A Continuous Data Assimilation Algorithm for the 2D Bénard Convection Model and Other Models of Turbulence

Jornades d'Interacciò entre Sistemes Dinàmics i Equacions en Derivades Parcials

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Data Assimilation

- The goal of data assimilation and signal synchronization is to use low spatial solution resolution observational measurements to find a corresponding reference solution from which future predictions can be made.
- One motivating application is weather forecasting.

- In late 1960's satellite observation systems began producing data on the climate.
- Charney, Halem and Jastraw (1969) proposed that the equations of the atmosphere equations be used to process this data and obtain improved estimates of the current atmospheric state.
- Their method, called *continuous data assimilation*, is to insert the observational measurements, at coarse scales, directly into a model as the later being integrated in time.

- In general, it is neccessary to seperate slow and fast parts of the solution before inserting the observations into the model.
- One way to exploit this is to insert low mode observables from a time series into the equation for the high modes.
- This was the approach taken for the 2D Navier-Stokes in Browning, Henshaw and Kreiss (1998) and Henshaw, Kreiss and Yström (2003).
- The question becomes, how do we find $u(t)$ from a measurement $P_\lambda u(t)$.

- Let $u(t)$ be the exact solution of the 2D NSE

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla\Pi = f,$$
$$\nabla \cdot u = 0,$$

with initial data $u(0,x) = u_0(x)$, on L- periodic torus $\Omega = [0,L]^2$.

- Find $v(t)$, a good asymptotic approximation of $u(t)$.
- Set $v(t) = p(t) + q(t)$, where $p(t) = P_\lambda v(t)$ and $q(t) = Q_\lambda v(t)$.

$$\frac{\partial p}{\partial t} + P_\lambda((p+q)\cdot\nabla(p+q)) - \nu\nabla p + \nabla P_\lambda\Pi = P_\lambda f, \quad \nabla\cdot p = 0,$$
$$\frac{\partial q}{\partial t} + Q_\lambda((p+q)\cdot\nabla(p+q)) - \nu\nabla q + \nabla Q_\lambda\Pi = Q_\lambda f, \quad \nabla\cdot q = 0.$$

- $p(t)$ is given directly by the measurement. We need to integrate the second equation to find $v(t)$.
- Compute an approximate solution $q_2(t)$ by integrating the second equation with the initial condition $q_2(0) = \eta = Q_\lambda \eta$, some initial guess.
- Set $u_1 = p + q$ to be the exact solution to the 2D NSE and set $u_2 = p + q_2$ to be the approximate solution.

Theorem (Olson and Titi, 2003)

Given $\lambda > c_1 Gr(f)$ and provided that u_2 exists globally, in time, it follows that $\|u_1(t) - u_2(t)\| \to 0$ as $t \to \infty$, at an exponential rate.

Connection to Determining Modes

- It was shown first by Foias and Prodi (1967) (and independently later proved by Ladyzhenskaya (1975)) that the 2D NSE posses a finite number of determining modes.
- The best estimate on the number of determining modes of the 2D NSE in periodic case is given by Jones and Titi (1992, 1993).

Theorem

Let u_1 and u_2 be two solutions of the 2D NSE on the L-periodic torus with possibly different initial conditions. Then there exists a constant c_1 independent of ν, L, f or of any initial conditions such that for every $\lambda(L/2\pi)^2 > c_1 Gr(f)$ the limit

$$|P_\lambda u_1(t) - P_\lambda u_2(t)| \to 0, \text{ as } t \to \infty,$$

implies that

$$\|u_1(t) - u_2(t)\| \to 0, \text{ as } t \to \infty.$$

A New Algorithm

- The classical method of continuous data assimilation is, in concept, simple but special care has to be taken inserting the observations into the model.
- In applications, the measured data is usually obtained as the values of the exact solutions at a discrete set of spatial nodal points.
- It is impossible to obtain exact values of spacial derivatives.
- A new Algorithm was introduced recently by Azouani and Titi 2013 and Azouani, Olson and Titi 2014.

- Suppose that $u(t)$ represents a solution of some dynamical system

$$\frac{du}{dt} = F(u),$$

with missing initial condition $u(0) = u_0$.

- $I_h(u(t))$ represent the observations of the reference solution u at a coarse spatial resolution of size h.
- Use $I_h(u)$ is a feedback control term

$$\frac{dv}{dt} = F(v) - \mu(I_h(v) - I_h(u)),$$
$$v(0) = v_0,$$

where $\mu > 0$ is a relaxation (nudging) parameter.

