

Presentation of the book *The Gröbner Cover*

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Abstract. The aim of this paper is to present the research book “The Gröbner Cover” [1] that will be published very soon. This book is divided into two parts, one theoretical and one focusing on applications, and offers a complete description of the Canonical Gröbner Cover, the most accurate algebraic method for discussing parametric polynomial systems. It also includes applications to the Automatic Deduction of Geometric Theorems, Loci Computation and Envelopes.

The theoretical part is a self-contained exposition on the theory of Parametric Gröbner Systems and Bases. It begins with Weispfenning’s introduction of Comprehensive Gröbner Systems (CGS) in 1992, and provides a complete description of the Canonical Gröbner Cover (GC), which includes a canonical discussion of a set of parametric polynomial equations developed by Michael Wibmer and the author [2].

In turn, the application part selects three problems for which the Gröbner Cover offers valuable new perspectives. The automatic deduction of geometric theorems (ADGT) becomes fully automatic and straightforward using GC, representing a major improvement on all previous methods. In terms of loci and envelope computation, GC makes it possible to introduce a taxonomy of the components and automatically compute it. The book also generalizes the definition of the envelope of a family of hypersurfaces, and provides algorithms for its computation, as well as for discussing how to determine the real envelope.

All the algorithms described in the book have also been included in the software library `grobcov.lib` implemented in Singular by the author, and serve as a User Manual for it.

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Keywords. Gröbner Cover, parametric polynomial systems, canonical discussion.

1. Introduction

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Bibliography

All the algorithms described in the book are implemented in the “*grobcov.lib*” library of SINGULAR. The book can also be used as software Manual for the library.

2. The Gröbner Cover

The existence of the Gröbner Cover of a parametric polynomial $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal is a consequence of Wibmer’s Theorem. It was described in (2010) by A. Montes and M. Wibmer [2].

Wibmer’s theorem establishes that for an *homogeneous parametric polynomial ideal* $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal, it exists

- a canonical partition of the parameter space into locally closed segments $S_i = \mathbf{V}(\mathbf{p}_i) \setminus \mathbf{V}(\mathbf{q}_i)$;
- each segment accepts a set of I -regular functions B_i that specializes to the reduced Gröbner basis for every point $\mathbf{a} \in S_i$,
- preserving the set of leading power products (lpp_i) of the basis,
- and each segment S_i having a different set of lpp_i .

The set of triplets (lpp_i, B_i, S_i) constitutes the *Gröbner Cover* of the ideal I .

For a *non-homogeneous parametric ideal* we proceed as follows:

- homogenize the ideal,
- determine its Gröbner Cover,
- and dehomogenize.

The result is the canonical *Gröbner Cover* of the non-homogeneous ideal. It can contain segments with the same set of lpp_i , but deriving from different sets of lpph_i of the homogenized ideal, it is still canonic.

2.1. Example: The singular points of a conic

We give an example to show the reduced number of segments that GC generates, and the canonical character of the output. We consider a conic

$$x^2 + 2cxy + by^2 + 2dx + 2ey + f = 0$$

and determine which are their singular points, depending on the parameter values. We can see that the result represents the canonical classification of a conic by its singular points. We consider the ideal

$$\begin{cases} x^2 + 2cxy + by^2 + 2dx + 2ey + f = 0 \\ 2x + 2cy + 2d = 0 \\ 2cx + 2by + 2e = 0. \end{cases}$$

Applying the `grobcov` command of the SINGULAR “*grobcov.lib*” library and using the option “`showhom`”, 1 in order to obtain information about the behaviour of the conic at infinity, which as we shall see, is significant for this problem. The Gröbner Cover determines four distinct segments, depending on the parameter values. We also obtain information about the `lpp` of the homogenized system (due to the option “`showhom`”, 1). The result can be summarized in the following table:

1. Generic case: `lpp = {1}`
 Basis: `{1}`
 Segment: $\mathbb{C}^5 \setminus \mathbf{V}(bd^2 - bf + c^2f - 2cde + e^2)$
 Description: Non-degenerate conic without singular points.
`lpph=1`

