Regularity of the free boundary in problems with distributed sources.

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Lesson 1

Outline

- *Introduction and examples*
- *One phase problems. Viscosity solutions*
- *Statement of the theorems: "flat implies smooth", "Lipschitz implies flat". Proof of Lipschitz implies flat.*

1.1. Introduction and examples

In these series of lectures we shall consider two typical model free boundary problems. The first one is a so called *one phase* problem, whose formulation is as follows.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we look for a *nonnegative* function u satisfying the system

$$\begin{cases} \Delta u = f & \text{in } \Omega^+(u) = \{x \in \Omega : u(x) > 0\} \\ |\nabla u| = g & \text{on } F(u) = \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{1}$$

As one can see, other than u also the set $F(u)$, called the *free boundary*, is an unknown, actually, quite often, **the unknown** and indeed we shall focus on its regularity properties.

A typical example comes from classical hydrodynamics. A travelling two-dimensional gravity wave moves with constant speed on the surface of an incompressible, inviscid, heavy fluid. The bottom is horizontal. With respect to a reference domain moving with the wave speed, the motion is steady and occupies a fixed region Ω, delimited from above by an unknown free line S, representing the wave profile.

Since the flow is incompressible, the velocity can be expressed by the gradient of a *stream function* ψ. Under suitable assumpions on the flow speed, ψ and the *vorticity*,$\omega = \Delta\psi$ are functionally dependent. Assuming furthermore that the bottom and S are *streamlines,* from Bernoully law on S, we derive the following model:

$$\begin{array}{ll} 0 \le \psi \le B & \text{in } \bar{\Omega} \\ \Delta\psi = -\gamma(\psi) & \text{in } \Omega = \{0 < \psi < B\} \\ \psi = B & \text{on } y = 0 \\ |\nabla\psi|^2 + 2gy = Q, \quad \psi = 0 & \text{on } S. \end{array}$$

Here Q is constant, B, g are positive constants and $\gamma : [0, B] \to \mathbb{R}$, called *vorticity function.*

The problem is to find S such that there exists a function ψ satisfying the above system.

Several papers have been recently devoted to solve this problem. Of particular interest is the proof of the so called *Stokes conjecture,* according to which at points where the gradient vanishes (*stagnation points*) the wave profile presents a $120°$ corner. Away from stagnation points the free boundary is Lipschitz and moreover $Q - 2gy > 0$. We refer to [V] and the reference therein, for more details and known results. Among the various problems left open there was the regularity of S away from stagnation points. The answer is given in [D], where the author shows that in this regions S is a smooth curve.

The second model is a two phase problem:

$$\begin{cases} \Delta u = f, & \text{in } \Omega^+(u) \cup \Omega^-(u) \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 & \text{on } F(u)\,. \end{cases} \tag{2}$$

Here

$$\Omega^+(u) = \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) = \{x \in \Omega : u(x) \le 0\}^\circ,$$

and u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively.

A significant eample in 2-d is the so called Prantl-Batchelor flow. A bounded domain is delimited by two simple cloded curves γ, Γ. Let Ω_1, Ω_2 be as in figure below.

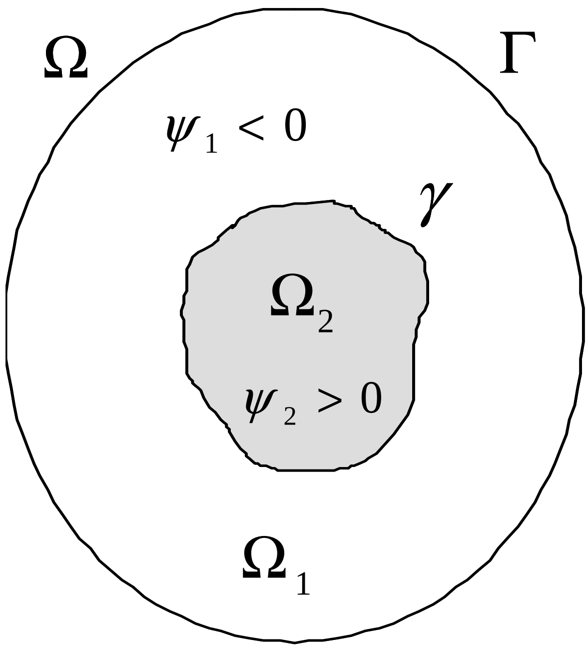

FIGURE 1.

For given constant $\mu < 0, \omega > 0$, consider functions ψ_1, ψ_1 satisfying

$$\Delta\psi_1 = 0 \ \text{ in } \Omega_1, \quad \psi_1 = 0 \text{ on } \gamma, \ \psi_1 = \mu \text{ on } \Gamma,$$

$$\Delta\psi_2 = \omega \ \text{ in } \Omega_2, \quad \psi_2 = 0 \text{ on } \gamma.$$

The two functions ψ_1, ψ_1 are interpreted as stream functions of an irrotational flow in Ω_1 and of a constant vorticity flow in Ω_2. In the model proposed by Batchelor, coming by limit of large Reynold number in the steady Navier-Stokes equation, is hypothesized a flow of this type in which there is a jump in the tangential velocity along γ, namely

$$|\nabla\psi_1|^2 - |\nabla\psi_1|^2 = \sigma$$

for some positive constant. In this problem γ is to be determined and plays the role of a free boundary.

There is no satisfactory theory for this problem. Viscosity solutions (see Lesson 4) are Lipschitz across γ as shown in [CJK], but neither existence nor regularity is known (uniqueness fails already in the radial case, where two explicit solution can be found).

Here we shall prove that flat or Lipschitz free boundaries are smooth (see [DFS1]).

Similar problems comes from singular perturbation problems with forcing terms in flame propagation theory (see [LW]) or from magnetohydrodysamics as in [FL].

The homogeneous case $f \equiv 0$ was settled in the classical works of Caffarelli [C1,C2]. A key step in these papers is the construction of a family of continuous supconvolution deformations that act as comparison subsolutions.

The results in [C1,C2] have been widely generalized to different classes of homogeneous elliptic problems.

In [D], De Silva introduced a new strategy to investigate inhomogeneous free boundary problems, motivated by a classical one phase problem in hydrodynamic. The first three lessons are devoted to the description of this technique.

In the last three lessons we extend this technique to two phase problems, describing the results in [DFS1]. Actually, in this paper, general second order uniformly elliptic linear operators with Hoelder coefficients operators are considered, with more general free boundary conditions. For the extension to fully nonlinear operators, see [DFS2].

1.2. One phase problems. Viscosity solutions

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we examine the following one phase problem:

$$\begin{cases} \Delta u = f & \text{in } \Omega^+(u) = \{x \in \Omega : u(x) > 0\} \\ \\ |\nabla u| = g & \text{on } F(u) = \partial\Omega^+(u) \cap \Omega \end{cases} \tag{3}$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C^{0,\beta}(\Omega), g \geq 0$.

By a classical subsolution (resp. super solution) of (3) we mean a function v such that $v \in C^2(\Omega)$, $\Delta v \geq f$ (resp. $\leq$) in $\Omega^+(v)$ and $|\nabla v| \geq g$ (resp. $|\nabla v| \leq g$) on $F(v)$, with $|\nabla v| > 0$.

Strict inequalities correspond to *strict* sub and supersolutions. Note that $v \in C^2(\Omega)$ but *only its positive part* plays a role on $F(v)$.

Viscosity sub/super solutions are defined in the usual way. Given $u, \varphi \in C(\Omega)$, we say that φ *touches* u *by below* (resp. *above*) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly by below (resp. above).

