On the canonical discussion of polynomial systems with parameters*

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Abstract

Given a parametric polynomial ideal I, the algorithm DISPGB, introduced by the author in 2002, builds up a binary tree describing a dichotomic discussion of the different reduced Gröbner bases depending on the values of the parameters, whose set of terminal vertices form a Comprehensive Gröbner System (CGS). It is relevant to obtain CGS's having further properties in order to make them more useful for the applications. In this paper the interest is focused on obtaining a canonical CGS. We define the objective, show the difficulties and formulate a natural conjecture. If the conjecture is true then such a canonical CGS will exist and can be computed. We also give an algorithm to transform our original CGS in this direction and show its utility in applications.

Keywords: canonical discussion, comprehensive Gröbner system, parametric polynomial system.

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1 Introduction

There are many authors [Be94, BeWe93, De99, DoSeSt06, Du95, FoGiTr01, Gi87, Gom02, GoTrZa00, GoTrZa05, HeMcKa97, Ka97, Kap95, MaMo06, Mo02, Mor97, Pe94, SaSu03, SuSa06, Sc91, Si92, We92, We03, Wi06] who have studied the problem of specializing parametric ideals into a field and determining the specialized Gröbner bases. Many other authors [Co04, Em99, GoRe93, GuOr04, Mo95, Mo98, Ry00] have applied some of these methods to solve concrete problems. In the previous paper [Mo02] we give more details of their contributions to the field. In the following we only refer to the papers directly related to the present work.

Let $I \subset K[\overline{a}][\overline{x}]$ be a parametric ideal in the variables $\overline{x} = x_1, \ldots, x_n$ and the parameters $\overline{a} = a_1, \ldots, a_m$, and $\succ_{\overline{x}}$ and $\succ_{\overline{a}}$ monomial orders in variables and parameters respectively.

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Denote $A = K[\overline{a}]$. Weispfenning [We92] proved the existence of a *Comprehensive Gröbner Basis* (CGB) of I and gave an algorithm for computing it. It exists a modern implementation in REDUCE of CGB algorithm due to T. Sturm et al. [DoSeSt06]. Let K be a computable field (for example $\mathbb{Q}$) and K' an algebraically closed extension (for example $\mathbb{C}$). A CGB of $I \subset A[\mathbf{x}]$ wrt (with respect to) the termorder $\succ_{\mathbf{x}}$ is a basis of I that specializes to a Gröbner basis of $\sigma_{\overline{a}_0}(I)$ for any specialization $\sigma_{\overline{a}_0} : K[\overline{a}][\overline{x}] \to K'[\overline{x}]$, that substitutes the parameters by values $\overline{a}_0 \in K'^m$.

In most applications of parametric ideals the related object called *Comprehensive Gröbner System* (CGS) is more suitable. A CGS of the ideal $I \subset A[\mathbf{x}]$ wrt $\succ_{\overline{x}}$ is a set

$$\begin{aligned} \mathrm{CGS}(I, \succ_{\overline{x}}) &= \{(S_i, B_i) \ : \ 1 \le i \le s,\ S_i \subset K'^m,\ B_i \subset A[\overline{x}],\ \textstyle\bigcup_i S_i = K'^m, \\ & \quad \forall \overline{a}_0 \in S_i,\ \sigma_{\overline{a}_0}(B_i) \text{ is a Gröbner basis of } \sigma_{\overline{a}_0}(I) \text{ wrt } \succ_{\overline{x}}\}. \end{aligned}$$

The sets S_i are often called *segments* and it is always assumed that they are *constructible sets*. A CGB is a special CGS with a unique segment K'^m. In a CGB the polynomials in the basis are *faithful*, i.e. they belong to I. Further properties are required to obtain more powerful CGS.

Definition 1 (Disjoint CGS)**.** A CGS is said to be *disjoint* if the sets S_i form a partition of K'^m.

Definition 2 (Reduced basis)**.** A subset $B \subset A[\overline{x}]$ is a reduced basis for a segment S if it verifies the following properties:

(i) the polynomials in B are normalized to have content 1 wrt $\overline{x}$ over A (in order to work with polynomials instead of rational functions);

(ii) the leading coefficients of the polynomials in B are different from zero on every point of S;

(iii) B specializes to the reduced Gröbner basis of $\sigma_{\overline{a}_0}(I)$, keeping the same lpp (leading power product set) for each $\overline{a}_0 \in S$, i.e. its lpp set remains stable under specializations within S.

Reduced bases are *not faithful*, i.e. they do not, in general, belong to I. They are not unique for a given segment, but the number of polynomials as well as the lpp are unique.

Definition 3 (Reduced CGS)**.** A CGS is said to be *reduced* if its segments have reduced bases.

As it is known, the lpp of the reduced Gröbner basis of an ideal determine the cardinal or dimension of the solution set over an algebraically closed field. This is the reason why disjoint reduced CGS are very useful for applications as they characterize the different kind of solutions of $\mathbb{V}(I)$.

Using Weispfenning's suggestions the author [Mo02] obtained an efficient algorithm (DISPGB) for Discussing Parametric Gröbner Bases to compute a disjoint reduced CGS. Actually this algorithm is called BUILDTREE.

BUILDTREE builds up a dichotomic binary tree, whose branches at each vertex correspond to the annihilation or not of a polynomial in $K[\overline{a}]$. It places at each vertex v a specification $\Sigma_v = (N_v, W_v)$ of the included specializations, that summarizes the null and non-null decisions taken before reaching v, and a specialized basis B_v of $\sigma_{\overline{a}}(I)$ for the specializations $\sigma_{\overline{a}_0} \in \Sigma_v$. The set of terminal vertices form a disjoint reduced CGS where the segments S_v are characterized by reduced specifications determined by (N_v, W_v).