<p>2. $\text{lpp} = \{y, x\}$ Basis: $\left\{ y - \frac{d^2 - f}{cd - e}, x - \frac{cf - de}{cd - e} \right\}$ Segment: $\mathbf{V}(bd^2 - bf + c^2f - 2cde + e^2) \setminus \mathbf{V}(cd - e, b - c^2)$ Description: A unique singular point: two intersecting lines. $\text{lpph}=y,x$</p>
<p>3. $\text{lpp} = \{x\}$ Basis: $\{x + cy + d\}$ Segment: $(\mathbf{V}(d^2 - f, cf - de, cd - e, b - c^2))$ Description: Singular double line. $\text{lpph}=x$</p>
<p>4. $\text{lpp} = \{1\}$ Basis: $\{1\}$. Segment: $= \mathbf{V}(cd - e, b - c^2) \setminus \mathbf{V}(d^2 - f, cf - de, cd - e, b - c^2)$ Description: Special case without singular points: two parallel lines. $\text{lpph}=@t,x$</p>

The above table gives a precise and complete discussion, classifying the conics by their singular points into 4 segments given as the difference between two varieties. The default option determines the P-representation (PREP) of the segment, which in this example is almost the same as the C-representation (CREP), because the segments have only one single component with one single hole. (Using option "rep", 1 we would obtain the CREP). The lpph represents the set of lpp of the homogenized ideal that uses the homogenizing variable @t.

Even if segments 1 and 4 have the same basis $\{1\}$, showing that the system has no singular points, they correspond to two different types of conics. In segment 1 there is no singular point even considering the points at infinity, as shown by $\text{lpph}=1$, so that the homogenized ideal does not have any solution either. Thus the conic has no singular point at infinity.

For segment 4 the basis is also 1, showing that the conic has no singular points. But $\text{lpph}=@t,x$ shows that at infinity it has a solution. Substituting the values of $b = c^2$ and $e = cd$, determined by the segment, the variety becomes

$$\begin{aligned} V &= \mathbf{V}(x^2 + 2cxy + c^2y^2 + 2dx + 2cdy + f) \\ &= \mathbf{V}\left((x + cy + d + \sqrt{d^2 - f})(x + cy + d - \sqrt{d^2 - f})\right) \\ &= \mathbf{V}\left(x + cy + d + \sqrt{d^2 - f}\right) \cup \mathbf{V}\left(x + cy + d - \sqrt{d^2 - f}\right), \end{aligned}$$

showing that, for $d^2 - f \geq 0$, it consists of two real parallel lines. The two lines intersect at infinity in the projective space, thus the conic has a singular point at infinity. This is the reason why segments 1 and 4 are separated in the Gröbner Cover, for which the different segments correspond to different lpph. (The segments are lpph-segments).

In the Application part, the book selects three problems for which the Gröbner Cover offers valuable new perspectives: Automatic Deduction of Geometric Theorems, Geometric Loci and Geometric Envelopes.

3. Automatic Deduction of Geometric Theorems

Consider a geometric proposition of the form

$$(H \wedge \neg H_1) \Rightarrow (T \wedge \neg T_1).$$

The book provides an *automatic algorithm* (ADGT) for obtaining *supplementary conditions* for transforming the proposition into a theorem. Consider as example the orthic triangle.

Let $A(-1, 0), B(1, 0), C(x, y)$ be the vertices of a triangle, and let $A'(x_1, y_1), B'(x_2, y_2), C'(x, 0)$ be the feet of the heights. The *orthic triangle* of ABC is $A'B'C'$. See Figure 1.

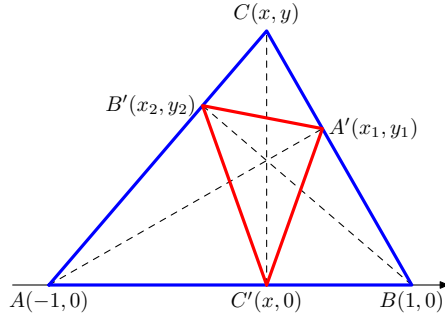


FIGURE 1. Orthic triangle

Associated to the orthic triangle, consider the following ideals of hypothesis and thesis.

Hypothesis H: The triangle $A'B'C'$ is the orthic triangle of ABC

$$H = -yx_1 + (x-1)y_1 + y, (x-1)(x_1+1) + yy_1, \\ -yx_2 + (x+1)y_2 - y, (x+1)(x_2-1) + yy_2$$

Hypothesis H₁: The triangle ABC is *degenerate* (we shall deny it):

$$H_1 = y$$

Thesis T: The orthic triangle $A'B'C'$ is *isosceles* ($\overline{A'C'} = \overline{B'C'}$):

$$T = (x_1 - x)^2 + y_1^2 - (x_2 - x)^2 - y_2^2$$

Thesis T₁: The orthic triangle $A'B'C'$ is *degenerate* (points aligned, we shall deny it):

$$T_1 = x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1$$

We consider now two propositions.

