A polynomial generalization of the power-compositions determinant*

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July 12, 2005.

Abstract

Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$, and $\mathbf{x} = (x_1, \dots, x_p)$ a vector of indeterminates. For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ two p-compositions of n, define $(\mathbf{x} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1 + \alpha_1)^{\beta_1} \cdots (x_p + \alpha_p)^{\beta_p}$. In this paper we prove an explicit formula for the determinant $\det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C(n,p)}((\mathbf{x} + \boldsymbol{\alpha})^{\boldsymbol{\beta}})$. In the case $x_1 = \cdots = x_p$ the formula gives a proof of a conjecture by C. Krattenthaler.

Key words. composition, polynomial determinant, power-composition, combinatorial determinant.
AMS subject classifications. 11C20, 15A36, 05A10, 05A19.

1 Introduction

Let us start with some notation. If $\mathbf{u} = (u_1, \dots, u_\ell)$ and $\mathbf{v} = (v_1, \dots, v_\ell)$ are two vectors of the same length, we define $\mathbf{u}^{\mathbf{v}} = u_1^{v_1} \cdots u_\ell^{v_\ell}$ (where, to be consistent $0^0 = 1$). In our case, the entries u_i and v_i of $\mathbf{u}$ and $\mathbf{v}$ will be nonnegative integers or polynomials. We use $\mathbf{x} = (x_1, \dots, x_p)$ to denote a vector of indeterminates and $\mathbf{1} = (1, \dots, 1)$. The lengths of $\mathbf{x}$ and $\mathbf{1}$ will be clear from the context. If $\mathbf{u} = (u_1, \dots, u_\ell)$, then $s(\mathbf{u})$ denotes the sum of the entries of $\mathbf{u}$, i.e. $s(\mathbf{u}) = u_1 + \cdots + u_\ell$, and $\bar{\mathbf{u}}$ denotes the vector obtained from $\mathbf{u}$ by deleting the last coordinate, $\bar{\mathbf{u}} = (u_1, \dots, u_{\ell-1})$.

Let $C(n,p)$ be the set of p-compositions of an integer n, i.e., the set of p-tuples $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ of nonnegative integers such that $\alpha_1 + \cdots + \alpha_p = n$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)$ are two p-compositions of n, using the above

*Work partially supported by the Ministerio de Ciencia y Tecnología under projects BFM2003-00368 and MTM2004-01728 and Ministerio de Ciencia y Tecnología and by the Generalitat de Catalunya under project 2005 SGR 00692

notation, we have $\boldsymbol{\alpha}^{\boldsymbol{\beta}} = \alpha_1^{\beta_1} \cdots \alpha_p^{\beta_p}$. In [1] the following explicit formula for the determinant $\Delta(n,p) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C(n,p)} \left(\boldsymbol{\alpha}^{\boldsymbol{\beta}}\right)$ was proved:

$$\Delta(n,p) = \prod_{k=1}^{\min\{n,p\}} \left(n^{\binom{n-1}{k}} \prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}} \right)^{\binom{p}{k}}. \tag{1.1}$$

In a complement [4] to his impressive *Advanced Determinant Calculus* [3], C. Krattenthaler mentions this determinant, and after giving the alternative formula

$$\Delta(n,p) = n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}} \tag{1.2}$$

he states as a conjecture a generalization to univariate polynomials. Namely, let x be an indeterminate and

$$\Delta(n,p,x) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C(n,p)} \left((x \cdot \mathbf{1} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} \right).$$

Note that $(x \cdot \mathbf{1} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x+\alpha_1)^{\beta_1} \cdots (x+\alpha_p)^{\beta_p}$.