- This approach works for a general class of interpolant observations without modification.
  1. One physical example are the volume elements in Ω. Let $h > 0$ be given and let $\Omega = \cup_{j=1}^{N} Q_j$, where Q_j are disjoint sets with $diam(Q_j) \leq h$, then, in this case
  $$I_h(\phi) = \sum_{j=1}^{n} \bar{\phi}_j \chi_{Q_j}(x), \text{ where } \bar{\phi}_j = \int_{Q_j} \phi(x)\, dx.$$
  2. Another example is measurements at discrete nodal points in Ω. In this case
  $$I_h(\phi) = \sum_{j=1}^{n} \phi(x_j) \chi_{Q_j}(x), \text{ where } x_j \in Q_j.$$

- The new approach was demonstrated in the case of 2D Navier-Stokes equations by Azouani, Olson and Titi (2014).
- It was shown that the approximate solution of the continuous data assimilation algorithm converges to the reference solution of the 2D Navier-Stokes equations.
- Analytic estimates were obtained on the relaxation parameter μ and on the spatial resolution $h > 0$.
- These estimates depend on physical parameters of the system a.k.a. *the Grashof (Reynolds) number Gr*.

- For volume elements measurements, with Dirichlet boundary conditions:

Theorem (Azouani, Olson and Titi 2014)

Let u be the solution of the 2D NSE and v be the approximate solution, with no-slip boundary conditions. Then $\|u(t) - v(t)\|_{L^2(\Omega)} \to 0$, at an exponential rate as $t \to \infty$, provided that $\mu > c_2 Gr^2 \nu \lambda_1$ and $\mu c_0 h^2 \leq \nu$.

- For discrete nodal set measurements, with periodic boundary conditions:

Theorem (Azouani, Olson and Titi 2014)

Let u be the solution of the 2D NSE and v be the approximate solution, on the L-periodic torus. Then $\|u(t) - v(t)\|_{H^1(\Omega)} \to 0$, at an exponential rate as $t \to \infty$, provided that $\mu > c_2 Gr(1 + \log(Gr)) \nu \lambda_1$ and $\mu c_0 h^2 \leq \nu$.

Remarks

- Numerical weather forecasting equations are three-dimensional equations involving variable density or temperature that is coupled to the some set of equations.
- It is important to analyze the validity and success of a data assimilation algorithm when some state variable observations are not available as an input on the numerical forecast model.

The Bénard Convection Problem

- The Bénard convection problem is a model of the convection of a fluid in a box $(0, L) \times (0, 1)$ which is heated from below.
- In this case, the non-dimensional Boussinesq system can be written as

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = (T - T_1)\mathbf{e}_2,$$
$$\frac{\partial T}{\partial t} - \kappa \Delta T + (u \cdot \nabla) T = 0,$$
$$\nabla \cdot u = 0,$$

- with boundary conditions

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1,$$
$$T = T_0 \quad \text{at} \quad y = 0 \quad \text{and} \quad T = T_1 \quad \text{at} \quad y = 1,$$

- The global regularity of he 2D Boussinesq system, with $\nu > 0$ and $\kappa > 0$, was established in Cannon and DiBenedetto(1980) following the classical methods for the Navier-Stokes equations.
- Recent results concerning the 2D Boussinesq equations, with $\nu = 0$ or $\kappa = 0$: Chae (2006), Hou and Li (2005), Hmidi and Keraani (2007), Danchin and Paicu (2008), Larios, Lunasin and Titi (2010), Hu, Kukavicka and Ziane (2013) and many others.

- After some change of variables, the Bénard convection problem in the box $\Omega = (0, L) \times (0, 1)$:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \theta \mathbf{e}_2, \\
&\frac{\partial \theta}{\partial t} - \kappa \Delta \theta + (u \cdot \nabla) \theta - u \cdot \mathbf{e}_2 = 0, \\
&\nabla \cdot u = 0, \\
&u(0; x) = u_0(x), \quad \theta(0; x) = \theta_0(x),
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
u &= 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1, \\
\theta &= 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = 1,
\end{aligned}
$$

and

$$u, \theta, p \text{ are periodic, of period } L, \text{ in the } x\text{-direction.}$$

- The 2D Bénard convection system has a finite dimensional global attractor $\mathcal{A}$ (Foias, Manley and Temam 1987).
- We observed that the values of the temperature (or the density) $\theta(t;x)$ in $\mathcal{A}$ are completely determined by the velocity vector field $u(t,x,y)$ for all time in $\mathcal{A}$.
- We propose a continuous data assimilation algorithm for the 2D Bénard convection *using velocity measurements alone*.

