Continuous Data Assimilation with Stochastic Data

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[1] joint work with A. Azouani, H. Bessaih and E. Olson

Outline

- Introduction and motivation
- Description of the method
- 2D Navier-Stokes equations
- Approximating problem
- Main results
- Numerical Implementation

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Introduction and Motivation

- Data assimilation is the process by which observations are incorporated into a computer model of a real system.
- Applications of data assimilation arise in many fields of geosciences, perhaps most importantly in weather forecasting and hydrology.
- The classical method of continuous data assimilation is to insert observational measurements directly into a computer model as the latter is being integrated in time.
- We propose a new approach based on ideas from control theory. Rather than inserting the measurements directly into the model, we introduce a feedback control term that forces/nudges the model toward the reference solution that is corresponding to the observations.

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Description of the method - Exact Observations

Let U be a solution lying on the attractor of the following dissipative dynamical system in the space H (finite or infinite dimensional)

$$\frac{dU}{dt} = F(U), \tag{1}$$

where the initial data, U_0, is missing.

Let $\mathcal{O}_h(U(t)) \in \mathbb{R}^D$, $t > 0$ be the exact observational measurements (without errors) of the true, unknown, solution U at time t.

Denote by $R_h(U(t))$ the interpolation of the observational data, namely,

$$R_h(U(t)) = \mathcal{L}_h \circ \mathcal{O}_h(U(t)),$$

where $\mathcal{L}_h \colon \mathbb{R}^D \to H$ is linear operator.

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Explicit examples of interpolant operators

[Foias & Titi 1991], [Jones & Titi 1992, 1993]

- **The volume elements interpolant:**

$$R_h(\varphi(x)) = \sum_{j=1}^{N} \bar{\varphi}_j \chi_{Q_j}(x) \quad \text{where} \quad \bar{\varphi}_j = \frac{N}{L^2} \int_{Q_j} \varphi(x)\, dx,$$

and the domain $D = [0, L]^2$ has been divided into N equal squares Q_j, with sides $h = L/\sqrt{N}$. This operator satisfies the approximate identity property (2).

- **Nodal values:**
Let $D = \cup_{j=1}^{N} Q_j$, where Q_j are disjoint subsets such that $\operatorname{diam} Q_j \leq h$ for $j = 1, 2, \ldots, N$, and let $x_j \in Q_j$ be arbitrary points. Then set

$$R_h(\varphi(x)) = \sum_{k=1}^{N} \varphi(x_k) \chi_{Q_j}(x).$$

This operator satisfies the approximate identity property (3).

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The interpolant observables R_h

We will be using two different interpolant operators (observables) that approximate identity

1. $R_h : H^1 \longrightarrow L^2$ that are linear and satisfy

$$\|\varphi - R_h(\varphi)\|_{L^2}^2 \leq c_1 h^2 \|\varphi\|_{H^1}^2 \tag{2}$$

for every $\varphi \in H^1(\Omega)$.

2. $R_h : H^2 \longrightarrow L^2$ such that

$$\|\varphi - R_h(\varphi)\|_{L^2}^2 \leq c_1 h^2 \|\varphi\|_{H^1}^2 + c_2 h^4 \|\varphi\|_{H^2}^2, \tag{3}$$

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Our Data Assimilation Algorithm

Our algorithm for constructing and approximate solutions $v(t)$ from the observational measurements $R_h(U(t))$ for $t > 0$ is given by

$$\frac{dv}{dt} = F(v) - \mu(R_h(v)) - R_h(U)),$$

$$v(0) = v_0,$$

where $\mu > 0$ is a positive relaxation parameter, which relaxes/nudgees the coarse spatial scales of v toward those of the observed data, and v_0 is taken to be arbitrary.

Our goal is to find a condition on the spatial resolution of the measurements, h, that guarantees that the approximating solution v converge to the unknown reference solution U as $t \longrightarrow \infty$.

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2D Navier-Stokes equations

Let $\mathcal{D} = \mathbb{T}^2$ (a two-dimensional torus identifiable with $[0, L]^2$ with periodic boundary conditions). We consider the 2D Navier-Stokes equations in $[0, T] \times \mathcal{D}$

$$\begin{cases} \frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla) U + \nabla p = f \\ \nabla \cdot U = 0 \\ U(0) = U_0. \end{cases} \tag{4}$$

Here, the unknowns are U, the velocity, and p the pressure. f is an external given force, and U_0 is the initial velocity that is missing.