Definition 1.1 $u \in C(\Omega)$, $u \geq 0$ in Ω, is a viscosity *subsolution (supersolution)* if the following conditions are satisfied:

i) if $\varphi \in C^2(\Omega)$ and φ touches u by above (below) at $x_0 \in \Omega^+(u)$ then $\Delta\varphi(x_0) \geq f(x_0)$ $(\leq)$.

ii) if $\varphi \in C^2(\Omega)$ and φ^+ touches u by above/below at $x_0 \in F(u)$ with $|\nabla\varphi(x_0)| > 0$, then

$$|\nabla\varphi(x_0)| \geq g(x_0) \quad ((\leq)).$$

We say that u is a viscosity solution if it is both a sub and a supersolution.

Notice that if v is a strict (classical) subsolution and $v^+ \leq u$ in Ω, then they cannot touch neither in $\Omega^+(v)$ nor on $F(v)$, therefore:

Lemma 1.1. *Let u, φ be a solution and a strict classical subsolution, respectively. If $u \geq \varphi^+$ in Ω then $u > \varphi^+$ in $\Omega^+(\varphi) \cup F(\varphi)$.*

1.3 Statement of the main theorems. Proof of Lipschitz implies $C^{1,\alpha}$

The flatness condition we impose is that the graph of u in B_1 is trapped between two hyperplane at distance ε: for some unit vector ν,

$$(x \cdot \nu - \varepsilon)^+ \leq u(x) \leq (x \cdot \nu + \varepsilon)^+ \qquad \text{in } B_1. \tag{4}$$

If (4) holds, we say that the graph of u is $\varepsilon-$*flat* in B_1 *in the direction* ν. The main theorems are the following two (see [D]). A constant depending only on n, $\|f\|_{L^\infty(B_1)}$ and on the Hölder norm of g is called *universal.*

Theorem 1.2 (Flatness implies $C^{1,\alpha}$). *Let u be a viscosity solution of our f.b.p in B_1. Assume that $0 \in F(u)$, $g(0) = 1$. Then, there is a universal constant $\bar{\varepsilon} > 0$ such that if the graph of u is $\bar{\varepsilon}-$flat in B_1 in the direction e_n and*

$$\|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \; [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

As a consequence of Theorem 1.2 we prove that Lipschitz free boundaries are $C^{1,\alpha}$. Namely:

Theorem 1.3 (Lipschitz implies $C^{1,\alpha}$). *Let u be a viscosity solution of our f.b.p in B_1. Assume that $0 \in F(u)$, $g(0) = 1$. If $F(u)$ is a Lipschitz graph in B_1 then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.*

Proof of Theorem 1.3.

For the proof of Theorem 1.3 we need the following consequence of [C1]. Let $u_0 \geq 0$ be a global Lipschitz solution of

$$\begin{cases} \Delta u_0 = 0 & \text{in } \{x \in \mathbb{R}^n : u_0(x) > 0\} \\ |\nabla u_0| = 1 & \text{on } F(u_0). \end{cases} \tag{5}$$

If $F(u_0)$ is a (global) Lipschitz graph, then up to a rotation, $u_0(x) = x_n^+$.

Lemma 1.4 (Lipschitz continuity and nondegeneracy). *Let u be as in Theorem 1.3. Assume that*

$$\|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \; \|g - 1\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \tag{6}$$

for $\bar{\varepsilon}$ small, universal. Then

$$c_0 d(x) \leq u(x) \leq C_0 d(x) \qquad \forall x \in B^+_{1/2}(u)$$

with $d(x) =$dist$(x, F(u))$, c_0, C_0 positive, universal.

Proof. Let $x_0 \in B^+_{1/2}(u)$, $d_0 = d(x_0)$ and $y_0 \in F(u)$ such that $d(x_0) = |x_0 = y_0|$. Set

$$w(z) = \frac{u(d_0(z + x_0))}{d_0} \qquad z \in B_{1/d_0}(0).$$

Then w satisfies (3) with right hand side $\tilde{f}(z) = d_0 f(d_0(z + x_0))$ and free boundary condition $|\nabla w(z)| = \tilde{g}(z) = g(d_0(z + x_0))$.

We show that

$$c_0 \le w(0) \le C_0.$$

Suppose $w(0) > C_0$, with C_0 to be chosen.

By Harnack inequality, in $B_{1/2}$ we get

$$w(z) \ge c\left\{w(0) - c_1 \|f\|_{L^\infty(B_1)}\right\} \ge c\{w(0) - c_1\bar{\varepsilon}\} \ge C_1 w(0).$$

Define

$$G(z) = C\left(|z|^{-\gamma} - 1\right).$$

We have $G = 0$ on ∂B_1 and we choose C such that $G = 1$ on $\partial B_{1/2}$.

In the anulus $A = B_1 \backslash \bar{B}_{1/2}$ we have

$$\Delta G(z) = C|z|^{-\gamma-2}\{-\gamma n + \gamma(\gamma + 2)\} \ge \bar{\varepsilon}$$

if γ is large enough.

Let $v(z) \equiv C_1 w(0) G(z)$. We have $(w(0) > C_0)$

$$\Delta v(z) \ge C_1 w(0)\bar{\varepsilon} > \tilde{f}(z).$$

Then the maximum principle gives

$$w(z) \ge v(z) \equiv C_1 w(0) G(z) \qquad \text{in } A$$

At the point $z_0 = \partial B_1 \cap F(w)$ corresponding to y_0, both w and v vanish. Since $\Delta v(z) \ge C_1 w(0)\bar{\varepsilon} > \tilde{f}(z)$ we must have at z a *supersolution condition,* or

$$\gamma C C_1 w(0) = |\nabla v(z_0)| \le \tilde{g}(z_0) \le 1 + \bar{\varepsilon} < 2$$

which contradicts $w(0) > C_0$ if C_0 is large enough. This proves the upper bound.

To prove the lower bound, let

$$G_0(z) = \eta(1 - G(z))$$

and choose η to make G_0 a *strict supersolution* on $\partial B_{1/2}$, precisely

$$|\nabla G_0| < 1 - \bar{\varepsilon}.$$

We may assume that $F(u) = \{x_n = \psi(x')\}$ and that $\mathrm{Lip}(\psi) \le 1$. Then, in the rescaled situation, we have: $w \equiv 0$ in $B_1(-e_n)$. Thus $G_0(z + e_n) \ge w(z)$ there.

Slide the graph of $G_0(z + e_n)$ along e_n until it touches the graph of w (from above). Since G_0 is a strict supersolution to our f.b.p., the touching point $\bar{z}$ can only occur at the level η in the positive phase of u and $|\bar{z}| \le C(L)$.

Note that $\bar{d} = \mathrm{dist}(\bar{z}, F(w)) \le 1$. On the other hand, since w is Lipschitz continuous, we have

$$w(\bar{z}) = \eta \le C\bar{d}$$

so that

$$\bar{d} \sim 1.$$

Since $F(w)$ is Lipschitz, we can costruct a Harnack chain with balls of radius comparable to 1, connecting 0 and $\bar{z}$.