Our algorithm for the construction of approximate solutions, $v(t)$ and $\eta(t)$, from the observational measurements $I_h(u(t))$ for the reference solution $u(t)$, for $t \in [0, T]$ is given by

$$\begin{aligned}
&\frac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v + \nabla\tilde{p} = \eta\mathbf{e}_2 - \mu(I_h(v) - I_h(u)),\\
&\frac{\partial \eta}{\partial t} - \kappa\Delta\eta - (v\cdot\nabla)\eta - v\cdot\mathbf{e}_2 = 0,\\
&\nabla\cdot v = 0,\\
&U(0,x,y) = v_0(x,y), \quad \eta(0,x,y) = \eta_0(x,y),
\end{aligned}$$

where v_0 and η_0 is arbitrary, and can be simply taken to be $v_0 = 0$ and $\eta_0 = 0$.

Theorem (Convergence to Reference Solution (Farhat, Jolly and Titi 2014))

Let $(u(t,x,y),\theta(t,x,y))$ *be a reference solution of the Bénard convection system and* $(v(t,x,y),\eta(t,x,y))$ *be a solution of the data assimilation algorithm. Let* $\mu>0$ *large enough and* $h>0$ *small enough such that*

$$\mu \geq C(\nu,\kappa,\lambda_1,L).$$

and $\mu c_0^2 h^2 \leq \nu$*. Then,* $\|u(t)-v(t)\|_{L^2(\Omega)} \to 0$ *and* $\|\theta(t)-\eta(t)\|^2_{L^2(\Omega)} \to 0$ *at an exponential rate as* $t\to\infty$*.*

A New Abridged Continuous Data Assimilation Algorithm for 2D NSE

- Inspired by the continuous data assimilation algorithm for the Bénard problem, we introduce an abridged and nominal approach to a dynamic continuous data assimilation for the 2D Navier-Stokes.

A New Abridged Continuous Data Assimilation Algorithm for 2D NSE

- Inspired by the continuous data assimilation algorithm for the Bénard problem, we introduce an abridged and nominal approach to a dynamic continuous data assimilation for the 2D Navier-Stokes.
- The 2D Navier-Stokes equations which can be written as

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \nu\Delta u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 + \partial_x p = f_1, \\ \frac{\partial u_2}{\partial t} - \nu\Delta u_2 + u_1\partial_x u_2 + u_2\partial_y u_2 + \partial_y p = f_2, \\ \partial_x u_1 + \partial_y u_2 = 0, \\ u_1(0,x,y) = u_1^0(x,y), \quad u_2(0,x,y) = u_2^0(x,y). \end{aligned}$$

- We propose algorithm for the construction of approximate solution, $U(t,x,y)$ from the observational measurements of only one component of velocity (horizontal or vertical).
- The algorithm is given by

$$
\begin{aligned}
\frac{\partial U_1}{\partial t} - \nu\Delta U_1 + U_1\partial_x U_1 + U_2\partial_y U_1 + \partial_x P &= f_1, \\
\frac{\partial U_2}{\partial t} - \nu\Delta U_2 + U_1\partial_x U_2 + U_2\partial_y U_2 + \partial_y P &= f_2 + \mu(I_h(u_2) - I_h(U_2)), \\
\partial_x U_1 + \partial_y U_2 &= 0, \\
U_1(0,x,y) = U_1^0(x,y), \quad U_2(0,x,y) &= U_2^0(x,y).
\end{aligned}
$$

- Taking advantage of the divergence free condition, $\nabla \cdot v = \nabla \cdot U = 0$, and integration by parts we prove: In the case of volume elements measurements with Dirichlet boundary conditions:

Theorem (Convergence to Reference Solution (Farhat, Lunasin and Titi 2014))

Let $u(t,x,y) = (u_1(t,x,y), u_2(t,x,y))$ be a refernce solution of the 2D NSE and $U(t,x,y) = (U_1(t,x,y), U_2(t,x,y))$ be a solution of the abridged continuous data assimilation system. Let $\mu > 0$ be chosen large enough such that

$$\mu \geq 2c(1 + \log(Gr) + Gr^4)Gr^2.$$

If $h > 0$ is chosen small enough such that $\mu c_0^2 h^2 \leq \nu$ then, $\|u(t) - U(t)\|_{L^2(\Omega)}^2 \to 0$ at an exponential rate as $t \to \infty$.