Since then, more advances have been made. Inspired by BUILDTREE, Weispfenning [We03] gave a constructive method for obtaining a canonical CGB (CCGB) for parametric polynomial ideals. Using this idea, Manubens and Montes [MaMo06] improved BUILDTREE showing that the tree T_0 built up by BUILDTREE, can be rewritten as a new tree T providing a more compact and effective discussion by computing a discriminant ideal that is easy to compute from T_0. The rebuilding algorithm given in [MaMo06] can be iterated to obtain a very compact new tree organized as a right-comb tree. It builds an ascending chain of discriminant ideals that orders the segments defined as differences of the varieties of two consecutive discriminants, and provides a very compact disjoint reduced CGS. Nevertheless this rebuilding algorithm does not always produce the canonical CGS. This method will be presented in a forthcoming paper.

Having in mind the improvement of our disjoint reduced CGS to obtain a canonical CGS, in the present paper we adopt a different perspective. Instead of rebuilding the tree, we analyze all the different situations that can occur in the BUILDTREE CGS, formulate a natural conjecture and, using it, show how the segments can be packed to obtain the largest possible segments allowing the same reduced basis. These segments become non-algorithmic dependent, i.e. intrinsic for the given ideal. The packed intrinsic partition contains the minimum number of segments corresponding to reduced bases. Algorithms to perform the discussion about which segments must be packed, to obtain the reduced basis for the packed segments and to describe segments defined by difference of two varieties in a canonical form are also given. It remains to give a canonical description of the union of the segments included in the packed segments as well as the algorithm to carry it out. This last step is described in [MaMo07a].

The whole set of algorithms, including those in [MaMo07a] have been implemented[1] and denoted MCCGS (Minimal Canonical Comprehensive Gröbner System) algorithm. It is yet operative and in experimental phase. It is promising and very useful for applications as can be seen through its applications to automatic geometric theorem proving and discovering [MoRe07]. It may be objected that canonicity pays a price in computing time. The implementation shows that in fact the time increases only about 20-30% whether the output becomes much more simpler, compact, easy to be understood and practical for applications.

Recently Sato and Suzuki [SuSa06] have developed a new simple algorithm for computing CGS based on Kalkbrenner's Theorem [Ka97]. Its interest lies in its simplicity (it is perhaps sometimes more efficient) but the output is not sufficiently clear and useful for applications.

Section 2 reviews the basic features of BUILDTREE. In Section 3 it is explained what is meant by canonical CGS, a conjecture is formulated and using it, it is shown how to obtain the canonical CGS. Section 4 give the basic theorems to obtain a canonical description of diff-specifications and gives the corresponding algorithms. Finally, in Section 5 further developments are pointed out giving some insight as an advance of the content of the final paper [MaMo07a], where the whole canonical description of the union of segments will

[1] Manubens and Montes implementation of the new algorithm MCCGS is available on the web http://www-ma2.upc.edu/∼montes. The library, called DPGB release 7, is implemented in *Maple* 8.

be given making the MCCGS the algorithm to provide a canonical representation of the intrinsic segments.

In this paper, we only give partial examples that illustrate the algorithms discussed here. A unique complete example is given in the final Section 5 to give an idea of how simple is the final output that has the minimum number of segments.

The algorithms work with ideals, as these are the algebraic objects that allow a Gröbner representation. But ideals do not represent varieties in a unique form. So we frequently adopt a geometrical view. We shall consider ideals defined in $A = K[\overline{a}]$, where K is the computable field (for example $\mathbb{Q}$), whereas the varieties will be considered in ${K'}^m$, where K' is an algebraically closed extension (for example $\mathbb{C}$). Let J be an ideal in $K[\overline{a}]$, and $J' = J \cdot K'[\overline{a}]$ be its extension to $K'[\overline{a}]$. The symbol $\mathbb{V}(J)$ will denote

$$\begin{aligned} \mathbb{V}(J) &= \{\overline{a} \in {K'}^m : \forall f \in J, f(\overline{a}) = 0\} \\ &= \{\overline{a} \in {K'}^m : \forall f \in J', f(\overline{a}) = 0\} = \mathbb{V}(J'). \end{aligned}$$

We emphasize the use of the non-standard notation $\mathbb{V}$ in the whole paper as used in the extended affine space, whereas the ideals are defined in the base field. Lemma 8 in Section 4 will justify that decision.

2 Reviewing BUILDTREE

A *specification of specializations* is a subset Σ of specializations determined by a constructible set of the parameter space.

BUILDTREE uses *reduced specifications* for the segments. Different definitions have been given in [Mo02] and [MaMo06]. The reduced specification used in release 4 of the DPGB package described in [Mo02] does not require N to be radical nor to obtain a prime decomposition of N. In this approach, when we need to test whether a polynomial in $K[\overline{a}]$ vanishes for $\sigma \in \Sigma$, it is not sufficient to divide it by N. Instead, we must test if it belongs to $\sqrt{\langle N \rangle}$. But this is simpler than computing the radical and its prime decomposition. This makes REDSPEC more efficient but does not ensure all the nice properties that we want to have. Nevertheless, even if this is a good practical solution, for theoretical purposes we need to replace the concept of *reduced specification*[2].

Definition 4 (Red-specification)**.** Given the pair (N, W) of null and not null conditions denote

$$h = \prod_{w \in W} w \in K[\overline{a}], \quad \text{and} \quad \mathbb{V}(h) = \bigcup_{w \in W} \mathbb{V}(w) \subset {K'}^m.$$

They determine a *reduced specification of specializations* (*red-specification*) whenever

1. N is a radical ideal described by its reduced Gröbner basis wrt $\succ_{\overline{a}}$)

2. W is a set of distinct irreducible polynomials in $K[\overline{a}]$,

3. Let N_i be the prime components of $\langle N \rangle$ over $K[\overline{a}]$. Then $h \notin N_i$ for all i.

[2]Definition 7 in [MaMo06].

Note that properties (ii), (iii) of the definition in [MaMo06] are simple consequences of Definition 4. Nevertheless property (3) of Definition 4 is stronger, and REDSPEC (denoted CANSPEC in previous papers) is supposed here to verify this new definition of red-specification.