Harnack inequality gives, for $\bar{\varepsilon}$ small, $w(0) \geq c\eta \equiv c_0$. $\square$

We are now ready to prove Theorem 1.3. Consider the blow-up sequence

$$u_k(x) = \frac{u(\rho_k x)}{\rho_k}$$

with $\rho_k \to 0$ as $k \to \infty$. Each u_k is a solution of our f.b.p. with $f_k(x) = \rho_k f(\rho_k x)$ and $g_k = g_k(\rho_k x)$. For k large, in B_1 we have

$$\|f_k\|_{L^\infty(B_1)} \leq \rho_k \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}$$

and

$$|g_k(x) - 1| = |g(\rho_k x) - g(0)| \leq \rho_k^\beta [g]_{0,\beta} \leq \bar{\varepsilon}$$

so that (6) are satisfied.

From Lemma 1.4 (up to passing to a subsequence) we deduce that.

1. $u_k \to u_0$ in $C^{0,\alpha}_{loc}(\mathbb{R}^n)$ for all $0 < \alpha < 1$ (by uniform Lipschitz continuity);
2. $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ locally in Haussdorf distance[1] (by nondegeneracy).[2]

Now, u_0 is a global Lipschitz solution of (5) and $F(u_0)$ is a global Lipschitz graph.

We infer that, up to a rotation, $u_0(x) = x_n^+$. This implies that, say, in $B_{1/2}$, for k large enough, u_k is $\bar{\varepsilon}$ flat in the e_n direction.

The conclusion follows from Theorem 1.2 $\square$

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[1] If K_1, K_2 are two compact sets, their Hausdorff distance is defined by

$$d^H(K_1, K_2) = \inf\{\alpha > 0, K_1 \subset N_\alpha(K_2) \text{ and } K_2 \subset N_\alpha(K_1)\}$$

where

$$N_\alpha(K) = \{x \in \mathbb{R}^n; d(x, K) \leq \alpha\}.$$

Equivalently,

$$d^H(K_1, K_2) = \|d(x, K_1) - d(x, K_2)\|_{L^\infty(\mathbb{R}^n)}.$$

[2] Suppose that a ball $\bar{B}_r$ does not intersect $F(u_0)$. Then either $u_0 > 0$ or $u \equiv 0$ in $\bar{B}_r$. In the first case, $u_k > 0$ in $\bar{B}_r$ so that $\bar{B}_{r/2}$ does not intersects $F(u_k)$ for k large. In the second case, $u_k^+ \leq \sigma$ for any chosen $\sigma > 0$, if k is large. Thus, by nondegeneracy, $\bar{B}_{r/2}$ still does not intersects $F(u_k)$ for k large.

Conversely, if $\bar{B}_r$ does not intersect $F(u_k)$ for any large k, then either $u_k > 0$ or $u_k \equiv 0$ in $\bar{B}_r$.In the first case, u_k is harmonic in B_r so that u_0 is harmonic too. Hence, $u_0 > 0$ or $u_0 \equiv 0$ in B_r and B_r does not intersects $F(u_0)$. In the second case, $u_0 \equiv 0$ in B_r and again B_r does not intersects $F(u_0)$.

Lesson 2

Outline

- *Flat implies smooth. De Silva strategy*
- *The basic Harnack inequality*

2.1 Flat implies smooth. De Silva strategy

The proof of Theorem 1.2 goes along 3 main steps.

1. Consider the normalized function

$$\tilde{u}_{\varepsilon}(x) = \frac{u(x) - x_n}{\varepsilon} \qquad \varepsilon \leq \bar{\varepsilon}$$

   and prove a *Harnack inequality* implying that $\tilde{u}_{\varepsilon}$ has a uniform Holder modulus of continuity at each point $x_0 \in \Omega^+(u) \cup F(u)$ outside a ball $B_{\varepsilon/\bar{\varepsilon}}(x_0)$.

2. A basic geometric improvement of flatness, from

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \qquad \text{in } B_1$$

   to

$$(x \cdot \nu - r\varepsilon/2)^+ \leq u(x) \leq (x \cdot \nu + r\varepsilon/2)^+ \qquad \text{in } B_r \tag{7}$$

   for $r \leq r_0$, universal, $\varepsilon \leq \varepsilon_0(r)$ and moreover

$$|\nu - e_n| \leq C\varepsilon.$$

   In this step, a contradiction argument leads to a sequence of normalized $\tilde{u}_{\varepsilon_k}$ converging, locally uniformly thanks to step 1, to a solution $\tilde{u}$ of a *Neumann problem* in a half ball, which is, in practice, a linearization of the original f.b.p.. The regularity properties of $\tilde{u}$ are transfered to $\tilde{u}_{\varepsilon_k}$ for k large, giving a contradiction.

3. Iteration of step 2 gives, for $r = \bar{r}$, suitably chosen and $\varepsilon_k = 2^{-k}\varepsilon_0(\bar{r})$,

$$\left(x \cdot \nu_k - \bar{r}^k \varepsilon_k\right)^+ \leq u(x) \leq \left(x \cdot \nu_k + \bar{r}^k \varepsilon_k\right)^+ \qquad \text{in } B_{\bar{r}^k}$$

   with $|\nu_{k+1} - \nu_k| \leq C\varepsilon_k$. This implies that $F(u)$ is $C^{1,\alpha}$ at the origin. Repeating the procedure for points in a neighborhood of $x = 0$, since all estimates are universal, we conclude that there exists a unit vector ν_∞ and $C > 0$, $\alpha \in (0, 1)$ both universal, such that, in the coordinate system $e_1, ..., e_{n-1}, \nu_\infty,$, $\nu_\infty \perp e_j$, $e_j \cdot e_k = \delta_{jk}$, $F(u)$ is a graph, $C^{1,\alpha}$ graph, say $x_n = f(x')$, with $f(0') = 0$ and

$$|f(x') - \nu_\infty \cdot x'| \leq C|x'|^{1+\alpha}$$

   in a neighborhood of $x = 0$.

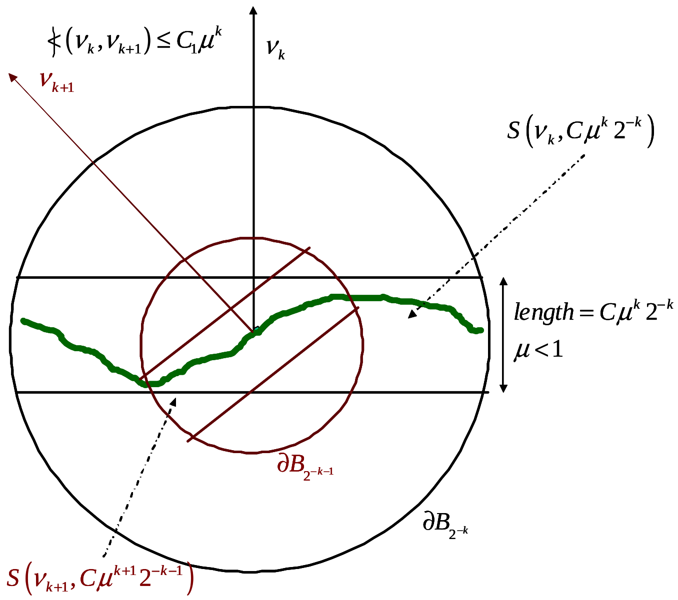

FIGURE 2. Improvement of flatness

2.2 The basic Harnack inequality

Theorem 2.1 (Harnack inequality). *Let u be a viscosity solution of our f.b.p in Ω. There exists a universal $\bar{\varepsilon}$ such that if $\varepsilon \leq \bar{\varepsilon}$,*

$$\|f\|_{L^\infty(\Omega)} \leq \varepsilon^2, \|g-1\|_{L^\infty(\Omega)} \leq \varepsilon^2,$$

and, for some $x_0 \in \Omega^+(u) \cup F(u)$,

$$(x_n + a_0)^+ \leq u(x) \leq (x_n + b_0)^+ \qquad \text{in } B_r(x_0) \subset \Omega \tag{8}$$

with

$$0 < b_0 - a_0 \leq \varepsilon r,$$

then

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \qquad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \qquad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$, universal.