- In the case of volume elements measurements with periodic boundary conditions, enough to take

$$\mu \geq 2c\nu\lambda_1(1+\log(Gr))\,Gr^2.$$

- In the case of discrete nodal measurements with periodic boundary conditions, enough to take

$$\mu > 2c\nu\lambda_1(Gr^2 + Gr^3).$$

A New Abridged Continuous Data Assimilation Algorithm for α-Models of Turbulenece

- Our analytical approach assumes the global existence of the underlying model and uses previously known estimates.
- It is for this reason that we are not able to prove similar results for the 3D NSE case.
- The α-models are are simplified models through an averaging process that is designed to capture the large scale dynamics of the flow and at the same time provide reliable closure model to the averaged equations.

- The first α-model that was proposed is the Navier-Stokes-α model (LANS-α or Camassa-Holm eqs.):

$$\frac{\partial v}{\partial t} + (u \cdot \nabla) v + \nabla u \cdot v + \nabla p = \nu \Delta v + f,$$
$$\nabla \cdot u = 0, \text{ and } v = u - \alpha^2 \Delta u \, ,$$

- Many other α-models, such as the Leray-α, the Clark-α, the Navier-Stokes-Voigt (NSV) equation, the modified Leray-α (ML-α), and the simplified Bardina model (SBM) were inspired by this regularization technique.
- All of the models just mentioned have global regular solutions and posses fewer degrees of freedom than the NSE.

- In Albanez, Nussenzveig-Lopes and Titi (2014), it was shown that approximate solutions constructed using observations on all three components of the unfiltered velocity field converge in time to the reference solution of the 3D NS-α model.
- We apply our data assimilation algorithm for the case of Leray-α model:

$$
\begin{aligned}
\partial_t v - \nu \Delta v + (u \cdot \nabla) v &= -\nabla p + f, \\
\nabla \cdot u = \nabla \cdot v &= 0, \\
v &= u - \alpha^2 \Delta u, \\
v(0, x, y, z) &= v^0(x, y, z).
\end{aligned}
$$

- The proposed algorithm for reconstructing $u(t)$ and $v(t)$ from only the horizontal observational measurements $I_h(v_1(t))$ and $I_h(v_2(t))$ is given by the system

$$
\begin{aligned}
\partial_t V_1 - \nu\Delta V_1 + (U\cdot\nabla) V_1 &= -\partial_x P + \mu\,(I_h(v_1) - I_h(V_1)) + f_1,\\
\partial_t V_2 - \nu\Delta V_2 + (U\cdot\nabla) V_2 &= -\partial_y P + \mu\,(I_h(v_2) - I_h(V_2)) + f_2,\\
\partial_t V_3 - \nu\Delta V_3 + (U\cdot\nabla) V_3 &= -\partial_x P + f_3,\\
\nabla\cdot U &= \nabla\cdot V = 0,\\
V &= U - \alpha^2\Delta U,\\
V(0,x,y,z) &= V^0(x,y,z).
\end{aligned}
$$

Theorem (Convergence to Reference Solution (Farhat, Lunasin and Titi 2014))

Let $v(t,x,y,z)$ be a strong solution of the Leray-α model and $V(t,x,y,z)$ be a strong solution of the continuous data assimilation system subject to periodic boundary conditions in $\Omega=[0,L]^3$. If $\mu>0$ is large enough such that

$$\mu \geq 2c\mathbf{Gr}^2,$$

and $h>0$ is small enough such that $\mu c_0^2 h^2 \leq \nu$. Then, $\|v(t)-V(t)\|_{\dot{\mathbf{L}}^2(\Omega)} \to 0$ at exponential rate as $t\to\infty$.

Proof for the 2D NSE

- Our data assimilation algorithm is given by

$$\frac{\partial U_1}{\partial t} - \nu\Delta U_1 + U_1\partial_x U_1 + U_2\partial_y U_1 + \partial_x P = f_1,$$
$$\frac{\partial U_2}{\partial t} - \nu\Delta U_2 + U_1\partial_x U_2 + U_2\partial_y U_2 + \partial_y P = f_2 + \mu(I_h(u_2) - I_h(U_2)),$$
$$\partial_x U_1 + \partial_y U_2 = 0,$$
$$U_1(0,x,y) = U_1^0(x,y), \quad U_2(0,x,y) = U_2^0(x,y).$$

Proof for the 2D NSE

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$$\partial_x U_1 + \partial_y U_2 = 0,$$
$$U_1(0,x,y) = U_1^0(x,y), \quad U_2(0,x,y) = U_2^0(x,y).$$