The segment associated to a red-specification is $S_{(N,W)} = \mathbb{V}(N)\setminus\mathbb{V}(h)$, and the included specifications are $\Sigma_{(N,W)} = \{\sigma_{\overline{a}} : \overline{a} \in \mathbb{V}(N) \setminus \mathbb{V}(h) \in K'^m\}$. Let $W = \{\omega_1, \ldots, \omega_s\} \subset K[\overline{a}]$ and $\overline{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{Z}^s_{\geq 0}$. Define

$$W(\overline{\lambda}) = \overline{\omega}^{\overline{\lambda}} = \prod_{i=1}^{s} \omega_i^{\lambda_i}.$$

The set of all non-null polynomials of $K[\overline{a}]$ as a consequence of W is

$$W^* = \{kW(\overline{\lambda}) \ : \ k \in K, \ \overline{\lambda} \in (\mathbb{Z}^+_{\geq 0})^s\}.$$

Definition 5 (Reduced polynomial)**.** A polynomial f is *reduced* over the segment S determined by the red-specification (N, W) if $\overline{f}^N = f$, $\mathrm{cont}_{\overline{x}}(f) = 1$ and $\mathrm{lc}(f) \in W^*$.

Definition 6 (Good specialization)**.** We say that the polynomial F *specializes well* to the reduced polynomial f over the segment S determined by the red-specification (N, W), if $\overline{F}^N$ and f are proportional except for non-null normalization, i.e. if $a\overline{F}^N = bf$ with $a, b \in W^*$, (i.e. the coefficients a, b do not become 0 on any point of S).

BUILDTREE is a Buchberger-like algorithm. Applied to the ideal I it builds up a rooted binary tree with the following properties:

1. At each vertex v a dichotomic decision is taken about the vanishing or not of some polynomial $p(\overline{a}) \in K[\overline{a}]$.

2. Each vertex is labelled by a list of zeroes and ones; the root label is the empty list. At the null child vertex $p(\overline{a})$ is assumed null and a zero is appended to the parent's label, whereas $p(\overline{a})$ is assumed non-null at the non-null son vertex, in which a 1 is appended to the father's label.

3. At each vertex v, the tree stores (N_v, W_v) and B_v, where

   - (N_v, W_v) determines a reduced specification Σ_v of the specializations summarizing all the decisions taken in the preceding vertices starting from the root.
   - B_v is reduced wrt Σ_v (not faithful) and specializes to a basis of $\sigma_{\overline{a}_0}(I)$ for every $\sigma_{\overline{a}_0} \in \Sigma_v$, preserving the lpp.

4. The set of terminal vertices form a disjoint reduced CGS in the sense of Definitions 1 and 3:

   - B_v specializes to the reduced Gröbner basis of $\sigma_{\overline{a}_0}(I)$ for every $\sigma_{\overline{a}_0} \in \Sigma_v$ and has the same lpp set. The polynomials g in the bases are normalized having $\mathrm{cont}_{\overline{x}}(g) = 1$.

- The specifications of the set of terminal vertices t_i determine subsets $S_{t_i} \subset K'^m$ forming a partition of the whole parameter space K'^m:

$$\mathcal{X} = \{S_{t_0}, S_{t_1}, \dots, S_{t_p}\},$$

and the sets S_{t_i} have characteristic lpp sets that do not depend on the algorithm.

5. The unique vertex having as label a list of 1 ($[1, \dots, 1]$) corresponds to the *generic case* as it is determined by only non-null conditions. It does not necessarily contain the whole generic case, as we will see next.

Thus the terminal vertices of BUILDTREE form a disjoint reduced CGS.

3 Finding a canonical CGS

The objective of the paper is to advance in the definition and computation of a unique (canonical) CGS. In order to reach this objective we need to obtain an intrinsic family of subsets S_i for the CGS, uniquely determined.

Denote $\Gamma = (C_1, \dots, C_s)$ the disjoint reduced CGS built by BUILDTREE, i.e. the list of terminal cases $C_i = (B_i, S_i)$. We shall always set the generic case as the first element of Γ. We group them by lpp.

$$\Gamma = ((C_{11}, \dots, C_{1s_1}), \dots, (C_{k1}, \dots, C_{ks_k})) = (\Gamma_1, \dots, \Gamma_k)$$

where the first index denotes the lpp and, as usual, C_{11} corresponds to the fundamental segment of the generic case. Obviously the sets in each group $(C_{i1}, \dots, C_{is_i})$ are canonically separated because it cannot exist a common reduced basis for them, as reduction implies preservation of the lpp. Thus if it is possible to obtain a unique reduced basis for each group then we will have a canonical CGS. But even when this is not possible and one or more groups must be split into several subgroups forming canonical equivalence classes where each class admits a common reduced basis, we will have a canonical CGS. Thus our objective is to obtain this classification.

Conjecture 7. *Let $\Gamma_i = (C_{i1}, \dots, C_{is_i})$ be the set of all segments of a disjoint reduced CGS having reduced bases with a common* lpp_i*. If C_{ij} and C_{ik} admit a common reduced basis and C_{ik} and C_{il} also, then it exists a common reduced basis to C_{ij}, C_{ik} and C_{il}.*

If the conjecture is true then we have an equivalent relation between the segments in Γ_i that is independent of the algorithm and thus shows the existence of the canonical CGS. This canonical CGS will be also minimal in the sense that it contains the minimum number of segments of a disjoint reduced CGS. We are now concerned with the task of giving algorithms to carry out the task of summarizing the C_{ij} varying j forming the equivalent class with a unique reduced basis.

3.1 Using sheaves

Before tackle that task we need to know that in some special cases we will need to use sheaves. We are indebted to Michael Wibmer [Wi06] for the idea of using sheaves for

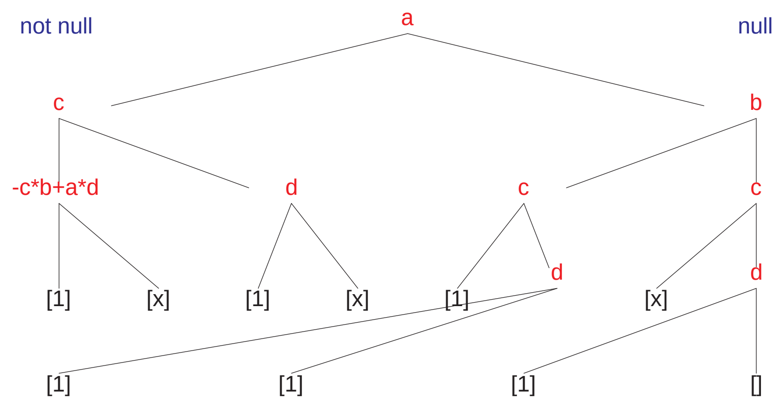

Figure 1: BUILDTREE for $I = \langle ax + b, cx + d\rangle$.

summarizing some kind of segments, as they are needed for some special problems. Let us give an example from him.