Conjecture [C. Krattenthaler]:

$$\Delta(n,p,x) = (px+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}. \tag{1.3}$$

As $(n-i+1)\binom{n+p-i-1}{p-2} = (p-1)\binom{n+p-i-1}{p-1}$, formula (1.2) can be written in the form

$$\Delta(n,p) = n^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}$$

and Krattenthaler's Conjecture (1.3) in the form

$$\Delta(n,p,x) = (px+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}. \tag{1.4}$$

The main goal of this paper is to prove a generalization of formula (1.4) for p indeterminates. For this, let $\mathbf{x} = (x_1, \ldots, x_p)$ be a vector of indeterminates, and let

$$\Delta(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C(n,p)} \left((\mathbf{x} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} \right).$$

(Recall that $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}} = (x_1+\alpha_1)^{\beta_1} \cdots (x_p+\beta_p)^{\beta_p}$). Then, we prove the following formula (Theorem 5.1):

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}. \tag{1.5}$$

As $s(\mathbf{x}) = x_1 + \cdots + x_p$, if $x_1 = \cdots = x_p = x$, then $s(\mathbf{x}) = px$ and the conjectured identity (1.4) follows.

We also prove a variant of this result for proper compositions. A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$ of n such that $\alpha_i \geq 1$ for all $i = 1, \ldots, n$. Denote by $C^*(n,p)$ the set of proper p-compositions of n and define

$$\Delta^*(n,p,\mathbf{x}) = \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C^*(n,p)} \left((\mathbf{x} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} \right).$$

The determinant $\Delta^*(n,p,\mathbf{x})$ has the following factorization (Theorem 6.1):

$$\Delta^*(n,p,\mathbf{x}) = (s(\mathbf{x}) + n)^{\binom{n-1}{p}} \left(\prod_{j=1}^{p} \prod_{i=1}^{n-p+1} (x_j + i)^{\binom{n-i-1}{p-2}} \right) \prod_{i=1}^{n-p+1} i^{(p-1)\binom{n-i-1}{p-1}}. \quad (1.6)$$

The paper is organized as follows. In the next section we collect some combinatorial identities for further reference. In Section 3 we prove the equivalence between the formula (1.2) given by Krattenthaler and (1.1). In Section 4 we prove two lemmas. The first one is a generalization of the determinant $\Delta(n,2,\mathbf{x})$. The second lemma uses the first and corresponds to a property of a sequence of rational functions which appear in the triangulation process of the determinant $\Delta(n,p,\mathbf{x})$. Section 5 contains the proof of the main result, Theorem 5.1. Finally, Section 6 is devoted to proving (1.6).

2 Auxiliary summation formulas

Lemma 2.1. *Let a, b, c, d, m and n be nonnegative integers. Then, the following equalities hold.*

(i) $\sum_{k \in \mathbb{Z}} \binom{a}{c+k}\binom{b}{d-k} = \binom{a+b}{c+d}$;

(ii) $\sum_{k \leq n} \binom{a+k}{a} = \sum_{k \leq n} \binom{a+k}{k} = \binom{n+a+1}{a+1}$;

(iii) $\sum_{r=1}^{n} r\binom{n+a-r}{a} = \binom{n+a+1}{a+2}$;

Proof. (i) is the well known Vandermonde's convolution, see [2, p. 169]. The formulas in (ii) are versions of the parallel summation [2, p. 159]. Part (iii) follows from

$$\begin{aligned} \sum_{r=1}^{n} r\binom{n+a-r}{a} &= \sum_{r=1}^{n} r\binom{n+a-r}{n-r} = \sum_{k=0}^{n-1}\sum_{i=0}^{k}\binom{a+i}{a} \\ &= \sum_{k=0}^{n-1}\binom{a+k+1}{a+1} = \binom{a+n+1}{a+2}. \end{aligned}$$

□

3 Equivalence between the two formulas for x = 0

Here we prove the equivalence beetween the formulas (1.1) and (1.2) for $\Delta(n,p)$. Obviously, the result of substituting $x=0$ in formula (1.3) of the Conjecture gives formula (1.2) for $\Delta(n,p)$.