- We will use the following logarithmic estimate proved in Titi (1986): For every $u, v, w \in H^1(\Omega)$, with $w \neq 0$, we have

$$|((u\cdot\nabla)v, w)| \leq$$
$$c_T \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \left(1 + \log\left(\frac{\|\nabla w\|_{L^2(\Omega)}}{\lambda_1^{1/2}\|w\|_{L^2(\Omega)}}\right)\right)^{1/2}.$$

- Define $\tilde{u} = u - U$ and $\tilde{p} = p - P$. Then $\tilde{u}_1$ and $\tilde{u}_2$ satisfy the equations

$$\frac{\partial \tilde{u}_1}{\partial t} - \nu \Delta \tilde{u}_1 + U_1 \partial_x \tilde{u}_1 + U_2 \partial_y \tilde{u}_1 + \tilde{u}_1 \partial_x u_1 + \tilde{u}_2 \partial_y u_1 + \partial_x \tilde{p} = 0,$$
$$\frac{\partial \tilde{u}_2}{\partial t} - \nu \Delta \tilde{u}_2 + U_1 \partial_x \tilde{u}_2 + U_2 \partial_y \tilde{u}_2 + \tilde{u}_1 \partial_x u_2 + \tilde{u}_2 \partial_y u_2 + \partial_y \tilde{p} = -\mu I_h(\tilde{u}_2)$$
$$\partial_x \tilde{u}_1 + \partial_y \tilde{u}_2 = 0.$$

- We obtain using the divergence free condition and integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{u}_1\|^2_{L^2(\Omega)} + \nu \|\nabla \tilde{u}_1\|^2_{L^2(\Omega)} &\le |I_{1a}| + |I_{1b}| - (\partial_x \tilde{p}, \tilde{u}_1),\\ \frac{1}{2}\frac{d}{dt}\|\tilde{u}_2\|^2_{L^2(\Omega)} + \nu \|\nabla \tilde{u}_2\|^2_{L^2(\Omega)} &\le |I_{2a}| + |I_{2b}| - (\partial_y \tilde{p}, \tilde{u}_2)\\ &\quad - \mu(I_h(\tilde{u}_2), \tilde{u}_2),\end{aligned}$$

- where

$$I_{1a} := (\tilde{u}_1 \partial_x u_1, \tilde{u}_1), \qquad I_{1b} := (\tilde{u}_2 \partial_y u_1, \tilde{u}_1),$$
$$I_{2a} := (\tilde{u}_1 \partial_x u_2, \tilde{u}_2), \qquad I_{2b} := (\tilde{u}_2 \partial_y u_2, \tilde{u}_2).$$

- Using integration by parts twice, we have

$$\begin{aligned} I_{1a} = -\left(\partial_x u_1, (\tilde{u}_1)^2\right) &= -2\left(u_1 \tilde{u}_{,} \partial_x \tilde{u}_1\right) \\ &= 2\left(u_1 \tilde{u}_1, \partial_y \tilde{u}_2\right) \\ &= -2\left(\tilde{u}_1 \partial_y u_1, \tilde{u}_2\right) - 2\left(u_1 \partial_y \tilde{u}_1, \tilde{u}_2\right) \\ &=: -2(I_{1a1}) - 2(I_{1a2}), \end{aligned}$$

- By the logarithmic estimate and Young's inequality, we show that

$$
\begin{aligned}
|I_{1a1}| :=& \, |\, (\tilde{u}_1 \partial_y u_1, \tilde{u}_2)\, | \\
\leq & \, c_T \, \|\nabla \tilde{u}_1\|_{L^2(\Omega)} \, \|\nabla u_1\|_{L^2(\Omega)} \, \|\tilde{u}_2\|_{L^2(\Omega)} \\
& \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|_{L^2(\Omega)}}{\lambda_1^{1/2}\, \|\tilde{u}_2\|_{L^2(\Omega)}}\right)\right)^{1/2} \\
\leq & \, \frac{\nu}{32} \, \|\nabla \tilde{u}_1\|_{L^2(\Omega)} + \frac{c}{\nu} \, \|\nabla u_1\|^2_{L^2(\Omega)} \\
& \left(1 + \log\left(\frac{\|\nabla \tilde{u}_2\|_{L^2(\Omega)}}{\lambda_1^{1/2}\, \|\tilde{u}_2\|_{L^2(\Omega)}}\right)\right) \|\tilde{u}_2\|^2_{L^2(\Omega)} \,,
\end{aligned}
$$