Proposition 1: $H \Rightarrow T$. The orthic triangle is isosceles.

Calling sequence: ADGT($H, T, 1, 1$); Result:

$$(x, y) \in S_1 \cup S_2 \cup S_3 \\ S_1 = \mathbf{V}(x^2 - y^2 - 1) \setminus \mathbf{V}(y^2 + 1, x) \\ S_2 = \mathbf{V}(x^2 + y^2 - 1) \\ S_3 = \mathbf{V}(x) \setminus \mathbf{V}(y^2 + 1, x)$$

We obtain three components, except complex points that are irrelevant, for the supplementary conditions for Proposition 1 to become a theorem (See Figure 2, left). But part of the supplementary conditions correspond to degenerate triangles ABC or degenerate orthic triangles $A'B'C'$. Thus we consider a new proposition, in which we eliminate degenerate triangles.

Proposition 2: $H \wedge \neg H_1 \Rightarrow T \wedge \neg T_1$. The orthic triangle $A'B'C'$ of a non-degenerate triangle ABC is isosceles but non-degenerate.

Calling sequence: $\text{ADGT}(H, T, H_1, T_1)$; Result:

$$(x, y) \in (S_1 \setminus (A \cup B)) \cup (S_3 \setminus O)$$

The result is that the points of the x -axis, which correspond to degenerate ABC are eliminated from the supplementary conditions, as well as the component S_2 which corresponds to degenerate $A'B'C'$. (See Figure 2, right).

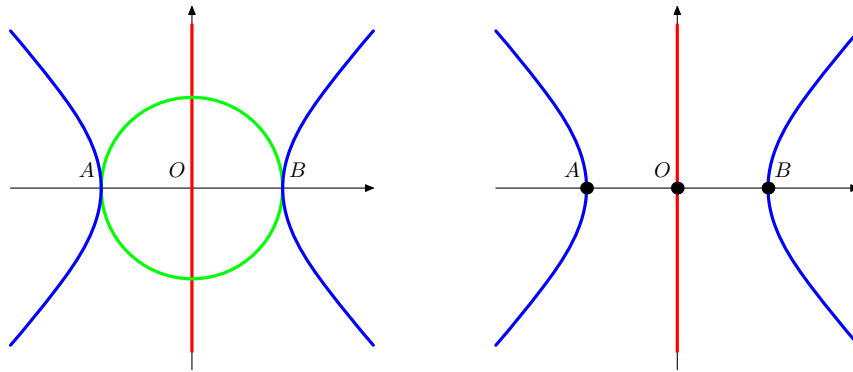


FIGURE 2. Proposition 1,

Proposition 2

4. Geometric Loci

Loci problems are usually considered in 2d or at most 3d, and are obtained by elimination techniques with unprecise definition.

Using the Gröbner Cover we are able to

- generalize them for n -dimensional space,
- precise their irreducible components,
- and assign a *taxonomy* to every component.

A locus problem has

- A *tracer point* $T(\mathbf{x}) = T(x_1, \dots, x_n)$, corresponding to the coordinates \mathbf{x} of the locus points.
- A set of *auxiliary variables* $\mathbf{u} = (u_1, \dots, u_m)$ usually containing a *mover point* $M(w_1, \dots, w_n)$,
- An *ideal* $F[\mathbf{x}, \mathbf{u}]$ expected to have $n - 1$ degrees of freedom.

`locus` allows to define and determine the *taxonomy* of its components.

Consider the solution of the system:

$$\mathbf{V}(F) = \{(\mathbf{x}, \mathbf{u}) \in \mathbb{C}^{n+m} : \forall f \in F, f(\mathbf{x}, \mathbf{u}) = 0\}.$$

Denote π_1, π_2 the *projections* onto the \mathbf{x} and the \mathbf{u} spaces respectively:

$$\begin{array}{ccc} \pi_1: & \mathbb{C}^{n+m} & \rightarrow & \mathbb{C}^n \\ & (\mathbf{x}, \mathbf{u}) & \mapsto & \mathbf{x} \end{array} \quad \begin{array}{ccc} \pi_2: & \mathbb{C}^{n+m} & \rightarrow & \mathbb{C}^m \\ & (\mathbf{x}, \mathbf{u}) & \mapsto & \mathbf{u} \end{array}$$

and the *anti-image* $\mathcal{A}(\mathbf{x})$ of a locus point \mathbf{x} on the \mathbf{u} space:

$$\begin{aligned} \mathcal{A}: \mathbb{C}^n &\rightarrow \mathbb{C}^m \\ \mathbf{x} &\mapsto \pi_2(\mathbf{V}(F) \cap \pi_1^{-1}(\mathbf{x})) \end{aligned}$$

We can give a formal generic definition of locus in algebraic terms.