Exemple 1. Let $I = \langle ax + b, cx + d\rangle$. Applying BUILDTREE with $\succ_{\overline{x}} = \text{lex}(x, y)$ and $\succ_{\overline{a}} = \text{lex}(a, b, c, d)$ we obtain the tree of Figure 1, that provides the following segments (ordered by lpp):

lpp	basis	null cond.	non-null cond
$[1]$	$[1]$	$[\,]$	$\{a, ad - cb, c\}$
$[1]$	$[1]$	$[c]$	$\{a, d\}$
$[1]$	$[1]$	$[a]$	$\{b, c\}$
$[1]$	$[1]$	$[c, a]$	$\{b, d\}$
$[1]$	$[1]$	$[d, c, a]$	$\{b\}$
$[1]$	$[1]$	$[c, b, a]$	$\{d\}$
$[x]$	$[cx + d]$	$[ad - cb]$	$\{a, c\}$
$[x]$	$[ax + b]$	$[d, c]$	$\{a\}$
$[x]$	$[cx + d]$	$[b, a]$	$\{c\}$
$[\,]$	$[\,]$	$[d, c, b, a]$	$\{\,\}$

Obviously, the six cases with basis [1] can be summarized into a single case. We must add the six corresponding segments, and thus we will need a method to do this in a canonical form. But in any case, the union of the six segments is intrinsic to the problem and corresponds to the total generic case having basis [1]. You can see in Section 5 how these segments are grouped in the canonical tree build by the MCCGS algorithm.

The three cases with lpp $= [x]$ can also be summarized into a unique basis but now instead of a single polynomial we must use a sheaf with two polynomials. Effectively, the polynomial $cx + d$ specializes well in the first and third segments with lpp $= [x]$ and specializes to 0 in the second segment. And the polynomial $ax + b$ that forms the reduced basis of the second segment is proportional (and thus equivalent) to $cx + d$ in the first segment, but specializes to 0 in the third segment. The common basis for the three segments in this case is given by one sheaf $[\{cx + d, ax + b\}]$ instead by a polynomial: at least one of the two polynomials in the sheaf specializes well in the union of the segments whether the other either specializes also or goes to zero. Thus we see that for our objective we must admit sheaves also for the bases instead of single polynomials. The three segments are grouped in the canonical tree.

When a reduced basis of a segment S contains a sheaf, then we need that, for all $\overline{a} \in S$, at least one of the polynomials in the sheaf specializes to the corresponding polynomial of reduced Gröbner basis of the specialized ideal and the others either specialize also, either to it or to 0.

Thus the canonical CGS for this example will contain only three segments, namely

lpp	basis	sets of pairs (N, W)
[1]	[1]	$([\ ], \{a, ad-cb, c\})$, $([c], \{a, d\})$, $([a], \{b, c\})$, $([c, a], \{b, d\})$, $([d, c, a], \{b\})$, $([c, b, a], \{d\})$
$[x]$	$[\{cx+d, ax+b\}]$	$([ad-cb], \{a, c\})$, $([d, c], \{a\})$, $([b, a], \{c\})$
[ ]	[ ]	$([d, c, b, a], \{\ \})$

Sheaves will appear only in over-determined systems with generic basis [1]. For these kind of systems, and for a combination of the parameter values making compatible the redundance with some degree of freedom as is the case in the previous example, sheaves may appear. Nevertheless this is not so for other kind of systems. An example having a larger sheaf for the basis of one of his segments is $I = \langle ax+b, cx+d, ex+f \rangle$.

3.2 Obtaining common reduced bases

In most common situations where the BUILDTREE CGS presents multiple segments with the same lpp it will exist one subsegment (the most generic one) whose basis already specializes well in the other segments, and then we only have to pack them.

There are also problems where it does not exist a common basis for segments having the same lpp. A new example from Wibmer [Wi06] shows this situation.

Exemple 2. Consider the following simple system: $I = \langle u(ux+1), (ux+1)x \rangle$. REBUILDTREE gives the following GCS with two unique segments having the same lpp.

lpp	basis	null cond.	non-null cond
$[x]$	$[ux+1]$	[ ]	$\{u\}$
$[x]$	$[x]$	$[u]$	$\{\ \}$

It is easy to convince oneself that it does not exist a common reduced basis for both segments as the leading term of the generic segment specializes to 0 for $u = 0$ whether the independent term is always different from zero.

A fourth possibility arises when we have two segments with the same lpp sets, characterized by (B_1, N_1, W_1) and (B_2, N_2, W_2) that do not directly specialize one to the other by reducing the basis, but nevertheless it can exist a more generic reduced basis specializing to both. Let us explore that possibility. We want to test if it exists a basis B_{12} such that

(i) $\text{lpp}(B_{12}) = \text{lpp}(B_1) = \text{lpp}(B_2)$.

(ii) $\sigma_{(N_1,W_1)}(B_{12}) = B_1$ and $\sigma_{(N_2,W_2)}(B_{12}) = B_2$

$F \leftarrow$ **GENIMAGE**$(f_1, N_1, W_1, f_2, N_2, W_2)$

`Input:`

$(f_1 = \sum a_\alpha x^\alpha, N_1, W_1)$: basis and red-spec of a terminal case

$(f_2 = \sum b_\alpha x^\alpha, N_2, W_2)$: basis and red-spec of a terminal case

$L, M \in \mathbb{Z}$ bounds for the tests.