Corollary 2.2. Let $r = 1$ in (8). *Then, the function*

$$\tilde{u}_\varepsilon(x) = \frac{u(x) - x_n}{\varepsilon} \qquad \varepsilon \leq \bar{\varepsilon}$$

satisfies

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma$$

for all $x \in B_1(x_0) \cap [\Omega^+(u) \cup F(u)]$ such that $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$.

Proof. From Theorem 2.1, we have

$$(x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \qquad \text{in } B_{1/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \qquad b_1 - a_1 \leq (1-c)\varepsilon.$$

We reapply Theorem 2.1, with $r = 1/20$. To do this we must have

$$b_1 - a_1 \leq \varepsilon'/20 \qquad \varepsilon' \leq \bar{\varepsilon}.$$

We have

$$b_1 - a_1 \leq 20(1-c)\varepsilon/20 \equiv \varepsilon'/20$$

and we require

$$\varepsilon' = 20(1-c)\varepsilon \leq \bar{\varepsilon}.$$

Theorem 2.1 gives

$$(x_n + a_2)^+ \leq u(x) \leq (x_n + b_2)^+ \qquad \text{in } B_{1/20^2}(x_0)$$

with

$$a_0 \leq a_1 \leq a_2 \leq b_2 \leq b_1 \leq b_0,$$

and

$$b_2 - a_2 \leq (1-c)\varepsilon'/20 = (1-c)^2\varepsilon.$$

Iterating, we get

$$(x_n + a_m)^+ \leq u(x) \leq (x_n + b_m)^+ \qquad \text{in } B_{1/20^m}(x_0) \tag{9}$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon$$

as long as

$$20^m (1-c)^m \varepsilon \leq \bar{\varepsilon}$$

or

$$20^{-m}(1-c)^{-m} \geq \frac{\varepsilon}{\bar{\varepsilon}}.$$

Set $|x - x_0|^m \sim r_m = 20^{-m}$ and $(1-c) = 20^{-\gamma}$. Then we deduce that

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq (1-c)^m = r_m^\gamma \leq C|x - x_0|^\gamma$$

as long as $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$. □

The proof of Harnack inequality relies on the following Lemma, which states that a pointwise gain in flatness away from $F(u)$ gives a little less gain, uniformly in half ball $B_{1/2}$.

Lemma 2.3. *Let u be a viscosity solution of our f.b.p in B_1. Set*

$$p(x) = x_n + \sigma, \; |\sigma| \leq \tfrac{1}{10}, \; \text{ and } \; \bar{x} = \tfrac{1}{5}e_n.$$

Assume that

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2, \|g-1\|_{L^\infty(B_1)} \leq \varepsilon^2.$$

There exists a universal $\bar{\varepsilon}$ such that if $\varepsilon \leq \bar{\varepsilon}$,

$$p(x)^+ \leq u(x) \leq (p(x) + \varepsilon)^+ \qquad \text{in } B_1 \tag{10}$$

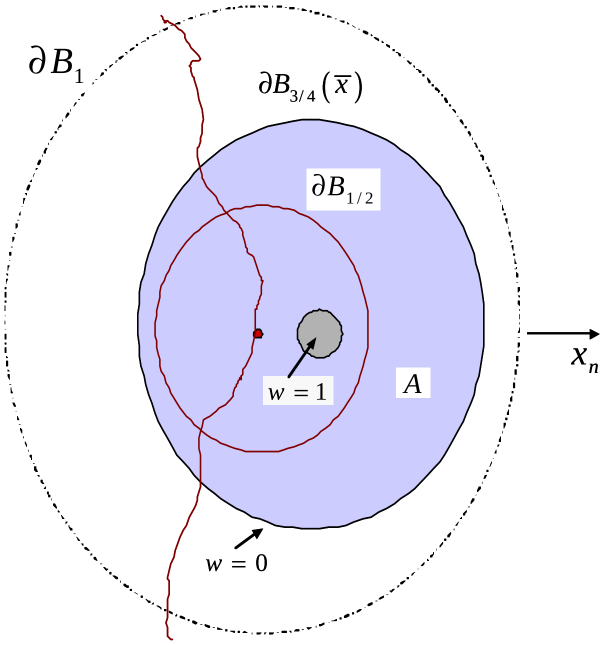

FIGURE 3.

and

$$u(\bar{x}) \geq (p(\bar{x}) + \varepsilon/2)^+ \qquad (\text{resp. } \leq) \tag{11}$$

then, in $B_{1/2}$,

$$u(x) \geq (p(x) + c\varepsilon)^+ \qquad (\text{resp. } u(x) \leq (p(x) + (1-c)\varepsilon)^+)$$

for some universal $1 < c < 1$.

Proof. First we show that the interior gain (11) propagates into a neighborhood of $\bar{x}$. Clearly we have $u \geq p$ in B_1. Let

$$A = B_{3/4}(\bar{x}) \setminus B_{1/20}(\bar{x}).$$

Note that, since $|\sigma| \leq 1/10$ we have (see figure 2 below)

$$B_{1/10}(\bar{x}) \subset B_1^+.$$

Also

$$B_{1/2} \subset\subset B_{3/4}(\bar{x}) \subset\subset B_1.$$

Define

$$w(x) = c\left[|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}\right] \qquad \text{in } A$$

and

$$w \equiv 1 \text{ in } B_{1/20}(\bar{x}),$$

with the constant c chosen such that $w = 1$ on $\partial B_{1/20}(\bar{x})$ and γ (large) so that

$$\Delta w \geq \delta > 0, \quad \delta \text{ universal}. \tag{12}$$

Note that $w \leq 1$ in A.

By Harnack inequality in $B_{1/10}(\bar{x})$, we get

$$u(x) - p(x) \geq c_1 (u(\bar{x}) - p(\bar{x})) - c_2 \|f\|_{L^\infty(B_{1/10})} \qquad \text{in } \bar{B}_{1/20}(\bar{x}).$$

Thus

$$u(x) - p(x) \geq \frac{c_1}{2}\varepsilon - c_2\varepsilon^2 \geq c_0\varepsilon \qquad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{13}$$

To propagate this gain up to $F(u)$ we construct a family of subsolutions.

Set, for $t \geq 0$, $x \in B_{3/4}(\bar{x})$

$$v_t(x) = p(x) + c_0\varepsilon(w-1) + t.$$

Observe that

$$\Delta v_t \geq c_0\varepsilon\delta \geq \varepsilon^2 \geq f \qquad \text{in } A.$$

Moreover, in $B_{3/4}(\bar{x})$,

$$v_0 \leq p \leq u.$$

Thus we can define $\bar{t}$ the largest $t > 0$ such that

$$v_t \leq u \quad \text{in } B_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0\varepsilon$. Then, in $B_{1/2}$

$$\begin{aligned} u(x) &\geq p(x) + c_0\varepsilon(w-1) + \bar{t} \\ &\geq p(x) + c_0\varepsilon w \\ &\geq p(x) + c\varepsilon \end{aligned}$$

since there $w \geq C > 0$, C universal, and we have done.