Definition 4.1 (Algebraic locus). The locus L associated to the parametric polynomial system $F(\mathbf{x}, \mathbf{u})$, is the set

$$L = \pi_1(\mathbf{V}(F)) \subset \mathbb{C}^n.$$

Definition 4.2 (Normal and non-normal locus). We define two class of locus points $\mathbf{x} \in \mathbb{C}^n$:

$$\begin{aligned} \text{Normal point } \mathbf{x} \text{ if} & \quad \dim(\mathcal{A}(\mathbf{x})) = 0. \\ \text{Non-normal point } \mathbf{x} \text{ if} & \quad \dim(\mathcal{A}(\mathbf{x})) > 0 \end{aligned}$$

The set of all normal points is called the *normal locus* and the set of all non-normal points is called the *non-normal locus*.

Proposition 4.3. The normal locus L_{nor} and the non-normal locus L_{nonor} are disjoint constructible sets. We have $L = L_{\text{nor}} \oplus L_{\text{nonor}}$, and each of both subsets can be decomposed into irreducible components

$$C_i = \mathbf{V}(\mathfrak{p}_i) \setminus \left(\bigcup_j \mathbf{V}(\mathfrak{p}_{ij}) \right) \quad (4.1)$$

where all the \mathfrak{p}_i and all the \mathfrak{p}_{ij} are prime ideals, being

$$L_{\text{nor}} = \bigcup_j C_{\text{nor},j}, \quad L_{\text{nonor}} = \bigcup_j C_{\text{nonor},j}$$

We can assign a *taxonomy* to each component of a locus

Definition 4.4 (“Normal” and “Special” components). Let C be a component of the *normal locus* and let \mathbf{w} be the subset of the auxiliary variables \mathbf{u} representing the mover coordinates (if they exist), or the last n' auxiliary variables \mathbf{u} conveniently chosen. A component C of the normal locus is “*Normal*” if $\dim(C) = \dim(\mathcal{A}_m(C))$. But it can happen that $\dim(C) > \dim(\mathcal{A}_m(C))$, and then the component is “*Special*”.

Definition 4.5 (“Degenerate” and “Accumulation” components). A component of the emf non-normal locus is “*Degenerate*” if its dimension is $n - 1$. If its dimension is smaller than $n - 1$, then it is an “*Accumulation*” component.

Example 4.6. *Richard Serra surfaces.*

Richard Serra is an artist that has constructed very nice mathematical sculptures, and particularly in he Guggenheim museum of Bilbao. They are generated by two curves located in two parallel planes in the 3-dimensional space, generating a surface. The surface is obtained in the following way

Consider a parabola $y_1 = x_1^2$ on the plane $z_1 = -1$ (*floor*) and a parabola $x_2 = y_2^2$ on the plane $z_2 = 1$ (*ceiling*).

Determine the locus formed by the lines relying the points of both parabolas having parallel tangents. The tangent vectors are respectively $(-2x_1, 1, 0)$ and $(1, -2y_2, 0)$. The condition of being parallel is $4x_1y_2 - 1 = 0$. The system is:

$$\begin{aligned} F = & x_1^2 - y_1, z_1 + 1, \\ & y_2^2 - x_2, z_2 - 1, \\ & 4x_1y_2 - 1, \\ & x - x_1 - \lambda(x_2 - x_1), y - y_1 - \lambda(y_2 - y_1), z - z_1 - \lambda(z_2 - z_1) \end{aligned}$$