Proposition 3.1. *Formulas* (1.1) *and* (1.2) *are equivalent.*

Proof. We derive formula (1.2) from (1.1), which was already proved in [1]. First, note that if $p<k\le n$, the binomial coefficient $\binom{p}{k}$ is zero. Thus, we can replace $\min\{p,n\}$ by n in formula (1.1). Analogously, if $n-k+1<i\le n$, the binomial coefficient $\binom{n-i-1}{k-2}$ is zero, and we can replace the upper value $n-k+1$ by n in the inner product. Second, the case $a=n-1$, $b=d=p$ and $c=0$ of Lemma 2.1 (i) yields

$$\sum_{k=1}^{n}\binom{n-1}{k}\binom{p}{k}=-1+\sum_{k=0}^{n}\binom{n-1}{k}\binom{p}{p-k}=\binom{n+p-1}{p}-1,$$

and, if $i\ge 1$, by taking $a=n-i-1$, $b=d=p$ and $c=-2$ in Lemma 2.1 (i), we obtain

$$\sum_{k=1}^{n-1}\binom{n-i-1}{k-2}\binom{p}{k}=\sum_{k}^{n-1}\binom{n-i-1}{k-2}\binom{p}{p-k}=\binom{n+p-i-1}{p-2}.$$

Therefore,

$$\begin{aligned}
\Delta(n,p) &= \prod_{k=1}^{\min\{n,p\}}\left(n^{\binom{n-1}{k}}\prod_{i=1}^{n-k+1} i^{(n-i+1)\binom{n-i-1}{k-2}}\right)^{\binom{p}{k}}\\
&= \prod_{k=1}^{n}\left(n^{\binom{n-1}{k}}\prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}}\right)^{\binom{p}{k}}\\
&= \left(\prod_{k=1}^{n} n^{\binom{n-1}{k}\binom{p}{k}}\right)\left(\prod_{k=1}^{n}\prod_{i=1}^{n} i^{(n-i+1)\binom{n-i-1}{k-2}\binom{p}{k}}\right)\\
&= n^{\binom{n+p-1}{p}-1}\left(\prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}}\right) n^{\sum_{k=1}^{n}\binom{-1}{k-2}\binom{p}{k}}\\
&= n^{\binom{n+p-1}{p}+p-1}\prod_{i=1}^{n-1} i^{(n-i+1)\binom{n+p-i-1}{p-2}}\\
&= n^{\binom{n+p-1}{p}}\prod_{i=1}^{n} i^{(n-i+1)\binom{n+p-i-1}{p-2}}.
\end{aligned}$$

□

4 A recurrence

The next lemma evaluates the determinant

$$D_r(n,y,z)=\det_{0\le i,j\le r}\left((y-i)^{n-j}(z+i)^{j}\right),$$

by reducing it to a Vandermonde determinant. Note that $D_n(n, x_1 + n, x_2) = \Delta(n, 2, \mathbf{x})$.

Lemma 4.1.

$$D_r(n, y, z) = (y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r} (y-i)^{n-r} \right) \left(\prod_{i=1}^{r} i^{r-i+1} \right).$$

Proof.

$$\begin{aligned}
D_r(n,y,z) &= \begin{vmatrix} (y-0)^n(z+0)^0 & (y-0)^{n-1}(z+0)^1 & \cdots & (y-0)^{n-r}(z+0)^r \\ (y-1)^n(z+1)^0 & (y-1)^{n-1}(z+1)^1 & \cdots & (y-1)^{n-r}(z+1)^r \\ \vdots & \vdots & & \vdots \\ (y-r)^n(z+r)^0 & (y-r)^{n-1}(z+r)^1 & \cdots & (y-r)^{n-r}(z+r)^r \end{vmatrix} \\
&= \left(\prod_{i=0}^{r} (y-i)^n \right) \begin{vmatrix} 1 & (z+0)/(y-0) & \cdots & (z+0)^r/(y-0)^r \\ 1 & (z+1)/(y-1) & \cdots & (z+1)^r/(y-1)^r \\ \vdots & \vdots & & \vdots \\ 1 & (z+r)/(y-r) & \cdots & (z+r)^r/(y-r)^r \end{vmatrix} \\
&= \left(\prod_{i=0}^{r} (y-i)^n \right) \prod_{0 \le i < j \le r} \left(\frac{z+j}{y-j} - \frac{z+i}{y-i} \right) \\
&= \left(\prod_{i=0}^{r} (y-i)^n \right) \prod_{0 \le i < j \le r} \frac{(y+z)(j-i)}{(y-j)(y-i)} \\
&= \left(\prod_{i=0}^{r} (y-i)^n \right) (y+z)^{\binom{r+1}{2}} \frac{\prod_{i=1}^{r} i^{r-i+1}}{\prod_{i=0}^{r} (y-i)^r} \\
&= (y+z)^{\binom{r+1}{2}} \left(\prod_{i=0}^{r} (y-i)^{n-r} \right) \left(\prod_{i=1}^{r} i^{r-i+1} \right).
\end{aligned}$$