`Output:`

F: when it exists, F returns a polynomial such that $\sigma_{(N_1,W_1)}(F) = f_1$ and $\sigma_{(N_2,W_2)}(F) = f_1$ else it returns $F =$ **false**.

begin

test:= **false**; $F :=$ **false**

$N :=$ **GBEX**$(N_1 + N_2)$ (returns the Gröbner basis and also the matrix M expressing the polynomials in N in terms of the polynomials in N_1 and N_2)

for all $\overline{\lambda} \in \mathbb{Z}^s_{\geq 0}$, $|\overline{\lambda}| \leq L$ **while not** test **do** $w_1 = W_1(\overline{\lambda})$

for all $\overline{\mu} \in \mathbb{Z}^r_{\geq 0}$, $|\overline{\mu}| \leq M$ **while** test **do** $w_2 = W_2(\overline{\mu})$

{ HT is the index of the leading term)

$h := k_1 w_1 a_{HT} - k_2 w_2 b_{HT}$

test := **true**

if $\overline{h}^N$ has a factor $Ak_1 + Bk_2$ with $A, B \in K$ **then**

$k'_1 := B$, $k'_2 := -A$

for all terms α of f_1 or f_2 **while** test **do**

$h := k'_1 w_1 a_\alpha - k'_2 w_2 b_\alpha$

$r := \overline{h}^N$

if $r \neq 0$ **then** test := **false** **end if**

end for

if test **then**

F:=0

for all indices α of terms of f_1 or f_2 **do**

$h := k'_1 w_1 a_\alpha - k'_2 w_2 b_\alpha$

$q_i :=$ list of quotients of the exact division $\overline{h}^N$

$$F := F + \left(k'_1 w_1 a_\alpha - \sum_{i=1}^{|N|} \sum_{j=1}^{|N_1|} N_{1j} q_i M_{ij} \right) x^\alpha$$

end for

else

test := **false**

end if

end do

end do

end

in order that both cases can be summarized into a single one conserving the lpp. Denote $f_1 \in B_1$ and $f_2 \in B_2$ two corresponding polynomials with the same lpp. Then we must test if it exists a F_{12} such that $\sigma_{(N_1,W_1)}(F_{12}) = f_1$ and $\sigma_{(N_2,W_2)}(F_{12}) = f_2$. For this it must exist

$w_1 \in W_1^*$, $n_1 \in \langle N_1 \rangle \cdot K[\overline{x}]$ and $w_2 \in W_2^*$, $n_2 \in \langle N_2 \rangle \cdot K[\overline{x}]$ such that

$$F_{12} = w_1 f_1 + n_1 = w_2 f_2 + n_2.$$

Let $f_1 = \Sigma_\alpha a_\alpha x^\alpha$, $f_2 = \Sigma_\alpha b_\alpha x^\alpha$ and $F_{12} = \Sigma_\alpha c_\alpha x^\alpha$. For every index α of a term in f_1 or f_2 the coefficients must verify

$$c_\alpha = w_1 a_\alpha + n_{1\alpha} = w_2 b_\alpha + n_{2\alpha}.$$

with fixed $w_1 \in W_1^*$ and $w_2 \in W_2^*$ and appropriate $n_{1\alpha} \in \langle N_1 \rangle$ and $n_{2\alpha} \in \langle N_2 \rangle$. This implies

$$w_1 a_\alpha - w_2 b_\alpha = n_{2\alpha} - n_{1\alpha} \in \langle N_1 \rangle + \langle N_2 \rangle = \langle N \rangle$$

that can be solved by testing for all the possible $w_1 \in W_1^*$ and $w_2 \in W_2^*$ when it exists a left hand side belonging to $\langle N \rangle$. The semi-algorithm GENIMAGE does it and will obtain an F_{12} if it exists. It is a semi-algorithm in the sense that the possible choices of w_1 and w_2 are in fact not finite and the algorithm must set a bound on the possible total degree ($|\overline{\lambda}| = \sum_i^s \lambda_i \leq L$) of the terms tested for which no bound is known. Even if this depends on a luck, and little combinatorics is used, in practice it does not cause big problems because this does not occur often and when it does the result is easily found.

The semi-algorithm is self understanding. Also, when two or more segments given by red-specifications $(N_1, W_1), \ldots, (N_s, W_s)$ have been generalized to a generic basis B_0 and we must test if a new segment $(B, (N, W))$ admits a common pre-image, we can also use GENIMAGE for each polynomial $f_0 \in B_0$ and $f \in B$ taking for f_0 as null and non-null common conditions $(\cap_i^s N_i, \cap_i^s W_i)$. If GENIMAGE obtains a pre-image it will reduce well to all the segments. Let us give an example:

Exemple 3. Consider the following example from Sato-Suzuki [SaSu03]: $I = \langle ax^2y + a + 3b^2, a(b-c)xy + abx + 5c \rangle$ wrt $\succ_{\overline{x}} = \text{lex}(x, y)$, $\succ_{\overline{a}} = \text{lex}(a, b, c)$. REBUILDTREE obtains a CGS with three segments with basis [1] that obviously can be directly added, three cases with lpp set $[y, x]$ that do not specialize one to the other, and five other segments with distinct lpp namely $[y^2, x], [y, x^2], [yx, x^2], [yx^2], [\ ]$.

The question arises for the three segments with $[y, x]$ as lpp. Let us detail these segments:

lpp	basis	null cond.	non-null cond
$[y, x]$	$[y, 3b^3x - 5c]$	$[a + 3b^2]$	$\{b - c, c, b\}$
$[y, x]$	$[a^2y + 25, 5x + a]$	$[b]$	$\{c, a\}$
$[y, x]$	$[25y + 3ac^2 + a^2, ax + 5]$	$[b - c]$	$\{c, a\}$

We can verify that none of the bases reduces to the others. Applying GENIMAGE first to $(B_1, (N_1, W_1))$ and $(B_2, (N_2, W_2))$ and then to $(B_{12}, (N_{12} = [b(a + 3b^2)], W_{12} = \{c\}))$ and $(B_3, (N_3, W_3))$ a common reduced basis is found

$$\begin{aligned} B_{123} \quad = \quad & [(25bc - 25a^3b - 75b^3a^2 + 25ca^3 + 75a^2b^2c)y \\ & -625ab + 1875cb^2 + 625ac + a^2bc + 3ab^3c - 1875b^3, \\ & (ba^2 - 15ab + 15ac - 9b^5 + 9b^4c - 45b^3 + 45cb^2)x \\ & -3ba^2 + 3a^2c + 5ab + 27b^5 - 27b^4c + 15b^3 - 15bc^2] \end{aligned}$$

that reduces to the three bases in the respective segments.