Applying `locus` we have the following calling sequence:

```
> LIB "grobcov.lib";
> ring R = (0, x, y, z), (lambda, x2, y2, z2, x1, y1, z1), lp;
> ideal F = x1^2 - y1, y2^2 - x2, 4x1y2 - 1, z1 + 1, z2 - 1,
           x - x1 - lambda(x2 - x1), y - y1 - lambda(y2 - y1),
           z - z1 - lambda(z2 - z1);
> locus(F);
```

and obtain the result

$$S = \mathbf{V}(2048x^3z + 2048x^3 - 4096x^2y^2 + 1152xyyz^2 - 1152xy \\ - 2048y^3z + 2048y^3 + 27z^4 - 54z^2 + 27) \\ \setminus (\mathbf{V}(z + 1, y) \cup \mathbf{V}(z - 1, x))$$

$T = \text{Special}$

$A = \mathbf{V}(z_1 + 1, x_1^2 - y_1)$

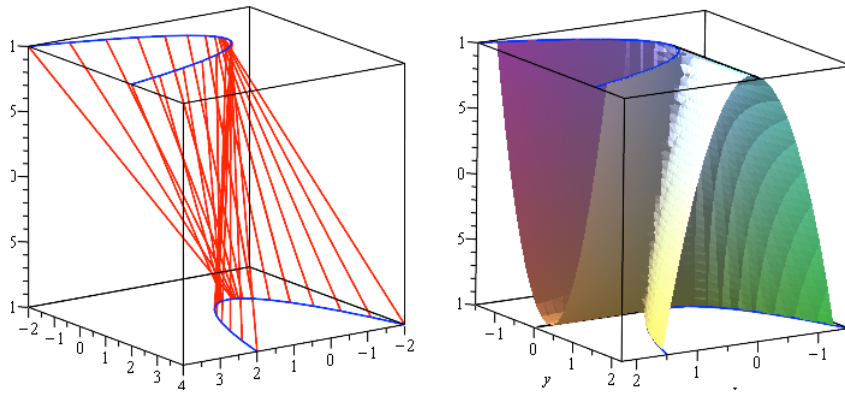


FIGURE 3. Richard Serra surface

We highlight two questions.

(1) `locus` obtains the equation of the surface and characterizes that it is a “Special” component. This is obvious because it is generated by a mover point on a parabola which has dimension 1, whereas the surface has dimension 2.

(2) `locus` also determines that the lines $\mathbf{V}(z + 1, x)$ and $\mathbf{V}(z - 1, x)$ do not form part of the locus. The reason is that each of these lines, included in the top of the envelope, would be generated by the tangent to one of the parabolas at the vertex, whose corresponding point in the second parabola is at infinity but not at any real point of them.

5. Envelopes

Usually the definition of *Envelope* concerns a family of curves or a family of surfaces with $n - 1$ degrees of freedom. We generalize the problem to an n dimensional space with higher degrees of freedom. We have the following definitions and theorems.

Definition 5.1 (Family of hyper-surfaces). We say that

$$F(\mathbf{x}, \mathbf{u}) = F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$$

and the independent *restrictions* $C = \{g_1, \dots, g_s\}$ (with $s < m$)

$$\begin{cases} g_1(u_1, \dots, u_m) = 0 \\ \dots \\ g_s(u_1, \dots, u_m) = 0 \end{cases}$$

represent a *family of hyper-surfaces* of \mathbb{C}^n , if F depends at least on 1 *parameter* u . Thus, if the g_i are independent

$$d = \dim(C) = m - s \geq 1.$$

Definition 5.2 (Envelope). Given a family of hyper-surfaces $F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$ with m parameters \mathbf{u} , constrained by the $s < m$ independent equations $C = \langle g_1(\mathbf{u}), \dots, g_s(\mathbf{u}) \rangle$, let

$$J = \frac{\partial(F, g_1, \dots, g_s)}{\partial(u_1, \dots, u_m)}; \quad J_c = \{\text{minors}(J) \text{ of order } (s+1) \times (s+1)\}.$$

The set of $\binom{m}{s+1}$ equations J_c , imply that the *rank of the Jacobian* is less than or equal than s . Consider the ideal S

$$S = \langle F, C, J_c \rangle,$$

The *algebraic envelope* of F, C , if it exists, is

$$L = \text{locus}(S).$$

Theorem 5.3 (Associated Tangent element). Let $F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$ be a family of hyper-surfaces with m parameters \mathbf{u} , constrained by the $s < m$ equations $C = \langle g_1(\mathbf{u}), \dots, g_s(\mathbf{u}) \rangle$.