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Let us introduce the following Hilbert spaces:

$$H = \left\{ u \in [\dot{L}^2_{\text{per}}(\mathcal{D})]^2, \quad \nabla \cdot u = 0 \quad \& \quad \int_{\mathcal{D}} u(x)dx = 0 \right\}$$

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$$Au = -\Pi \Delta u, \qquad \forall u \in D(A) = [\dot{H}^2_{\text{per}}(\mathcal{D})]^2 \cap V.$$

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Preliminaries and Notations – continue

By the classical spectral theorems there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of eigenvalues of the Stokes operator with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, corresponding to the eigenvectors $e_j \in D(A)$; $\{e_j\}_j$ form an orthonormal basis in H.

We have the following Poincaré inequalities:

$$\|u\|_{L^2}^2 \leq \lambda_1^{-1} \|u\|_{H^1}^2 \qquad \forall u \in V \tag{5}$$

$$\|u\|_{H^1}^2 \leq \lambda_1^{-1} \|Au\|_{L^2}^2 \qquad \forall u \in D(A) \tag{6}$$

Let $b(\cdot,\cdot,\cdot) : V \times V \times V \longrightarrow \mathbb{R}$ be the continuous trilinear form defined as

$$b(u,v,w) = \int_{\mathcal{D}} ([u(x)\cdot\nabla]v(x))\cdot w(x)\,dx.$$

$B(\cdot,\cdot) : V \times V \longrightarrow V'$ such that

$$\langle B(u,v), w\rangle = b(u,v,w), \quad forall \quad w \in V.$$

Projecting the NSE onto H, we obtain the abstract formulation

$$\begin{cases} \frac{dU}{dt} + \nu AU + B(U,U) = f, \\ U(0) = U_0. \end{cases} \tag{7}$$

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Properties of the nonlinear term

Lemma

for all u, v, w *with appropriate regularity*

- $$\langle B(u,v), w\rangle = -\langle B(u,w), v\rangle, \tag{8}$$

- $$\langle B(u,v), v\rangle = 0, \tag{9}$$

- $$\langle B(v,v), Av\rangle = 0. \tag{10}$$

- $$\langle B(u,v), Av\rangle + \langle B(v,u), Av\rangle = -\langle B(v,v), Au\rangle. \tag{11}$$

Lemma

$$\langle B(u,v),w\rangle \leq C_\star \|u\|_{L^4}\|v\|_{H^1}\|w\|_{L^4}. \tag{12}$$

Moreover, using the Ladyzhenskaya interpolation inequality

$$\|u\|_{L^4}^2 \leq |u|_{L^2}\|u\|_{H^1} \tag{13}$$

The 2D NSE are well-posed and possess a compact finite-dimensional global attractor:

Theorem

Let $U_0 \in H$ and $f \in H^{-1}$. Then the 2D system of Navier-Stokes equations has a unique weak solution that satisfies

$$U \in C([0,T]; L^2) \cap L^2([0,T]; H^1), \quad \text{for any} \quad T > 0.$$

$$\langle U(t), \phi\rangle + \nu \int_0^t \langle U(s), A\phi\rangle - \int_0^t \langle B(U(s), \phi), U(s)\rangle ds = \int_0^t \langle f, \phi\rangle ds$$

for all $\phi \in D(A)$ and $t \in [0,T]$. Moreover, the solution U depends continuously on the initial data U_0 in the H norm.

Known results on the deterministic NSE

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Strong solutions to the NSE

Theorem

Let $U_0 \in V$ *and* $f \in H$. *Then* (7) *has a unique strong solution that satisfies*

$$U \in C([0,T];H^1) \cap L^2([0,T];H^2), \quad \textit{forany} \quad T > 0.$$

Moreover, the solution U *depends continuously on the initial data* U_0 *in the* H^1 *norm.*

Let us denote by G the Grashof number

$$G = \frac{1}{\nu^2 \lambda_1} \limsup_{t \to \infty} |f(t)|_H. \tag{14}$$

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Theorem

Let $T > 0$ and G the Grashof number given by (14). There exists a time t_0 such that for $t \geq t_0$ we have

$$|U(t)|_H^2 \leq 2\nu^2 G^2 \quad \text{and} \quad \int_t^{t+T} \|U(s)\|_V^2 ds \leq 2(1 + T\nu\lambda_1)\nu G^2. \tag{15}$$