□

Lemma 4.2. *Define $f_r \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Q}(y, z)$ recursively by*

$$\begin{aligned}
f_0(i,j) &= (z+i)^j; \\
f_{r+1}(i,j) &= f_r(i,j) \quad \text{if} \quad j \le r; \\
f_{r+1}(i,j) &= f_r(i,j) - \left(\frac{y-i}{y-r} \right)^{j-r} \frac{f_r(i,r) f_r(r,j)}{f_r(r,r)} \quad \text{if} \quad j > r.
\end{aligned}$$

Then

(i) $f_{r+1}(r, j) = 0$ *for* $j \ge r+1$;

(ii) $f_r(r, r) = (y+z)^r \dfrac{r!}{\prod_{i=0}^{r-1}(y-i)}$.

Proof. Part (i) is trivial using induction. To obtain $f_r = f_r(r, r)$, we take $n \ge r$ and calculate $D(n, y, z) = D_n(n, y, z)$ by Gauss triangulation method.

The entry (i,j) of $D(n,y,z)$ is $(y-i)^{n-j}(z+i)^j = (y-i)^{n-j} f_0(i,j)$. If $j \geq 1$, add to the column j the column 0 multiplied by

$$-\frac{1}{(y-0)^{j-0}}\frac{f_0(0,j)}{f_0(0,0)}.$$

Then, the entry (i,j) with $j \geq 1$ is modified to

$$\begin{aligned}
& (y-i)^{n-j} f_0(i,j) - (y-i)^{n-0} f_0(i,0)\frac{1}{(y-0)^{j-0}}\frac{f_0(0,j)}{f_0(0,0)} \\
= \;& (y-i)^{n-j}\left\{ f_0(i,j) - \left(\frac{y-i}{y-0}\right)^{j-0} \frac{f_0(i,k)f_0(k,j)}{f_0(0,0)} \right\} \\
= \;& (y-i)^{n-j} f_1(i,j).
\end{aligned}$$

Therefore, $D(n,y,z) = \det_{0\leq i,j\leq r}\left((y-i)^{n-j} f_1(i,j)\right)$ and $f_1(0,j)=0$ for $j \geq 1$.

Now, assume that $D(n,y,z) = \det_{0\leq i,j\leq n}\left((y-i)^{n-j} f_k(i,j)\right)$ for $k \geq 1$ with $f_k(i,j) = 0$ for $k,j > i$. Add to the column $j \geq k+1$ the column k multiplied by

$$-\frac{1}{(y-k)^{j-k}}\frac{f_k(k,j)}{f_k(k,k)}.$$

The entry (i,j) is modified to

$$\begin{aligned}
& (y-i)^{n-j} f_k(i,j) - (y-i)^{n-k} f_k(i,k)\cdot\frac{1}{(y-k)^{j-k}}\cdot\frac{f_k(k,j)}{f_k(k,k)} \\
= \;& (y-i)^{n-j}\left\{ f_k(i,j) - \left(\frac{y-i}{y-k}\right)^{j-k} \frac{f_k(i,k)f_k(k,j)}{f_k(k,k)} \right\} \\
= \;& (y-i)^{n-j} f_{k+1}(i,j).
\end{aligned}$$

