C(n,p)\rangle$, for which the set $\{x^\beta:\beta\in C(n,p)\}$ is a universal Gröbner basis. The same is true for Corollary 7.3 relative to proper compositions. We note that the natural extension of the order $\succ$ defined in section 2 does not have the monotony property and, consequently, is not a monomial order.

8. Polynomials as difference of convex functions. In this section we show how our result can be of interest in the context of global optimization. We refer to [12] for details about convex sets and functions and difference of convex (d.c.) functions.

A subset $\Omega\subset\mathbb{R}^p$ is *convex* if $(1-\lambda)x+\lambda y\in\Omega$ whenever $x,y\in\Omega$, $\lambda\in\mathbb{R}$, $0\leq\lambda\leq 1$. Let $\Omega\subset\mathbb{R}^p$ be a convex set. A function $f\colon\Omega\to\mathbb{R}$ is *convex* if $f((1-\lambda)x+\lambda y)\leq(1-\lambda)f(x)+\lambda f(y)$ whenever $x,y\in\Omega$, $\lambda\in\mathbb{R}$, $0\leq\lambda\leq 1$. A function

$f\colon \Omega \to \mathbb{R}$ is a *difference of two convex functions* or a *d.c.* function if there exists two convex functions $f_1(x)$ and $f_2(x)$ on Ω such that $f(x) = f_1(x) - f_2(x)$ for all $x \in \Omega$. The class $DC(\Omega)$ of d.c. functions on Ω is the smallest vector space containing all convex functions on Ω. H. Tuy [12, p. ix] has pointed out that "...virtually every nonconvex optimization problem can be described in terms of functions representable as differences of convex functions...." A theorem by Hiriart-Urruty [6] proves that every function $f \in C^2(\mathbb{R}^p)$ is a d.c. function on any compact convex set $\Omega \subset \mathbb{R}^p$. In particular, any polynomial function is a d.c. function on any compact convex set. Nevertheless, how to represent effectively a polynomial as a difference of two convex polynomials has been considered a difficult problem, which arises, for instance, in the short-term hydrothermal coordination of electricity generation problem [2, 5].

Let $\mathbb{R}_+$ be the set of nonnegative real numbers. The linear functions $L_\alpha(x) = \alpha_1 x_1 + \cdots + \alpha_p x_p$, where $\alpha \in C(n,p)$, are convex and nonnegative in the convex set $\mathbb{R}_+^p$. Since the functions $t \mapsto t^n$ are increasing in $\mathbb{R}_+$, the functions $p_\alpha(x) = (L_\alpha(x))^n$ are convex in $\mathbb{R}_+^p$. Corollary 1.2 implies that, given a homogeneous polynomial $h(x) \in H_n[x_1,\ldots,x_p]$, the polynomial $h(x)$ can be expressed as

$$h(x) = \sum_{\alpha \in C(n,p)} \lambda_\alpha p_\alpha(x),$$

where the coefficients λ_α can be found from the coefficients of $h(x)$ and the inverse of $M(n,p)$. Let P and N be the set of $\alpha \in C(n,p)$ such that $\lambda_\alpha > 0$ and $\lambda_\alpha < 0$, respectively. Then

$$h(x) = \left(\sum_{\alpha \in P} \lambda_\alpha p_\alpha(x) \right) - \left(\sum_{\alpha \in N} (-\lambda_\alpha) p_\alpha(x) \right)$$

is an explicit representation of $h(x)$ as a difference of two convex functions on $\mathbb{R}_+^p$.

If $f(x) = f(x_1,\ldots,x_p)$ is a polynomial of total degree m, then $f(x) = h_0(x) + \cdots + h_m(x)$, where $h_j(x) \in H_j[x_1,\ldots,x_p]$. From the above representation of each $h_j(x)$ as a d.c. function, we obtain an expression of $f(x)$ as a difference of two convex functions on $\mathbb{R}_+^p$.

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