Suppose $\bar{t} < c_0\varepsilon$ and let $x^* \in B_{3/4}(\bar{x})$ such that

$$v_{\bar{t}}(x^*) = u(x^*).$$

Claim: $x^* \in \bar{B}_{1/20}(\bar{x})$. Indeed, since $w = 0$ on $\partial B_{3/4}(\bar{x})$, we deduce

$$v_{\bar{t}} < p \leq u \qquad \text{on } \partial B_{3/4}(\bar{x}).$$

Inside A we have

$$\Delta v_t \geq f$$

and also

$$|\nabla v_{\bar{t}}| \geq |D_n v_{\bar{t}}| = |1 + c_0\varepsilon D_n w|.$$

We want to show that $v_{\bar{t}}$ is a strict subsolution in A. For this we have to prove that

$$|\nabla v_{\bar{t}}| \geq g \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Observe that

$$\begin{aligned} \{v_{\bar{t}} \leq 0\} \cap A &= \{p(x) + c_0\varepsilon(w-1) + \bar{t} \leq 0\} \cap A \\ &\subset \{p(x) - c_0\varepsilon \leq 0\} \cap A = \{x_n \leq -\sigma + c_0\varepsilon\} \cap A \\ &\subset \{x_n < 3/20\} \end{aligned}$$

so that $\bar{B}_{1/20} \cap \{v_{\bar{t}} \leq 0\} = \varnothing$.

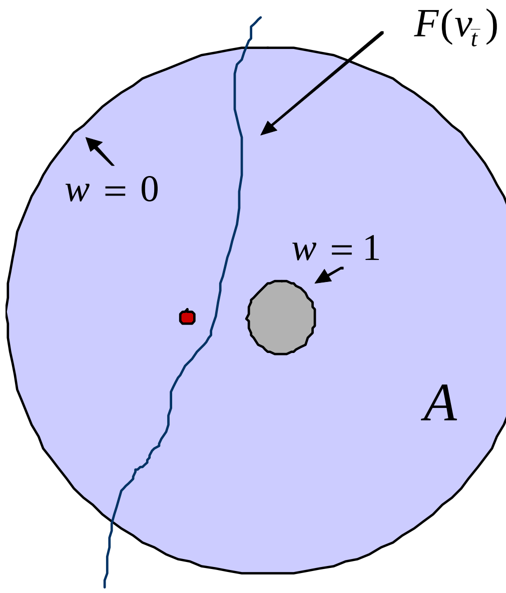

FIGURE 4.

This implies that

$$\nu_x \cdot e_n \equiv \frac{x-\bar{x}}{|x-\bar{x}|} \cdot e_n \geq c > 0 \qquad \text{on } \{v_{\bar{t}} \leq 0\} \cap A.$$

In particular, since $|\nabla w| \geq C > 0$ in A, we get

$$D_n w = \nabla w \cdot e_n = |\nabla w| \left(\nu_x \cdot e_n\right) \geq c_1 > 0 \text{ on } F\left(v_{\bar{t}}\right) \cap A.$$

Thus,

$$|\nabla v_{\bar{t}}| = |1 + c_0 \varepsilon D_n w| \geq 1 + c_2 \varepsilon \geq g \quad \text{ on } F\left(v_{\bar{t}}\right) \cap A$$

and $v_{\bar{t}}$ is a strict subsolution in A. Therefore $x^* \in \bar{B}_{1/20}\left(\bar{x}\right)$ and

$$u\left(x^*\right) = v_{\bar{t}}\left(x^*\right) = p\left(x^*\right) + \bar{t} \leq p\left(x^*\right) + c_0 \varepsilon$$

in contradiction with (13). □

Proof of Theorem 2.1. Let $x_0 = 0 \in \Omega^+\left(u\right) \cup F\left(u\right)$, $r = 1$. From (8) we have

$$p\left(x\right)^+ \leq u\left(x\right) \leq \left(p\left(x\right) + \varepsilon\right)^+ \qquad \text{in } B_1 \tag{14}$$

with $p\left(x\right) = x_n + a_0$. We distinguish 3 cases.

1. $|a_0| < 1/10$. Then the result follows directly from Lemma 4.

2. $a_0 > 1/10$. Then $u \geq x_n + a_0 > x_n + 1/10$ implies that $B_{1/10} \subset B_1^+\left(u\right)$ and the conclusion follows from interior Harnack inequality.

3. $a_0 < -1/10$. Then $u \leq x_n + a_0 + \varepsilon < \left(x_n - 1/10 + \varepsilon\right)^+$ implies that (ε small) $u = 0$ in a neighborhood of $x = 0$. Contradiction.

Lesson 3

Outline

- *A Neumann problem*
- *Improvement of flatness*
- *The final iteration*

3.1 A Neumann problem

We need to show that viscosity solutions of the problem

$$\begin{cases} \Delta v = 0 & \text{in } B_r \cap \{x_n > 0\} \\ v_n = 0 & \text{on } B_r \cap \{x_n = 0\} \end{cases} \tag{15}$$

are indeed classical. We recall the notion of viscosity solution.

Definition 3.1. We say that v, continuous in $B_r \cap \{x_n \geq 0\}$, is a viscosity solution of (15) if for every quadratic polynomial P touching v by below (resp. above) at $x^* \in B_r \cap \{x_n \geq 0\}$ we have:

$a)$ if $x^* \in B_r \cap \{x_n > 0\}$, then $\Delta P \leq 0$ (resp. $\Delta P \geq 0$) ;

$b)$ if $x^* \in B_r \cap \{x_n = 0\}$ then $P_n(x^*) \leq 0$ (resp. $P_n \geq 0$) .

Remarks 3.1.

1) It is enough to consider polynomials P touching v *strictly* by below or above. If not, one replaces P by $P_\eta(x) = P(x) \mp \eta (x_n - x_n^*)^2$, $\eta > 0$.

2) In the condition $b)$ it is enough to consider polynomials P with $\Delta P > 0$ (resp. $\Delta P < 0$). Indeed, assume P touches by *below* v at $x^* \in B_r \cap \{x_n = 0\}$. Consider

$$P^*(x) = P(x) - \eta(x_n - x_n^*) + C(\eta)(x_n - x_n^*)^2.$$

for $\eta > 0$. Then P^* touches by below and, if $C(\eta) > 0$ is large,

$$\Delta P^* > 0, \ P_n^*(x^*) = P_n(x^*) - \eta.$$

If $b)$ holds for strictly subharmonic polynomials, then $P_n(x^*) \leq \eta$. Letting $\eta \to 0$ we recover $P_n(x^*) \leq 0$.

Lemma 3.1. *Let v be a viscosity solution of problem (15), then $v \in C^\infty(B_r \cap \{x_n \geq 0\})$.*

Proof. Reflect v in an even way across $x_n = 0$, defining

$$v^*(x) = v(x) \text{ for } x_n \geq 0, \ v^*(x) = v(x', -x_n) \text{ for } x_n < 0.$$

We show that v^* is harmonic in B_r, in the viscosity sense. Since viscosity harmonic functions are harmonic in the classical sense, it follows that v^* is smooth in B_r.

Thus, let P be a quadratic polynomial touching v^* strictly by below at $x^* \in B_r$. We must show that $\Delta P \leq 0$.

It is clearly enough to consider $x^* \in \{x_n = 0\}$. Define

$$S(x) = \frac{P(x) + P(x', -x_n)}{2}.$$

Then S touches strictly by below v^* at x^* and

$$\Delta S = \Delta P, \ S_n(x', 0) = 0.$$

For $\varepsilon > 0$, let

$$S^\varepsilon(x) = S(x) + \varepsilon x_n + t.$$

If ε and t are small, S^ε touches v^* by below at some point x_ε.