We have explored four possible situations for a pair (or a collection) of segments (B_1, N_1, W_1) and (B_2, N_2, W_2) with the same lpp, namely

1. the polynomials of B_1 reduce to the polynomials of B_2 on (N_2, W_2), (most frequent case);

2. first case does not happen but it exists a pre-image basis B_{12} that reduces to both and can be computed by GENIMAGE;

3. both bases can be summarized using sheaves;

4. a reduced common basis does not exist.

Table 1 shows the algorithm DECIDE that decides if two corresponding polynomials of B_1 and B_2 have a common pre-image or a sheaf or it does not exist. If $\overline{S(f_1, f_2)}^{N_2} \neq 0$ then it calls GENIMAGE that will decide if a pre-image exists or not, but in this case the result cannot be a sheaf. If $\overline{S(f_1, f_2)}^{N_2} = 0$ as $\mathrm{lc}(f_1)f_2 - \mathrm{lc}(f_2)f_1$ specializes to 0 in the subset S_2, then f_1 specializes either to f_2 or to 0 in S_2. Then, if $\overline{\mathrm{lc}(f_1)}^{N_2} \in W_2^*$ then it is always non-null in S_2 and so f_1 specializes to f_2 and is the generic polynomial F. Else we carry out the symmetric comparisons and conclusions, and only when the S-polynomial specializes to 0 both in S_1 and in S_2 and none of the leading coefficients remain non-null in the other segment we will have a sheaf $\{f_1, f_2\}$.

4 Canonical specifications

The following Lemma plays an important role in the obtention of canonical specifications of diff-specifications.

Lemma 8. *Let K be a field of characteristic zero and K' an algebraically closed extension, P and Q ideals in $K[\overline{a}]$, P prime and $Q \not\subset P$. Then, on K'^m*

$$\overline{\mathbb{V}(P) \setminus \mathbb{V}(Q)} = \mathbb{V}(P).$$

Proof. Denote $P' = P \cdot K'[\overline{a}]$ and $Q' = Q \cdot K'[\overline{a}]$ the respective extensions of P and Q in $K'[\overline{a}]$. To prove the lemma we follow four steps:

(i) As P is prime and $Q \not\subset P$, we conclude that $P : Q = P$. We leave the proof as an exercise.

(ii) $(P : Q)' = P' : Q'$. See [ZaSa79], Vol II, p. 221.

(iii) As P is prime, P' is radical. See [ZaSa79],Vol II, p. 226.

(iv) Since K' is algebraically closed and P' is radical, $\overline{\mathbb{V}(P') \setminus \mathbb{V}(Q')} = \mathbb{V}(P' : Q')$. See [CoLiSh92], Theorem 7, p. 192.

Combining these four steps, we obtain

$$\overline{\mathbb{V}(P) \setminus \mathbb{V}(Q)} = \overline{\mathbb{V}(P') \setminus \mathbb{V}(Q')} = \mathbb{V}(P' : Q') = \mathbb{V}((P : Q)') = \mathbb{V}(P') = \mathbb{V}(P).$$

□

Using Definition 4 we can now prove the following

```
F ← DECIDE(f_1, N_1, W_1, f_2, N_2, W_2)
Input:
    (f_1, N_1, W_1): basis and red-spec of a terminal case
    (f_2, N_2, W_2): basis and red-spec of a terminal case
Output:
    F: if it exists a pre-image or a sheaf then F is a polynomial (or sheaf) such that
      σ_(N_1,W_1)(F) = f_1 and σ_(N_2,W_2)(F) = f_2 else it returns false
begin
  if overline(S(f_1, f_2))^N_2 ≠ 0 then
    F := GENIMAGE(f_1, N_1, W_1, f_2, N_2, W_2)
  else
    if overline(lc(f_1))^N_2 ∈ W_2^* then F := f_1
    else
      if overline(S(f_1, f_2))^N_1 ≠ 0 then
        F := GENIMAGE(f_1, N_1, W_1, f_2, N_2, W_2)
      else
        if overline(lc(f_2))^N_1 ∈ W_1^* then F := f_2
        else F := {f_1, f_2} # (sheaf)
        end if
      end if
    end if
  end if
end
```

Table 1:

Theorem 9. *Let* (N, W) *determine a red-specification. Then we have*

$$\overline{\mathbb{V}(N) \setminus \left(\bigcup_{w \in W} \mathbb{V}(w) \right)} = \overline{\mathbb{V}(N) \setminus \mathbb{V}(h)} = \mathbb{V}(N)$$

Proof. Decompose $\sqrt{\langle N \rangle} = \bigcap_i N_i$ into primes in $K[\overline{a}]$, so that

$$\mathbb{V}(N) \setminus \mathbb{V}(h) = \left(\bigcup_i \mathbb{V}(N_i) \right) \setminus \mathbb{V}(h) = \bigcup_i \left(\mathbb{V}(N_i) \setminus \mathbb{V}(h) \right).$$

As (N, W) determines a red-specification, $h \notin N_i$ for all i, and thus, applying Lemma 8 for each i it results

$$\overline{\mathbb{V}(N) \setminus \mathbb{V}(h)} = \bigcup_i \overline{\mathbb{V}(N_i) \setminus \mathbb{V}(h)} = \bigcup_i \mathbb{V}(N_i) = \mathbb{V}(N).$$

□

We have seen that if Conjecture 7 is true it exists an intrinsic canonical partition of the parameter space K'^m and a reduced basis for each segment and from the BUILDTREE output the algorithms DECIDE and GENIMAGE will obtain it. It is apparent the need of giving a canonical representation of the intrinsic partition because otherwise we cannot verify its uniqueness, for example if determined by another algorithm. So we focus now in the canonical description of the union of red-specifications.