Let E be the envelope, and $E_i = \mathbf{V}(p_i(\mathbf{x})) \setminus \mathbf{V}(q_i(\mathbf{x}))$ the C -representation of a “Normal” standard $(n-1)$ -dimensional component of E corresponding to a hyper-surface, and $\mathbf{x}^{(0)} \in E_i$ be a regular point of $p_i(\mathbf{x}) = 0$.

Then there exists one Associated Tangent hyper-surface $F(\mathbf{x}, \mathbf{u}^{(0)})$ of the family F (or at most a finite set) that passes at point $\mathbf{x}^{(0)}$ (i.e. $F(\mathbf{x}^{(0)}, \mathbf{u}^{(0)}) = 0$) and is tangent to $\mathbf{V}(p_i(\mathbf{x}))$ at point $\mathbf{x}^{(0)}$.

The SINGULAR `grobcov.lib` library has incorporated the following algorithms and commands related to envelopes:

1. `envelop`: Determines the envelope components and their taxonomies.
2. `AssocTanToEnv`: Determines the associated tangent element of the family passing at a regular point of a “Normal” component.
3. `FamElemsAtEnvComPoints` Determines all the elements of the family passing at a point of the envelope.
4. `discrim` Determines the discriminant with respect to a variable, when an equation is of degree 2.

Example 5.4. Consider the family of spheres of radius 1, centered at point (x_1, y_1, z_1) of a sphere of center $(0, 0, t)$ and radius \sqrt{t} . We have the following family and restrictions:

$$\begin{aligned} F &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - 1, \\ C &= \langle x_1^2 + y_1^2 + (z_1 - t)^2 - t \rangle. \end{aligned}$$

Applying `envelop(F, C)` we obtain 2 components:

$$E_1 = \mathbf{V}((16x^6 + 48x^4y^2 + 16x^4z^2 - 32x^4z - 56x^4 + 48x^2y^4 + 32x^2y^2z^2 - 64x^2y^2z - 112x^2y^2 - 32x^2z^3 - 24x^2z^2 + 29x^2 + 16y^6 + 16y^4z^2 - 32y^4z - 56y^4 - 32y^2z^3 - 24y^2z^2 + 29y^2 + 16z^4 - 24z^3 - 15z^2 + 38z - 15)) \setminus \mathbf{V}(z - 1, x^2 + y^2)$$

Normal

$$E_2 = \mathbf{V}(z - 1, y, x)$$

Accumulation

The envelope has a “Normal” component E_1 , and an ”Accumulation” point.

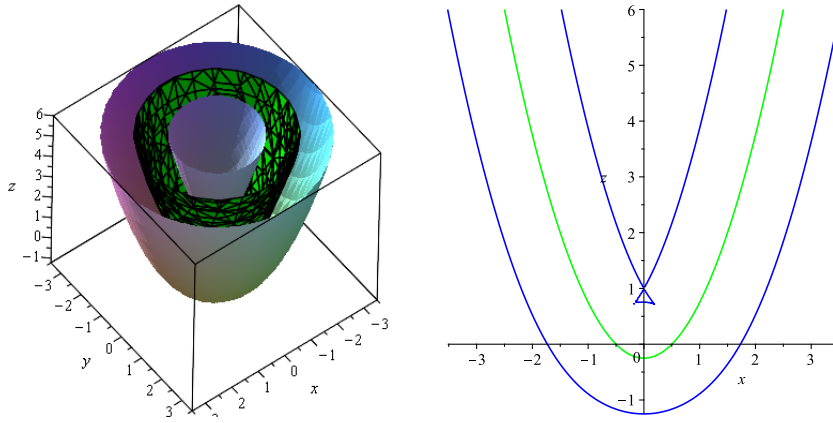


FIGURE 4. Envelope and surface $\Delta_t = 0$

2d-Section

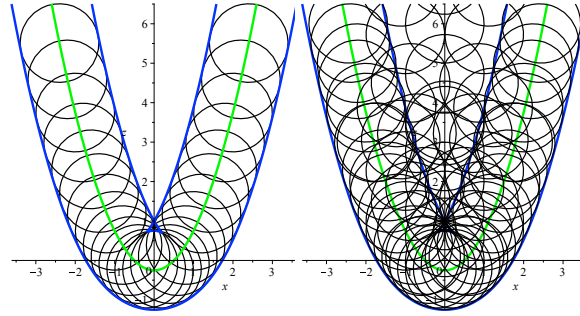
As the restriction C has degree 2 in t , we can compute its discriminant with respect to t

$$\Delta_t = 4z_1 - 1 - 4x_1^2 - 4y_1^2,$$

which divides the space into two regions. On the region $\Delta_t(x, y, z) \geq 0$ there can exist centers of family spheres, but they cannot exist on the region $\Delta_t(x, y, z) < 0$. These regions also separate both parts of the envelope, E_{ext} , for $\Delta_t(x, y, z) < 0$ and E_{int} for $\Delta_t(x, y, z) > 0$.