Since $v_n(x', 0) = 0$ in the viscosity sense and

$$S_n^\varepsilon(x_\varepsilon) = \varepsilon > 0$$

we deduce that $x_\varepsilon \in B_r \backslash \{x_n = 0\}$. Therefore $\Delta P = \Delta S \leq 0$.

3.2 Improvement of flatness

The key lemma is the following.

Lemma 3.2 (Improvement of flatness). *Assume that*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2, \|g - 1\|_{L^\infty(B_1)} \leq \varepsilon^2 \tag{16}$$

and

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \qquad \text{in } B_1. \tag{17}$$

There exists a (universal) r_0 such that, if $r \leq r_0$ and $\varepsilon \leq \varepsilon_0$, for some $\varepsilon_0(r)$, then we have

$$\left(x \cdot \nu - \tfrac{1}{2}\varepsilon r\right)^+ \leq u(x) \leq \left(x \cdot \nu + \tfrac{1}{2}\varepsilon r\right)^+ \qquad \text{in } B_r \tag{18}$$

for a suitable unit vector ν, with $|e_n - \nu| \leq c\varepsilon$, c universal.

Remark 3.2. The number r_0 will determine the rescaling parameter $\bar{r}$ in the final iteration. In turn, $\bar{\varepsilon} = \varepsilon_0(\bar{r})^2$, in order to insure the hypotheses (16).

Proof. We split it into 3 main steps. Introduce the notation:

$$\Omega_\rho(u) = \left[B_1^+(u) \cup F(u)\right] \cap B_\rho.$$

Step 1. *Compactness.* Fix r_0 universal (it will be chosen in **step 3**) and $r \leq r_0$. Assume that the theorem is not true. Then we can find a sequence $\varepsilon_k \to 0$ and a sequence of solutions u_k of our f.b.p. in B_1 with r.h.s. f_k and f.b. term g_k,

$$\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2, \|g_k - 1\|_{L^\infty(B_1)} \leq \varepsilon_k^2$$

such that $0 \in F(u_k)$,

$$(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \qquad \text{in } B_1, \tag{19}$$

but (18) is not true.

Consider the normalization

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k} \qquad x \in \Omega_1(u_k).$$

Then

$$|\tilde{u}_k| \le 1 \qquad \text{in } \Omega_1(u_k).$$

From corollary 2.2 we get

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \le C|x-y|^\gamma$$

for

$$|x-y| \ge \frac{\varepsilon_k}{\bar{\varepsilon}} \qquad \text{in } \Omega_{1/2}(u_k)$$

where $\bar{\varepsilon}$ is defined in Theorem 2.1.

From (19) it follows that $F(u_k)$ converges in Hausdorff distance to $B_1 \cap \{x_n = 0\}$. Then, using Ascoli-Arzelà Theorem, we infer that the graph of $\tilde{u}_k$ over $\Omega_{1/2}(u_k)$ converges (up to a subsequence) in Hausdorff distance to a graph of a Hölder continuous $\tilde{u}$ in $B_{1/2} \cap \{x_n \ge 0\}$.

Step 2. *The linearized (Neumann) problem.* We show that the limiting function $\tilde{u}$ satisfies in the viscosity sense the following conditions:

$$\begin{cases} \Delta\tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\} \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

We prove only the supersolution condition. The subsolution one is analogous.

Let P be a quadratic polinomial touching $\tilde{u}$ at $x^* \in B_{1/2} \cap \{x_n \ge 0\}$ *strictly by below.* We have to show that

a) if $x^* \in B_{1/2} \cap \{x_n > 0\}$ then $\Delta P \le 0$;

b) if $x^* \in B_{1/2} \cap \{x_n = 0\}$ then $P_n(x^*) \le 0$.

We have to carry the supersolution condition on the sequence u_k, on which we have information.

First, since $\tilde{u}_k \to \tilde{u}$ in the sense specified above, there exist points $x_k \in \Omega_{1/2}(u_k)$, $x_k \to x^*$, and constants $c_k \to 0$ such that

$$P(x_k) + c_k = \tilde{u}_k(x_k)$$

and

$$P(x) + c_k < \tilde{u}_k(x)$$

near x_k. In terms of u_k this says that the polynomial

$$Q_k(x) = \varepsilon_k(P(x) + c_k) + x_n.$$

touches by below u_k at x_k.

There are only two possibilities.

a) If $x^* \in B_{1/2} \cap \{x_n > 0\}$ then $x_k \in B^+_{1/2}(u_k)$ for k large and we get, since u_k is a viscosity solution,

$$\varepsilon_k \Delta P = \Delta Q_k \le f_k(x_k) \le \varepsilon_k^2$$

or

$$\Delta P \le \varepsilon_k$$

and in the limit $\Delta P \le 0$.

b) If $x^* \in B_{1/2} \cap \{x_n = 0\}$ we can assume that $\Delta P > 0$, as observed above. Then, $x_k \in F(u_k)$ for k large. In fact, if this is not true, we find a subsequence $x_{k_s} \in B^+_{1/2}(u_{k_s})$ for which, as in case a),

$$\Delta P \leq \varepsilon_{k_s}$$

in contraddiction with $\Delta P > 0$.

Thus, $x_k \in F(u_k)$ for k large. Since

$$\nabla Q_k(x) = \varepsilon_k \nabla P(x) + e_n,$$

we have, for k large, $|\nabla Q_k| > 0$. Since Q^+ touches u_k by below, we can write

$$|\nabla Q_k(x_k)| \leq g(x_k) \leq 1 + \varepsilon_k^2.$$

On the other hand, since

$$|\nabla Q_k(x_k)|^2 = |\varepsilon_k \nabla P(x_k) + e_n|^2 = \varepsilon_k^2 |\nabla P(x_k)|^2 + 2\varepsilon_k P_n(x_k) + 1$$

we get, after division by ε_k,

$$\varepsilon_k |\nabla P(x_k)|^2 + 2P_n(x_k) \leq 2\varepsilon_k + \varepsilon_k^3$$

from which $P_n(x^*) \leq 0$ as desired.

Step 3. *Basic improvement.* From step 2, $\tilde{u}$ solves the Neumann type problem and

$$|\tilde{u}| \leq 1 \quad \text{in } B_{1/2} \cap \{x_n \geq 0\}. \tag{20}$$

From the regularity of $\tilde{u}$ (Lemma 3.1) and (20) we can write, for any $r < 1/2$,

$$|\tilde{u}(x) + \nabla_{x'}\tilde{u}(0) \cdot x'| \leq C_0 r^2 \qquad \text{in } B_r \cap \{x_n \geq 0\}$$

since $\tilde{u}(0) = \tilde{u}_n(0) = 0$, with C_0 universal.