Definition 10 (Diff-specification)**.** Given two ideals $N \subset M$ whose associated varieties verify $\mathbb{V}(N) \supset \mathbb{V}(M)$, they define a *diff-specification* (N, M) describing the subset $S = \mathbb{V}(N) \setminus \mathbb{V}(M)$ of K'^m.

In particular a red-specification (N, W) is easily transformed into a diff-specification. Take $h = \prod_{w \in W} w$ and $M = N + \langle h \rangle$. Obviously (N, M) is a diff-specification.

We begin giving a canonical representation of the subsets of a diff-specification, and then we shall discuss how to add subsets defined by diff-specifications.

Definition 11 (Can-specification)**.** A *can-specification* of a subset C is a representation defined by the set of prime ideals (N_i, M_{ij}) varying i, j such that

$$C = \mathbb{V}(N) \setminus \mathbb{V}(M) = \bigcup_i \left(\mathbb{V}(N_i) \setminus (\cup_j \mathbb{V}(M_{ij})\right), \tag{1}$$

where $\mathcal{N} = \cap_i N_i$ and $\mathcal{M}_i = \cap_j M_{ij}$ are the prime decompositions over $K[\overline{a}]$ of the radical ideals $\mathcal{N}$ and $\mathcal{M}_i$ respectively, where $N_i \subsetneq \mathcal{M}_{ij}$.

We have the following

Theorem 12.

1. *Every set* $C = \mathbb{V}(N) \setminus \mathbb{V}(M) \subset K'^m$ *corresponding to a diff-specification admits a can-specification, and the algorithm* DIFFTOCANSPEC *given in Table 2 builds it.*

2. *Over* K'^m*, a can-specification verifies*

$$\overline{C} = \overline{\bigcup_i \left(\mathbb{V}(N_i) \setminus (\cup_j \mathbb{V}(M_{ij})\right)} = \bigcup_i \mathbb{V}(N_i) = \mathbb{V}(\mathcal{N}).$$

3. *The can-specification associated to a set* C *given by a diff-specification is unique.*

4. *All points in* $C \cap \mathbb{V}(N_i)$ *are in* $\mathbb{V}(N_i) \setminus (\cup_j \mathbb{V}(M_{ij}))$*.*

Proof. 1. Let $\sqrt{N} = \bigcap_i N_i$ be the prime decomposition of the radical ideal $\sqrt{N}$ over $K[\overline{a}]$. Then we have

$$C = \mathbb{V}(N) \setminus \mathbb{V}(M) = \left(\bigcup_i \mathbb{V}(N_i)\right) \setminus \mathbb{V}(M) = \bigcup_i \left(\mathbb{V}(N_i) \setminus \mathbb{V}(M + N_i)\right).$$

In this decomposition the variety to be subtracted from $\mathbb{V}(N_i)$ is contained in it.

$S \leftarrow$ **PRIMEDECOMP**(N)
`Input:`
  N: ideal (representing a variety)
`Output:`
  $S = (N_1, \dots, N_k)$: the set of irredundant prime ideals wrt K of the decomposition of $\sqrt{N} = \cap_j N_j$

$Y \leftarrow$ **DIFFTOCANSPEC**(N, M)
`Input:`
  N: the null-condition ideal of the diff-specification
  M: the non-null condition ideal of the diff-specification $M \supsetneq N$
`Output:`
  $Y = \{(N_i, (\{M_{ij} : 1 \leq j \leq \ell_i\})) : 1 \leq i \leq k\}$: the set of prime ideals corresponding to the canonical decomposition of $\mathbb{V}(N) \setminus \mathbb{V}(M)$ (Theorem 12)

begin
  $Y = \emptyset$
  $S :=$ **PRIMEDECOMP**(N)
  for $N_j \in S$ **do**
    if $N_j \neq \sqrt{M + N_j}$ **then**
      $T_j :=$ **PRIMEDECOMP**$(M + N_j)$
      $Y := Y \cup_j \{(N_j, T_j)\}$
    end if
  end for
end

Table 2:

It can happen that $\sqrt{M + N_i} = \langle 1 \rangle$, in which case nothing is to be subtracted from $\mathbb{V}(N_i)$. It can also happen that $\sqrt{M + N_i} = N_i$, in which case the term $\mathbb{V}(N_i) \setminus \mathbb{V}(N_i)$ disappears from the union. The above expression is simplified and for all the remaining terms we have $N_i \subsetneq \sqrt{M + N_i}$

Let now $\sqrt{M + N_i} = \bigcap_j M_{ij}$ be the prime decomposition of $\sqrt{M + N_i}$ over $K[\overline{a}]$. For each j we have $N_i \subsetneq M_{ij}$. The decomposition becomes

$$\mathbb{V}(N) \setminus \mathbb{V}(M) = \bigcup_i \left(\mathbb{V}(N_i) \setminus (\cup_j \mathbb{V}(M_{ij})\right), \tag{2}$$

where $\mathcal{N} = \bigcap_i N_i$ and $\mathcal{M}_i = \bigcap_j M_{ij}$ are the prime decompositions over $K[\overline{a}]$ of the radical ideals $\mathcal{N}$ and $\mathcal{M}_i$ respectively, proving part (i) of the theorem. (Observe that the algorithm DIFFTOCANSPEC is nothing else than the description done in this paragraph).

It should be noted that the prime decompositions in the computations are performed in $K[\overline{a}]$, as it is the computable field. Thus these decompositions can split over $K'[\overline{a}]$.

In the same sense we cannot ensure that the varieties $\mathbb{V}(N_i)$ nor $\mathbb{V}(M_{ij})$ are irreducible over K^m nor over K'^m as we cannot use the Nullstellensatz in K. Nevertheless the prime decompositions are canonically well defined over $K[\overline{a}]$.

2. Using Lemma 8 for each term in the decomposition given by formula (1) of C we have

$$\overline{C} = \bigcup_i \overline{\mathbb{V}(N_i) \setminus (\cup_j \mathbb{V}(M_{ij})} = \bigcup_i \mathbb{V}(N_i) = \mathbb{V}(\mathcal{N})$$

over K'^m, proving part (ii) of the theorem.