Clearly $f_{k+1}(k,j) = 0$ for $j > k$. After n iterations, we get the determinant of a triangular matrix. Hence

$$D(n,y,z) = \det_{0\leq k\leq n}\Big((y-k)^{n-k} f_k(k,k)\Big) = \prod_{r=0}^{n}(y-k)^{n-k} f_k.$$

The principal minor of order $r+1$ is $D_r(n,y,z) = \prod_{k=0}^{r}(y-k)^{n-k} f_k$. Therefore,

$$\frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} = (y-r)^{n-r} f_r. \tag{4.7}$$

On the other hand, by Lemma 4.1 we obtain

$$\begin{aligned}
\frac{D_r(n,y,z)}{D_{r-1}(n,y,z)} &= \frac{(y+z)^{\binom{r+1}{2}}\left(\prod_{i=0}^{r}(y-i)^{n-r}\right)\left(\prod_{i=1}^{r} i^{r-i+1}\right)}{(y+z)^{\binom{r}{2}}\left(\prod_{i=0}^{r-1}(y-i)^{n-r-1}\right)\left(\prod_{i=1}^{r-1} i^{r-i}\right)} \\
&= (y+z)^r \cdot r! \cdot \frac{(y-r)^{n-r}}{\prod_{i=0}^{r-1}(y-i)}.
\end{aligned}$$

Comparing with (4.7), we have arrived at

$$f_r = (y+z)^r \frac{r!}{\prod_{i=0}^{r-1}(y-i)}.$$

□

5 Proof of the main theorem

We sort $C(n,p)$ in lexicographic order. For instance, for $n=5$, and $p=3$, we obtain

$$\begin{aligned} C(5,3) = \quad & \{(5,0,0),(4,1,0),(3,2,0),(2,3,0),(1,4,0),(0,5,0),\\ & (4,0,1),(3,1,1),(2,2,1),(1,3,1),(0,4,1),\\ & (3,0,2),(2,1,2),(1,2,2),(0,3,2),\\ & (2,0,3),(1,1,3),(0,2,3),\\ & (1,0,4),(0,1,4),\\ & (0,0,5)\}. \end{aligned}$$

Let $M(n,p,\mathbf{x})$ be the matrix with rows and columns labeled by the p-compositions of n in lexicographic order and with the entry $(\boldsymbol{\alpha},\boldsymbol{\beta})$ equal to $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$. We have $\Delta(n,p,\mathbf{x}) = \det M(n,p,\mathbf{x})$.

An entry $(\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}$ in $M(n,p,\mathbf{x})$ can be written in the form $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}(x_p+\alpha_p)^{\beta_p}$. For $0 \le i,j \le n$, let S_{ij} be the matrix with entries $(\bar{\mathbf{x}}+\bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ satisfy $\alpha_p = i$ and $\beta_p = j$. Thus, the submatrix of $M(n,p,\mathbf{x})$ formed by the entries labeled $(\boldsymbol{\alpha},\boldsymbol{\beta})$ with $\alpha_p = i$ and $\beta_p = j$ can be written $\left(S_{ij}(x_p+i)^j\right)$. Note that

$$S_{kk} = M(n-k,p-1,\bar{\mathbf{x}}).$$

Define $f_0(i,j) = (x_p+i)^j$. Therefore, $M(n,p,\mathbf{x})$ admits the block decomposition

$$M(n,p,\mathbf{x}) = (S_{ij} f_0(i,j))_{0\le i,j\le n}.$$

The idea is to put $M(n,p,\mathbf{x})$ in block triangular form in such a way that at each step only the last factor of each block is modified.