Set $\tilde{\nu}' = \nabla_{x'}\tilde{u}(0)$ and note that $|\tilde{\nu}'| \leq \tilde{C}$, $\tilde{C}$ universal. Then for k large enough, depending on r, we have

$$\tilde{\nu}' \cdot x' - C_1 r^2 \leq \tilde{u}_k(x) \leq \tilde{\nu}' \cdot x' + C_1 r^2 \qquad \text{in } \Omega_r(u_k).$$

Going back to u_k, we can write

$$\varepsilon_k \tilde{\nu}' \cdot x' + x_n - C_1 r^2 \varepsilon_k \leq u_k(x) \leq \varepsilon_k \tilde{\nu}' \cdot x' - x_n + C_1 r^2 \varepsilon_k \qquad \text{in } \Omega_r(u_k).$$

Define

$$\nu = \frac{(\varepsilon_k \tilde{\nu}', 1)}{\sqrt{\varepsilon_k^2 |\tilde{\nu}'|^2 + 1}}$$

and note that, for k large,

$$1 \leq \sqrt{\varepsilon_k^2 |\tilde{\nu}'|^2 + 1} \leq 1 + \frac{\tilde{C}^2 \varepsilon_k^2}{2}$$

$$|\nu - e_n|^2 = \frac{\varepsilon_k^2 |\tilde{\nu}'|^2 + \left(\sqrt{\varepsilon_k^2 |\tilde{\nu}'|^2 + 1} - 1\right)^2}{\varepsilon_k^2 |\tilde{\nu}'|^2 + 1} \leq C\varepsilon_k^2.$$

Then, we have

$$\nu \cdot x - \frac{\tilde{C}^2 \varepsilon_k^2}{2} r - C_1 r^2 \varepsilon_k \leq u_k(x) \leq \nu \cdot x + \frac{\tilde{C}^2 \varepsilon_k^2}{2} r + C_1 r^2 \varepsilon_k \qquad \text{in } \Omega_r(u_k).$$

We want that

$$\frac{\tilde{C}^2\varepsilon_k}{2} + C_1 r \leq \frac{1}{2}.$$

Thus, choose $r \leq r_0$ with, say, $C_1 r_0 \leq 1/4$, and k large so that $\tilde{C}^2\varepsilon_k \leq 1/2$. Then

$$\nu \cdot x - \frac{\varepsilon_k}{2} r \leq u_k(x) \leq \nu \cdot x + \frac{\varepsilon_k}{2} r \qquad \text{in } \Omega_r(u_k).$$

Since

$$(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \qquad \text{in } B_1,$$

we infer

$$\left(\nu \cdot x - \frac{\varepsilon_k}{2} r\right)^+ \leq u_k(x) \leq \left(\nu \cdot x + \frac{\varepsilon_k}{2} r\right)^+ \qquad \text{in } B_r$$

which is a contraddiction. □

3.3 Final iteration

We are now ready to prove Theorem 1.1. Consider the blow up sequence

$$u_k(x) = \frac{u(\rho_k(x))}{\rho_k} \qquad x \in B_1.$$

We have to check that the hypotheses of the improvement of flatness lemma are iteratively satisfied.

We choose $\rho_k = \bar{r}^k$, where $\bar{r}^\beta \leq 1/4$, $\bar{r} \leq r_0$, r_0 as in Lemma 3.2. Moreover, let

$$\bar{\varepsilon} = \varepsilon_0(\bar{r})^2, \ \varepsilon_k = \varepsilon_0(\bar{r})\, 2^{-k} \qquad k \geq 0$$

with $\varepsilon_0(\bar{r})$ as in lemma 3.2.

Then

$$|f_k(x)| \equiv |\rho_k f_k(\rho_k x)| \leq \bar{\varepsilon}\bar{r}^k \leq \varepsilon_k^2$$

and

$$\begin{aligned} |g(\rho_k x) - 1| &= |g(\rho_k x) - g(0)| \leq [g]_{0,\beta}\, \rho_k^\beta = \bar{\varepsilon}\bar{r}^{\beta k} \\ &\leq \varepsilon_k^2. \end{aligned}$$

Thus, for $k = 0$ the flatness assumption of Lemma 3.2 is satisfied by u_0. By induction on k we conclude that u_k is ε_k-flat in B_1 and therefore that $F(u)$ is $C^{1,\alpha}$ at the origin, for some $\alpha \in (0,1)$. Since all the estimates are uniform if we replace the origin by any point on $F(u) \cap B_{1/2}$, it follows that $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

This concludes the proof of Theorem 1.2.

Lesson 4

Outline

- *Two phase problems and their viscosity solutions. Lipschitz continuity.*
- *Main Theorems and.preliminary results.*
- *Strategy for the improvement of flatness.*

4.1 Two phase problems and their viscosity solutions. Lipschitz continuity

To better emphazize ideas and techniques we consider the model problem

$$\begin{cases} \Delta u = f, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1, & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{21}$$

where, we recall,

$$\Omega^+(u) = \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) = \{x \in \Omega : u(x) \leq 0\}^\circ,$$

and u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively.

We assume that f is bounded in Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$. Let us introduce the notion of comparison subsolution/supersolution.

Definition 4.1. *We say that $v \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (21) in Ω, if and only if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:*

1. *$\Delta v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;*
2. *If $x_0 \in F(v)$, then, at x_0 :*

$$(v_\nu^+)^2 - (v_\nu^-)^2 > 1 \quad (\text{resp. } (v_\nu^+)^2 - (v_\nu^-)^2 < 1,\ v_\nu^+(x_0) \neq 0).$$

Notice that by the implicit function theorem, according to our definition the free boundary of a comparison subsolution/supersolution is C^2.

Finally we can give the definition of viscosity solution to the problem (21).

Definition 4.2. *Let u be a continuous function in Ω. We say that u is a viscosity solution to (21) in Ω, if the following conditions are satisfied:*

1. *$\Delta u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense;*
2. *Any strict comparison subsolution v (resp. supersolution) cannot touch u by below (resp. by above) at a point $x_0 \in F(v)$ (resp. $F(u)$).*

The next result states the optimal regularity of the solution of our free boundary problem (f.b.p. in the sequel).

Theorem 4.1. *A viscosity solution of (21) in Ω is Lipschitz continuous in every compact subset of Ω.*

The proof follows from the following monotonicity formula, as in [CJK], Theorem 4.5.

Theorem 4.2. *Let u, v be nonnegative, continuous functions in B_1, with*

$$\Delta w \geq -1, \Delta v \geq -1 \text{ in the sense of distributions}$$

and $u(0) = v(0) = 0$, $u(x)\, v(x) = 0$ in B_1. Then there exists $C = C(n)$ such that

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla v|^2}{|x|^{n-2}} \leq C \left(1 + \int_{B_1} u^2\right) \left(1 + \int_{B_1} v^2\right).$$

for $r \leq 1/2$.

4.2 Main Theorems and preliminary results

We now state our main results. Here constants depending only on n, $\|f\|_\infty$, and $Lip(u)$ will be called universal. We always assume that $0 \in F(u)$.

Theorem 4.3. *Let u be a (Lipschitz) viscosity solution to our f.b.p. in B_1. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\delta_0 > 0$ such that, if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \tag{22}$$

with $0 \leq \delta \leq \delta_0$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

As in the one phase case, the following consequence holds.

Theorem 4.4. *Let u be a (Lipschitz) viscosity solution to our f.b.p. in B_1. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

The proof of Theorem 4.3 is based on an iterative procedure that "squeezes" dyadically our solution around an optimal configuration $U_\beta(x \cdot \nu)$ where $U_\beta = U_\beta(t)$ is given by

$$U_\beta(t) = \alpha t^+ - \beta t^- \quad \beta \geq 0, \alpha = \sqrt{1 + \beta^2}$$

and ν is a unit vector, which play the role of normal vector at the origin. $U_\beta(x \cdot \nu)$ is a so-called *two plane solution* when $f = 0$. Indeed the first step is to check that the flatness condition (22) implies that u is close to $U_\beta(x_n)$ for some β (see Lemma 4.9).

The above plan works nicely as long as the two phases u^+, u^- are, say, comparable (*non-degenerate case*). The difficulties arise when the negative fase becomes very small but at the same time not negligeable (*degenerate case).* In this case the flatness assumption in Theorem 4.3 gives a control of the positive phase only, through the closedness to a *one plane solution* $U_0(x_n) = x_n^+$.