3. Suppose that C admits two diff-specifications characterized by the pairs of discriminant ideals (N, M) and (R, S) respectively. If we denote by $\mathcal{R} = \bigcap_k R_k$ and $\mathcal{S}_\ell = \bigcap_\ell S_{k\ell}$ the ideals in the decomposition obtained from the diff-specification with R and S using the method described in part (i) of this theorem they will verify $\overline{C} = \mathbb{V}(\mathcal{N}) = \mathbb{V}(\mathcal{S})$ by part (ii). As $\mathcal{N}$ and $\mathcal{S}$ are radical, they are also radical over $K'[\overline{a}]$ and thus by the Nullstellensatz they are both equal to $\mathbb{I}(C)$ over $K'[\overline{a}]$. Thus we have $N_i = R_i$ for each i, as the prime decomposition in $K[\overline{a}]$ is unique.

   Next we subtract from each $\mathbb{V}(N_i)$ all the points that are not in C as they are in $\mathbb{V}(M)$. We have already eliminated the components of $\mathbb{V}(N)$ that are also in $\mathbb{V}(M)$, so that the points in $\mathbb{V}(N_i)$ that are not in C are the points of the variety $\mathbb{V}(M+N_i) \subsetneq \mathbb{V}(N_i)$. Then by the Nullstellensatz $\mathcal{M}_i = \sqrt{M + N_i}$ is the variety ideal $\mathbb{I}(\mathbb{V}(\mathcal{M}_i))$. Carrying out the prime decomposition of $\mathcal{M}_i$ we are done with the canonical decomposition. Thus the decomposition of C given in part (i) of formula (1) is unique.

4. This is now obvious as we have subtracted from each $\mathbb{V}(N_i)$ all the points in $\mathbb{V}(N_i) \cap \mathbb{V}(M)$.

□

Note that the can-specification is canonical but the constructible sets whose union describes C do not have empty intersection. Nevertheless this does not affect C itself.

5 Further developments

If Conjecture 7 is true, it exists a minimal canonical CGS. The algorithms here described, start from the BUILDTREE CGS and regroup the segments to obtain the intrinsic segments having a reduced basis. The result for each segment is of the form:

$$C_i = (B_i, S_i) = (B_i, ((N_{i1}, W_{i1}), \ldots, (N_{ij_i}, W_{ij_i})))$$

The subsegments defined by (N_{ik}, W_{ik}) are described by red-specifications and thus as a difference of varieties $\mathbb{V}(N_{ik}) \setminus \mathbb{V}(M_{ik})$ where $h = \prod_{w \in W_{ik}} w$ and $M_{ik} = \langle N_{ik} \rangle + \langle h \rangle$ thus (N_{ik}, M_{ik}) is its diff-specification. In order to obtain a canonical description of the intrinsic partition of our disjoint reduced CGS it is apparent that we need to add diff-specified sets in a canonical form. This task is done in [MaMo07a].

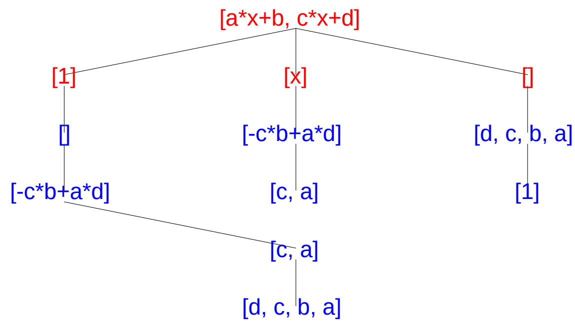

Figure 2: MCCGS tree for Exemple 1.

Let us outline how this works. We cannot assume that the simple form given by formula (1) will be sufficient. A more complex constructible set will be formed. There can exist different canonical forms for describing it, but in any case this will need prime decomposition of radical ideals. Our canonical form is given by an even level rooted tree called P-tree whose root defines level 0. At the nodes there are prime ideals $P_{i_1,\ldots,i_j}$ of $K[\overline{a}]$. These ideals verify $P_{i_1,\ldots,i_j} \subsetneq P_{i_1,\ldots,i_j,k}$ for every k and the set of $P_{i_1,\ldots,i_j,k}$ for every k are the prime decomposition of a radical ideal $\mathcal{P}_{i_1,\ldots,i_j}$. The set C defined by the P-tree has to be read

$$C = \bigcup_{i_1} \mathbb{V}(P_{i_1}) \setminus \left(\bigcup_{i_2} \mathbb{V}(P_{i_1 i_2}) \setminus \left(\bigcup_{i_3} \mathbb{V}(P_{i_1 i_2 i_3}) \setminus \left(\cdots \setminus \bigcup_{i_{2N}} \mathbb{V}(P_{i_1 \ldots i_{2N}}) \right) \right) \right).$$

Let us advance some results from [MaMo07a] and illustrate how is the final output of the MCCGS algorithm for Exemple 1. It is illustrated by a plot procedure in Figure 2, and the algebraic output summarized in the following table:

lpp	basis	segment
$[1]$	$[1]$	$\mathbb{C}^4 \setminus (\mathbb{V}(ad-bc) \setminus (\mathbb{V}(a,c) \setminus \mathbb{V}(a,b,c,d)))$
$[x]$	$[\{cx+d, ax+b\}]$	$V(ad-bc) \setminus \mathbb{V}(a,c)$
$[\,]$	$[\,]$	$\mathbb{V}(a,b,c,d)$

It can be seen that the MCCGS algorithm gives a very compact solution for the problem easy to interpret. This is generally so for many other problems. It has been successfully applied to geometrical theorem discovery [MoRe07] obtaining very simple answers for relatively complex problems.

There are two possible lacks coming from the Conjecture and the semi-algorithm GENIMAGE as we must set artificial bounds to make it algorithmic. Nevertheless, the use of MCCGS is at least useful to find examples where the minimal canonical CGS either does not exist or is not obtained by the actual algorithm, providing examples to test both the Conjecture and the semi-algorithm.

Finally it must be pointed out that the term order $\succ_{\overline{a}}$ chosen for the computation in A only affect the description by Gröbner bases of the varieties describing the segments but not to the varieties themselves.

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