- Thanks to the assumption $\mu c_0^2 h^2 \le \nu$ and Young's inequality,

$$
\begin{aligned}
-\mu(I_h(\tilde{u}_2), \tilde{u}_2) &= -\mu(I_h(\tilde{u}_2) - \tilde{u}_2, \tilde{u}_2) - \mu \|\tilde{u}_2\|_{L^2(\Omega)}^2 \\
&\le \mu \|I_h(\tilde{u}_2) - \tilde{u}_2\|_{L^2(\Omega)} \|\tilde{u}_2\|_{L^2(\Omega)} - \mu \|\tilde{u}_2\|_{L^2(\Omega)}^2 \\
&\le \mu c_0 h \|\tilde{u}_2\|_{L^2(\Omega)} \|\nabla \tilde{u}_2\|_{L^2(\Omega)} - \mu \|\tilde{u}_2\|_{L^2(\Omega)}^2 \\
&\le \frac{\mu c_0^2 h^2}{2} \|\nabla \tilde{u}_2\|_{L^2(\Omega)}^2 - \frac{\mu}{2} \|\tilde{u}_2\|_{L^2(\Omega)}^2 \\
&\le \frac{\nu}{2} \|\nabla \tilde{u}_2\|_{L^2(\Omega)}^2 - \frac{\mu}{2} \|\tilde{u}_2\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

- It follows that

$$\frac{d}{dt}\|\tilde{u}\|^2_{L^2_0(\Omega)} + \frac{\nu\lambda_1}{2}\|\tilde{u}\|^2_{L^2_0(\Omega)} + \frac{\nu\lambda_1}{2}\frac{\|\nabla\tilde{u}_2\|^2_{L^2(\Omega)}}{\lambda_1\|\tilde{u}_2\|^2_{L^2(\Omega)}}\|\tilde{u}_2\|^2_{L^2(\Omega)} \leq$$
$$\left(\frac{c}{\nu}\|\nabla u\|^2_{L^2_0(\Omega)}\left(1+\log\left(\frac{\|\nabla\tilde{u}_2\|^2_{L^2(\Omega)}}{\lambda_1\|\tilde{u}_2\|^2_{L^2(\Omega)}}\right)\right) - \mu\right)\|\tilde{u}_2\|^2_{L^2(\Omega)}.$$

- Using the following elementary inequality:
  Let $\phi(r) = r - \gamma(1+\log r)$ where $\gamma > 0$. Then

$$\min\{\phi(r) : r \geq 1\} \geq -\gamma\log(\gamma),$$

- we have

$$\frac{d}{dt}\|\tilde{u}\|^2_{L^2_0(\Omega)} + \frac{\nu\lambda_1}{2}\|\tilde{u}\|^2_{L^2_0(\Omega)} + \beta(t)\|\tilde{u}_2\|^2_{L^2(\Omega)} \leq 0,$$

  with

$$\beta(t) := \mu - \frac{c}{\nu}\|\nabla u\|^2_{L^2_0(\Omega)} \log\left(\frac{c}{\nu^2\lambda_1}\|\nabla u\|^2_{L^2_0(\Omega)}\right).$$

- Using the estimates on the solution of the 2D NSE, there exists $t_0 > 0$ and $\tau > 0$ such that

$$\liminf_{t\to\infty}\int_t^{t+\tau}\beta(s)\,ds \geq \frac{\mu}{2} > 0, \quad \text{and} \quad \limsup_{t\to\infty}\int_t^{t+\tau}\beta(s)\,ds \leq \frac{3\mu}{2} < \infty.$$

- Using the general lemma:

Lemma

Let $\tau > 0$ be arbitrary but fixed. Suppose that $Y(t)$ is an absolutely continuous function which is locally integrable and that it satisfies the following:

$$\frac{dY}{dt} + \beta(t)\,Y \le 0, \qquad \text{a.e. on } (0,\infty),$$

and

$$\liminf_{t\to\infty} \int_t^{t+\tau} \beta(s)\,ds \ge \gamma, \qquad \limsup_{t\to\infty} \int_t^{t+\tau} \beta^-(s)\,ds < \infty,$$

for some $\gamma > 0$, where $\beta^- = \max\{\beta, 0\}$. Then, $Y(t) \to 0$ at an exponential rate, as $t \to \infty$.

we can conclude that $\|\tilde{u}\|_{L^2_0(\Omega)} \to 0$, at an exponential rate as $t \to \infty$.

Thank You!