Theorem 5.1.

$$\Delta(n,p,\mathbf{x}) = (s(\mathbf{x})+n)^{\binom{n+p-1}{p}} \prod_{i=1}^{n} i^{(p-1)\binom{n+p-i-1}{p-1}}.$$

Proof. The proof is by induction on p. For $p=1$, $\Delta(n,p,x)$ is the determinant of the 1×1 matrix $((x+n)^n)$. Hence $\Delta(n,p,x) = (x+n)^n$. This value coincides with the right hand side of the formula for $p=1$.

Consider now the case $p=2$. Any 2-composition of n is of the form $(n-i,i)$ for some i, $0 \le i \le n$. The determinant to be calculated is $\Delta(n,2,\mathbf{x}) = \det_{0\le i,j\le n}\left((x_1+n-i)^{n-j}(x_2+i)^j\right)$. By taking $r=n$, $y=x_1+n$ and $z=x_2$ in Lemma 4.1, we get

$$\Delta(n,2,\mathbf{x}) = D_n(n,x_1+n,x_2) = (x_1+x_2+n)^{\binom{n+1}{2}} \prod_{i=1}^{n} i^{n-i+1}.$$

Therefore, the formula holds for $p = 2$.

Now, let $p > 2$ and assume that the formula holds for $p - 1$. Begin with the block decomposition of the matrix $M(n, p, \mathbf{x}) = (S_{ij} f_0(i, j))_{0 \le i,j \le n}$.

Assume $\Delta(n, p, \mathbf{x}) = \det(S_{ij} f_r(i, j))$ where $S_{ij} = ((\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\boldsymbol{\beta}}})$, with $\alpha_p = i$, $\beta_p = j$, and $f_r(i, j) = 0$ for $i < r$ and $j > i$.

Fix a column $\boldsymbol{\beta}$ with $\beta_p = j > r$. For each $\boldsymbol{\gamma} \in C(n, p)$ with $\gamma_p = r$ and $\gamma_k \ge \beta_k$ for $k \in [p - 1]$, add to the column $\boldsymbol{\beta}$ the column $\boldsymbol{\gamma}$ multiplied by

$$-\frac{1}{(s(\bar{\mathbf{x}}) + n - r)^{j-r}} \binom{j - r}{\bar{\boldsymbol{\gamma}} - \bar{\boldsymbol{\beta}}} \frac{f_r(r, j)}{f_r(r, r)}.$$

The differences $\bar{\boldsymbol{\delta}} = \bar{\boldsymbol{\gamma}} - \bar{\boldsymbol{\beta}}$ are exactly the $(p - 1)$-compositions of $j - r$. Also note that by the multinomial theorem,

$$(s(\bar{\mathbf{x}}) + n - i)^{j-r} = ((x_1 + \alpha_1) + \cdots + (x_{p-1} + \alpha_{p-1}))^{j-r} = \sum_{\bar{\boldsymbol{\delta}}} \binom{j - r}{\bar{\boldsymbol{\delta}}} (s(\bar{\mathbf{x}}) + \bar{\boldsymbol{\alpha}})^{\bar{\delta}}.$$

Then, a term of column $\boldsymbol{\beta}$ is modified to

$$\begin{aligned} & (\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\beta}} f_r(i, j) - \sum_{\bar{\gamma}} \frac{1}{(s(\bar{\mathbf{x}}) + n - r)^{j-r}} \binom{j - r}{\bar{\boldsymbol{\gamma}} - \bar{\boldsymbol{\beta}}} \frac{f_r(r, j)}{f_r(r, r)} (\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\gamma}} f_r(i, r) \\ = \ & (\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\beta}} \left\{ f_r(i, j) - \frac{1}{(s(\bar{\mathbf{x}}) + n - r)^{j-r}} \left(\sum_{\bar{\boldsymbol{\delta}}} \binom{j - r}{\bar{\boldsymbol{\delta}}} (\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\delta}} \right) \frac{f_r(r, j) f_r(i, r)}{f_r(r, r)} \right\} \\ = \ & (\bar{\mathbf{x}} + \bar{\boldsymbol{\alpha}})^{\bar{\beta}} \left\{ f_r(i, j) - \frac{(s(\bar{\mathbf{x}}) + n - i)^{j-r}}{(s(\bar{\mathbf{x}}) + n - r)^{j-r}} \frac{f_r(r, j) f_r(i, r)}{f_r(r, r)} \right\}. \end{aligned}$$