In the case of periodic boundary conditions we also have

$$\|U(t)\|_V^2 \leq 2\nu^2 \lambda_1 G^2 \quad \text{and} \quad \int_t^{t+T} |AU(s)|_H^2 ds \leq 2(1+T\nu\lambda_1)\nu\lambda_1 G^2. \tag{16}$$

If $f \in H$ is time independent then

$$|AU(t)|_H^2 \leq c\nu^2\lambda_1(1 + G)^8. \tag{17}$$

The approximating scheme the deterministic case

Theorem

Let v be a solution to equations

$$\begin{cases} \frac{dv}{dt} + \nu Av + B(v,v) = f - \mu R_h(v - U), \\ v(0) = v_0 \end{cases} \tag{18}$$

Assume h is small enough such that

$$1/h^2 \geq c_1 \lambda_1 G^2,$$

. Then there exists $\mu > 0$, given explicitly, such that $\|v - u\|_{L^2(\Omega)} \to 0$ exponentially, as $t \to \infty$.

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The approximating scheme Stochastic case

Let u be the approximation of U solution the stochastic NSE

$$\begin{cases} du + [\nu Au + B(u,u)]dt = [f + \mu R_h(U-u)]\,dt + \mu dR_h(W), \\ u(0) = u_0 \end{cases} \tag{19}$$

It is not difficult to prove that system (19) is well-posed.

Let u be the approximation of U solution the stochastic NSE

$$\begin{cases} du + [\nu A u + B(u,u)]dt = [f + \mu R_h(U - u)]\, dt + \mu d R_h(W), \\ u(0) = u_0 \end{cases} \tag{19}$$

It is not difficult to prove that system (19) is well-posed.

Theorem

Let us assume that U is the strong solution of the deterministic Navier-Stokes equations and $u_0 \in H$ and that trace$(Q) < \infty$. Moreover, assume that R_h satisfies (2) and that $\mu c_1 h^2 \leq \nu$. Then, for any $T > 0$ there exists a continuous stochastic process solution of (19) such that P-a.s.

$$u \in C([0,T];H) \bigcap L^2([0,T]);V).$$

And

$$E\left(\sup_{0\leq t\leq T} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) < \infty.$$

Moreover if $u_0 \in V$ then, P-a.s. the process u is such that

$$u \in C([0,T];V) \bigcap L^2([0,T]);D(A)).$$

Main result

Our goal is to prove that the approximating solution u converges to the true solution U when $t \longrightarrow \infty$, in some sense.

Theorem

Assume that U is a solution of (7) with period boundary conditions. If R_h satisfies assumption (2) and

$$\frac{1}{h^2} > 4c_1 C_\star \lambda_1 G^2$$

then there exists $\mu > 0$ such that

$$\lim_{t\to\infty} E|u(t) - U(t)|_H^2 \leq Tr[Q]\frac{\mu}{\mu - 2C_\star \nu \lambda_1 G^2}.$$

Let us also assume that

$$\|R_h(\varphi)\|_{H^1} \leq \frac{c}{h}\|\varphi\|_{L^2}. \tag{20}$$

Theorem

Assume that U is a solution of (7) with period boundary conditions. If R_h satisfies assumption (3) and (20) and

$$\frac{1}{h^2} > 2\mu c_1 \lambda_1 G \left(c_2 \log c^{1/2} + 4c_2 \log(1+G) \right)$$

then there exists $\mu > 0$ such that

$$\lim_{t\to\infty} E\|u(t) - U(t)\|_V^2 \leq \frac{1}{h}\left(\frac{\mu Tr[Q]}{\mu - \frac{2J^2}{\mu}\nu^2\lambda_1^2 G^2} \right)$$

where $J = c_2 \log c^{1/2} + 4c_2 \log(1+G)$.

Setting

$$v := U - u$$

The process v is solution of the following equation

$$\begin{cases} dv + [\nu Av + B(U,U) - B(u,u)]\, dt = -\mu R_h(v)dt + \mu R_h dW \\ v(0) = v_0 \end{cases}$$

Using the bilinearity of the operator B, we get v satisfies the following problem

$$dv + [\nu Av + B(U,v) + B(v,U) - B(v,v)]\, dt = -\mu R_h(v)dt + \mu dR_h(W) \tag{21}$$