In Figure 5 right is represented a section at $y = 0$ of the envelope, a set of family spheres, the separating paraboloid $\Delta_t(x, y, z) = 0$ (right), and a set of family spheres with centers in the paraboloid (left). We observe that the last spheres are tangent to both parts E_{ext} and E_{int} of the envelope.

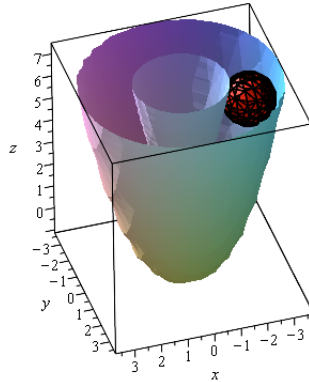
Considering a point $P = (x, y, z)$ of the envelop component E_1 , `AssocTanToEnv(F, C, E1)` gives:

FIGURE 5. Section for $y = 0$

$$\begin{aligned}
 t &= -\frac{12x^4 + 24x^2y^2 - 4x^2z^2 - 28x^2z - 31x^2 + 12y^4 - 4y^2z^2 - 28y^2z - 31y^2 - 16z^4 - 28z^3 - 3z^2 + 10z + 10}{16z^3 + 24z^2 + 12z - 25} \\
 z_1 &= -\frac{24x^4 + 48x^2y^2 - 8x^2z^2 - 56x^2z - 62x^2 + 24y^4 - 8y^2z^2 - 56y^2z - 62y^2 - 32z^4 - 40z^3 + 18z^2 + 32z - 5}{32z^3 + 48z^2 + 24z - 50} \\
 y_1 &= -\frac{6x^4yz + 32x^4y + 32x^2y^3z + 64x^2y^3 + 16x^2yz^3 - 16x^2yz^2 - 100x^2yz - 116x^2y + 16y^5z + 32y^5}{32z^4 + 16z^3 - 24z^2 - 74z + 50} \\
 &\quad - \frac{16y^3z^3 - 16y^3z^2 - 100y^3z - 116y^3 - 48yz^4 - 68yz^3}{32z^4 + 16z^3 - 24z^2 - 74z + 50} \\
 x_1 &= -\frac{16x^5z + 32x^5 + 32x^3y^2z + 64x^3y^2 + 16x^3z^3 - 16x^3z^2 - 100x^3z - 116x^3 + 16xy^4z + 32xy^4}{32z^4 + 16z^3 - 24z^2 - 74z + 50} \\
 &\quad - \frac{16xy^2z^3 - 16xy^2z^2 - 100xy^2z - 116xy^2 - 48xz^4 - 68xz^3 - 36xz^2 + 36xz + 35x}{32z^4 + 16z^3 - 24z^2 - 74z + 50}
 \end{aligned}$$

determining the values of (t, x_1, y_1, z_1) of the parameters of the associated tangent sphere to E_1 at $P(x, y, z)$.

For example, at point $P_1 = (-3.519505319, 0, 6)$, the associated tangent sphere has center $(-2.538358523, 0, 6.193264030)$. It is represented in red in Figure 6. This sphere is also the associated tangent sphere of the family at point $P_2 = (-1.557188274, 0, 6.386408905)$.

FIGURE 6. Envelope E_1 and the associated tangent sphere at point P_1

We conclude that through the points of E_{ext} there exist a unique sphere of the family passing through them, namely the associated tangent one, whereas they exist an infinity set of family spheres passing through E_{int} . Thus, the real envelope is

$$\text{Real envelope} = E_{ext} = \{(x, y, z) \in E_1 : \Delta_t(x, y, z) < 0\}$$

References

- [1] Antonio Montes, The Gröbner Cover, ed: Springer, series: Algorithms and Computation in Mathematics, To appear, January, 2019
- [2] Antonio Montes and Michael Wibmer, Gröbner bases for polynomial systems with parameters, Journal of Symbolic Computation, 45, 1391–1425, 2010

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