Now, define $f_{r+1}(i, j) = f_r(i, j)$ for $j \le r$ and

$$f_{r+1}(i, j) = f_r(i, j) - \frac{(s(\bar{\mathbf{x}}) + n - i)^{j-r}}{(s(\bar{\mathbf{x}}) + n - r)^{j-r}} \frac{f_r(r, j) f_r(i, r)}{f_r(r, r)}$$

for $j > r$. Note that $f_{r+1}(r, j) = 0$ for $j > r$. After n iterations, we arrive at the block matrix $(S_{ij} f_n(i, j))_{0 \le i,j \le n}$ where $f(i, j) = 0$ for $j > i$. Thus, the determinant $\Delta(n, p, \mathbf{x})$ is the product of the determinants of the diagonal blocks:

$$\Delta(n, p, \mathbf{x}) = \prod_{r=0}^{n} \det(S_{rr} f_r(r, r)).$$

Now, $S_{rr} = M(n - r, p - 1, \bar{\mathbf{x}})$, a square matrix of order $\binom{n-r+p-2}{p-2}$. Therefore

$$\Delta(n, p, \mathbf{x}) = \prod_{r=0}^{n} \left(\Delta(n - r, p - 1, \bar{\mathbf{x}}) f_r(r, r)^{\binom{n-r+p-2}{p-2}} \right).$$

Now, observe that the rational funcions f_r satisfy the hypothesis of Lema 4.2 with $y = s(\bar{\mathbf{x}}) + n = x_1 + \cdots + x_{p-1} + n$ and $z = x_p$. Thus,

$$f_r = f_r(r, r) = (s(\mathbf{x}) + n)^r \cdot \frac{r!}{\prod_{i=0}^{r-1} (s(\bar{\mathbf{x}}) + n - i)}.$$

By the induction hypothesis,

$$\Delta(n,p,\mathbf{x}) \;=\; \prod_{r=0}^{n}\left((s(\bar{\mathbf{x}})+n-r)^{\binom{n-r+p-2}{p-1}}\prod_{i=1}^{n-r} i^{(p-2)\binom{n-r+p-i-2}{p-2}}\right) \cdot\prod_{r=0}^{n}\left((s(\mathbf{x})+n)^{r}\cdot r!\cdot\frac{1}{\prod_{i=0}^{r-1}(s(\bar{\mathbf{x}})+n-i)}\right)^{\binom{n-r+p-2}{p-2}}$$

It remains to count how many factors of each type there are in the above product.

The number of factors $(s(\mathbf{x})+n)$ is $\sum_{r=1}^{n} r\binom{n+p-r-2}{p-2}$. From Lemma 2.1 (iii) for $a=p-2$ this coefficient is $\binom{n+p-1}{p}$.

The number of factors $s(\bar{\mathbf{x}})+n-i$, for $0\le i\le n-1$, is (by using Lemma 2.1 (ii) with $a=p-2$)

$$\binom{n-i+p-2}{p-1}-\sum_{r=i+1}^{n}\binom{n-r+p-2}{p-2}=\binom{n-i+p-2}{p-1}-\binom{n-i+p-2}{p-1}=0.$$

Finally, for $1\le i\le n$, the number of factors equal to i is

$$(p-2)\sum_{r=0}^{n-i}\binom{n+p-i-r-2}{p-2}+\sum_{r=i}^{n}\binom{n+p-r-2}{p-2}= (p-2)\binom{n+p-i-r-1}{p-1}+\binom{n+p-r-1}{p-1}=(p-1)\binom{n+p-r-1}{p-1}.$$