Using the Itô formula on $|v(t)|_H^2$

$$
\begin{aligned}
d|v(t)|_H^2 &= 2\langle v(t), dv(t)\rangle + \mu Tr[R_h(Q)]dt \\
&= -2\nu\langle v(t), Av(t)\rangle - 2\langle v(t), B(U(t), v(t))\rangle \\
&- 2\langle v(t), B(v(t), U(t))\rangle + 2\langle v(t), B(v(t), v(t))\rangle \\
&- 2\mu\langle v(t), R_h v(t)\rangle + 2\mu\langle v(t), dR_h W\rangle \\
&+ \mu Tr[R_h(Q)]dt
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d|v(t)|_H^2 + 2\nu\|v(t)\|_V^2 &= -2\langle v(t), B(v(t), U(t))\rangle \\
&- 2\mu\langle v(t), R_h v(t)\rangle + 2\mu\langle v(t), dR_h(W)\rangle \\
&+ \mu Tr[R_h(Q)]dt
\end{aligned}
$$

Using the properties of B Young inequality

$$|\langle v(t), B(v(t), U(t))\rangle| \leq \nu\|v(t)\|_V^2 + \frac{C_\star}{\nu}|v(t)|_H^2\|U(t)\|_V^2.$$

On the other side, using the Young inequality and the approximation (2) we obtain

$$\begin{aligned} -2\mu\langle R_h v(t), v(t)\rangle &= -2\mu\langle R_h v(t) - v(t), v(t)\rangle - 2\mu|v(t)|_H^2 \\ &\leq 2\mu|R_h v(t) - v(t)|_H|v(t)|_H - 2\mu|v(t)|_H^2 \\ &\leq \mu|v(t)|_H^2 + \mu|R_h v(t) - v(t)|_H^2 - 2\mu|v(t)|_H^2 \\ &\leq c_1\mu h^2\|v(t)\|_V^2 - \mu|v(t)|_H^2. \end{aligned}$$

Choose $(c_1\mu h^2) \leq \nu/2$, then taking the expected value we get

$$\frac{d}{dt}E|v(t)|^2 \le \left(\frac{C_\star}{\nu}\|U(t)\|^2 - \mu\right)E|v(t)|^2 + \mu Tr[R_h(Q)]$$

Using Gronwall lemma, we obtain

$$\begin{aligned}
E|v(t)|_H^2 &\le |v_0|_H^2 \mathrm{e}^{-t\left[\mu - \frac{C_\star}{\nu}\frac{1}{t}\int_0^t \|U(s)\|^2 ds\right]} \\
&\quad + \mu Tr[Q]\int_0^t \mathrm{e}^{-(t-s)\left[\mu - \frac{C_\star}{\nu}\frac{1}{t-s}\int_s^t \|U(r)\|^2 dr\right]} ds \\
&\le |v_0|_H^2 \mathrm{e}^{-t\left[\mu - 2C_\star\nu\lambda_1 G^2\right]} + \mu Tr[Q]\int_0^t \mathrm{e}^{-(t-s)\left[\mu - 2C_\star\nu\lambda_1 G^2\right]} ds. \\
&\le |v_0|^2 \mathrm{e}^{-t\left[\mu - 2C_\star\nu\lambda_1 G^2\right]} \\
&\quad + Tr[Q]\frac{\mu}{\mu - 2C_\star\nu\lambda_1 G^2}\left(1 - \mathrm{e}^{-t\left[\mu - 2C_\star\nu\lambda_1 G^2\right]}\right).
\end{aligned}$$

Now, choose

$$\mu > 2C_{\star}\nu\lambda_1 G^2.$$

Then, taking the limit $t \to \infty$ in the previous estimate completes the proof of the first main result.

Proof of the 2nd Theorem

Using the Itô formula on $\|v(t)\|_V^2 = |A^{1/2}v(t)|_H^2$

$$
\begin{aligned}
d|A^{1/2}v(t)|^2 &= 2\langle A^{1/2}v(t), dA^{1/2}v(t)\rangle + \mu Tr[A^{1/2}R_h(Q)]dt \\
&= -2\nu\langle Av(t), Av(t)\rangle - 2\langle Av(t), B(U(t), v(t))\rangle \\
&- 2\langle Av(t), B(v(t), U(t))\rangle + 2\langle Av(t), B(v(t), v(t))\rangle \\
&- 2\mu\langle Av(t), R_h v(t)\rangle + 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)
\end{aligned}
$$