□

6 Proper compositions

A *proper p-composition* of an integer n is a p-composition $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_p)$ of n such that $\alpha_i\ge 1$ for all $i=1,\ldots,n$. We denote by $C^*(n,p)$ the set of proper p-compositions of n. In [1] the following formula was given:

$$\Delta^*(n,p)=\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}(\boldsymbol{\alpha}^{\boldsymbol{\beta}})=n^{\binom{n-1}{p}}\prod_{i=1}^{n-p+1} i^{(n-i+1)\binom{n-i-1}{p-2}}.$$

Here, we study the corresponding generalization

$$\Delta^*(n,p,\mathbf{x})=\det_{\boldsymbol{\alpha},\boldsymbol{\beta}\in C^*(n,p)}\left((\mathbf{x}+\boldsymbol{\alpha})^{\boldsymbol{\beta}}\right).$$

Theorem 6.1. *If $p\le n$, then*

$$\Delta^*(n,p,\mathbf{x})=(s(\mathbf{x})+n)^{\binom{n-1}{p}}\left(\prod_{i=1}^{n-p+1}\prod_{j=1}^{p}(x_j+i)^{\binom{n-i-1}{p-2}}\right)\prod_{i=1}^{n-p} i^{(p-1)\binom{n-i-1}{p-1}}.$$

Proof. The mapping $C^*(n,p) \to C(n-p,p)$ defined by $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p) \mapsto \boldsymbol{\alpha} - \mathbf{1} = (\alpha_1 - 1, \ldots, \alpha_p - 1)$ is bijective. Thus, we have

$$\begin{aligned}
\Delta^*(n,p,\mathbf{x}) &= \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C^*(n,p)} \left((\mathbf{x} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} \right) \\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C^*(n,p)} \left((\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha} - \mathbf{1})^{\boldsymbol{\beta} - \mathbf{1} + \mathbf{1}} \right) \\
&= \det_{\boldsymbol{\alpha},\boldsymbol{\beta} \in C(n-p,p)} \left((\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha})^{\boldsymbol{\beta}} (\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha})^{\mathbf{1}} \right) \\
&= \Delta(n-p,p,\mathbf{x}+\mathbf{1}) \prod_{\boldsymbol{\alpha} \in C(n-p,p)} (\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha})^{\mathbf{1}}.
\end{aligned}$$

The number of times that an integer i, $0 \le i \le n-p$ appears as the first entry of p-compositions of $n-p$ is the number of solutions $(\alpha_2, \ldots, \alpha_{n-p})$ of $i + \alpha_2 + \cdots + \alpha_p = n - p$, which is $\binom{n-p-i+p-2}{p-2} = \binom{n-i-2}{p-2}$. The count is the same for every coordinate. Then, in the product $\prod_{\boldsymbol{\alpha} \in C(n-p,p)} (\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha})^{\mathbf{1}}$, the number of factors equal to $x_j + 1 + i$ is $\binom{n-i-2}{p-2}$; equivalently, for $1 \le i \le n-p+1$, the number of factors equal to $x_j + i$ is $\binom{n-i-1}{p-2}$. Therefore,

$$\begin{aligned}
\Delta^*(n,p,\mathbf{x}) &= \Delta(n-p,p,\mathbf{x}+\mathbf{1}) \prod_{\boldsymbol{\alpha} \in C(n-p,p)} (\mathbf{x} + \mathbf{1} + \boldsymbol{\alpha})^{\mathbf{1}} \\
&= (s(\mathbf{x}) + n)^{\binom{n-1}{p}} \left(\prod_{i=1}^{n-p+1} \prod_{j=1}^{p} (x_j + i)^{\binom{n-i-1}{p-2}} \right) \prod_{i=1}^{n-p} i^{(p-1)\binom{n-i-1}{p-1}}.
\end{aligned}$$

□

Acknowledgements

The authors would like to thank the referee very much for valuable suggestions, corrections and comments, which result in a great improvement of the original manuscript.

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