Using again the properties of B, we obtain that

$$
\begin{aligned}
d\|v(t)\|^2 + 2\nu|Av(t)|^2 &= 2\langle B(v(t), v(t)), AU(t)\rangle - 2\mu\langle Av(t), R_h v(t)\rangle \\
&+ 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)]dt
\end{aligned}
$$

Now, using the Brézis–Gallouet inequality due to Brezis(80) which may be written as

$$\|u\|_{L^\infty(\Omega)} \le c_2\|u\|\left\{1 + \log\frac{|Au|_H^2}{\lambda_1\|u\|_V^2}\right\}, \tag{22}$$

we get that

$$\begin{aligned}|\langle B(v,v), AU\rangle| &\le \|v\|_\infty\|v\|_V|AU|_H \\ &\le c_2\|v\|_V^2\left\{1 + \log\frac{|Av|_H^2}{\lambda_1\|v\|_V^2}\right\}|AU|_H.\end{aligned}$$

On the other side, using Young inequality and the approximation (3)

$$
\begin{aligned}
-2\mu\langle R_h v, Av\rangle &= 2\mu\langle v - R_h v, Av\rangle - 2\mu\|v\|_V^2 \\
&\le 2\mu|v - R_h v|_H|Av|_H - 2\mu\|v\|_V^2 \\
&\le \frac{\mu^2}{\nu}|v - R_h v|_H^2 + \frac{\nu}{2}|Av|_H^2 - 2\mu\|v\|_V^2 \\
&\le \frac{\mu^2 c_1^2 h^4}{\nu}|Av|_H^2 + \frac{\nu}{2}|Av|_H^2 - 2\mu\|v\|_V^2
\end{aligned}
$$

If we choose

$$\frac{\mu^2 c_1^2 h^4}{\nu} \leq \frac{\nu}{2}$$

then

$$d\|v(t)\|^2 + \nu|Av(t)|^2 dt + \left(2\mu - c_2\left\{1 + \log\frac{|Av|^2}{\lambda_1\|v\|^2}\right\}|AU|\right)\|v\|^2 dt$$
$$\leq 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)]dt$$

Now, if we assume that $r = \frac{|Av|^2}{\lambda_1 \|v\|^2}$ and $\beta = \frac{c_2|AU|}{\nu\lambda_1}$. Here, $r \geq 1$. Using the following Lemma due to Olson & Titi (2013)

Lemma

Let $\phi(r) = r - \beta(1 + \log r)$ *where* $\beta > 0$. *Then*

$$\min\{\phi(r) : r \geq 1\} \geq -\beta \log \beta.$$

We obtain that

$$d\|v(t)\|^2 + \left(2\mu - c_2|AU| \log \frac{c_2|AU|}{\nu\lambda_1}\right) \|v\|^2 dt \leq 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)]dt.$$

Using the estimate (16) yields that

$$c_2 \log \frac{c_2|AU|}{\nu\lambda_1} \le J$$

where $J = c_3 + c_4 \log(1+G)$, $c_3 = c_2 \log c^{1/2}$, $c_4 = 4c_2$, then

$$d\|v(t)\|^2 + (2\mu - J|AU|)\|v\|^2 dt \le 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)]dt$$

Furthermore, the inequality

$$J|AU| \leq \frac{J^2}{\mu}|AU|^2 + \mu$$

implies

$$d\|v(t)\|^2 + \left(\mu - \frac{J^2}{\mu}|AU|^2\right)\|v\|^2 dt \leq 2\mu\langle Av(t), dR_h W\rangle + \mu Tr[A^{1/2}R_h(Q)$$

Now integrate over $(0, t)$ and take the expectation value yields that

$$E\|v(t)\|^2 \leq E\|v(0)\|^2 e^{-\alpha(t)} + \mu Tr[A^{1/2}R_h(Q)] \int_0^t e^{-\alpha(t-s)} ds$$

where $\alpha(t) = \mu t - \frac{J^2}{\mu}\int_0^t |AU(s)|^2 ds$.

Now choose

$$\mu - \frac{2J^2}{\mu}\nu^2\lambda_1^2 G^2 > 0$$

implies that

$$\lim_{t\to\infty} E\|v(t)\|^2 \leq \frac{\mu Tr[A^{1/2}R_h(Q)]}{\mu - \frac{2J^2}{\mu}\nu^2\lambda_1^2 G^2} \leq \frac{1}{h}\frac{\mu Tr[Q]}{\mu - \frac{2J^2}{\mu}\nu^2\lambda_1^2 G^2}$$

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THANK YOU