As we shall see, this require to face a dycotomy in the final iteration.

Let us also state the following elementary lemma, that we give for a general continuous function and that translate "vertical" closedness between u and U_β into "horizontal" closedness, which is much more confortable for our purposes.

Lemma 4.5. *Let u be a continuous function. If, for a small $\eta > 0$,*

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \eta$$

and

$$\{x_n \le -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \eta\},$$

then:

- *If $\beta \ge \eta^{1/3}$,*
$$U_\beta\left(x_n - \eta^{1/3}\right) \le u(x) \le U_\beta\left(x_n + \eta^{1/3}\right) \qquad \text{in } B_{3/4}$$
- *If $\beta < \eta^{1/3}$,*
$$U_0\left(x_n - \eta^{1/3}\right) \le u^+(x) \le U_0\left(x_n + \eta^{1/3}\right) \qquad \text{in } B_{3/4}.$$

The proof of Theorem 4.3 is reduced to the following main Lemma.

Main Lemma 4.6. *Let u be a (Lipschitz) viscosity solution to our f.b.p. in B_1,* with *$Lip(u) \le L$. There exists a universal constant $\bar\eta > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \bar\eta \quad \text{for some } 0 \le \beta \le L, \tag{23}$$

and

$$\{x_n \le -\bar\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \le \bar\eta\},$$

and

$$\|f\|_{L^\infty(B_1)} \le \bar\eta,$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

The parameter $\bar\eta$ will be equal to $\tilde\varepsilon^3$, where $\tilde\varepsilon$ is universal, suitably chosen in the basic improvement lemma. In practice, the dichotomy *nondegenerate versus degenerate* translates (according to Lemma 4.6) into the two cases:

$$\beta \ge \tilde\varepsilon : \textbf{nondegenerate}, \qquad \beta < \tilde\varepsilon : \textbf{degenerate.}$$

The reduction of Theorem 4.3 to Lemma 4.6 is based on the following three lemmas. The first one is an "almost nondegeneracy" of u^+, $\delta-$away from $F(u)$. The proof parallel the second part of the proof of Lemma 1.4.

Lemma 4.7 (Almost nondegeneracy). *Let u be a solution to our f.b.p. in B_2 with $Lip(u) \le L$ and $\|f\|_{L^\infty(B_2)} \le L$. Let g be a Lipschitz function with,* $\text{Lip}(g) \le L, g(0) = 0$. *If*

$$\{x_n \le g(x') - \delta\} \subset \{u^+ = 0\} \subset \{x_n \le g(x') + \delta\},$$

then

$$u(x) \ge c_0(x_n - g(x')), \quad x \in \{x_n \ge g(x') + 2\delta\} \cap B_{\rho_0},$$

for some $c_0, \rho_0 > 0$ depending on n, L as long as $\delta \le c_1, .c_1$ universal.

Proof. It suffices to show that our statement holds for $\{x_n \ge g(x') + C\delta\}$ for a possibly large constant C. Then one can apply Harnack inequality to obtain the full statement.

Also it is enough to consider $x = de_n$ (recall that $g(0) = 0$). Precisely, we want to show that

$$u(de_n) \geq c_0 d, \quad d \geq C\delta.$$

After rescaling, we are reduced to prove that (keeping the same notations)

$$u(e_n) \geq c_0$$

as long as $\delta \leq 1/C$, and $\|f\|_{L^\infty(B_2)}$ is sufficiently small. Let $\gamma > 0$ and

$$w(x) = \frac{1}{2\gamma}(1 - |x|^{-\gamma})$$

be defined on the closure of the annulus $A = B_2 \setminus \overline{B}_1$, with $\|f\|_{L^\infty}$ small enough so that

$$\Delta w < -\|f\|_{L^\infty(B_2)} \quad \text{on } A.$$

Let

$$w_t(x) = w(x + te_n).$$

Notice that

$$|\nabla w_0| < 1 \quad \text{on } \partial B_1.$$

From our flatness assumption for $t >, 0$ sufficiently large (depending on the Lipschitz constant of g), w_t is strictly above u. We decrease t and let $\bar{t}$ be the first t such that w_t touches u by above.

Since $w_{\bar{t}}$ is a strict supersolution to $\Delta u = f$ in A the touching point z can occur only on the $\eta := \frac{1}{2\gamma}(1 - 2^{-\gamma})$ level set in the positve phase of u, and $|z| \leq C = C(L)$.

Since u is Lipschitz continuous, $0 < u(z) = \eta \leq Ld(z, F(u))$, that is a full ball around z of radius η/L is contained in the positive phase of u.

Thus, for $\bar{\delta}$ small depending on η, L we have that $B_{\eta/2L}(z) \subset \{x_n \geq g(x') + 2\bar{\delta}\}$. Since $x_n = g(x') + 2\bar{\delta}$ is Lipschitz we can connect e_n and z with a chain of intersecting balls included in the positive side of u, with radii comparable to $\eta/2L$. The number of balls depends on L . Then we can apply Harnack inequality and obtain

$$u(e_n) \geq cu(z) = c_0,$$

as desired. □

The second one is a compactness lemma (we skip the the proof that requires a rather standard viscosity argument).

Lemma 4.8 (Compactness). *Let u_k be a sequence of viscosity solutions to our f.b.p. with right-hand-side f_k satisfying $\|f_k\|_{L^\infty} \leq L$. Assume:*

(a) $u_k \to u^*$ uniformly on compact sets,

(b) $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance.:

Then

$$-L \leq \Delta u^* \leq L, \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

and

$$(u_\nu^{*+})^2 - (u_\nu^{*-})^2 = 1 \quad \text{on } F(u^*)$$

both in the viscosity sense.

The final lemma translates the flatness condition of the zero set of u^+ into closedness to a two plane (one plane if $\beta = 0$) solution. Precisely:

Lemma 4.9. *Let u be a solution to (21) in B_1 with $Lip(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. For any $\eta > 0$ there exist $\bar{\delta}, \bar{\rho} > 0$ depending only on η, n, and L such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then

$$\|u - U_\beta\|_{L^\infty(B_{\bar{\rho}})} \leq \eta\bar{\rho} \tag{24}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\eta > 0$ and $\bar{\rho}$ depending on η to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem *(21)* with right-hand-side f_k such that $Lip(u_k), \|f_k\| \leq L$ and

$$\{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\}, \tag{25}$$

but the u_k do not satisfy the conclusion (24).

Then, up to a subsequence, the u_k converge uniformly on every compact to a function u^*. In view of the flatness condition and of the non-degeneracy of u_k^+ $2\delta_k$-away from the free boundary (Lemma 4.7), we can apply our compactness lemma and conclude that

$$-L \leq \Delta u^* \leq L, \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

in the viscosity sense and also

$$(u_n^{*+})^2 - (u_n^{*-})^2 = 1 \quad \text{on } F(u^*) \tag{26}$$

with

$$u^* > 0 \quad \text{in } B_{1/2} \cap \{x_n > 0\}.$$

Thus,

$$u^* \in C^{1,\gamma}(B_{1/2} \cap \{x_n \geq 0\}) \cap C^{1,\gamma}(B_{1/2} \cap \{x_n \leq 0\})$$

for all γ and in view of (26) we have that (for any $\bar{\rho}$ small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{\rho}})} \leq C(n, L)\bar{\rho}^{1+\gamma}$$

with $\alpha^2 = 1 + \beta^2$. If $\bar{\rho}$ is chosen depending on η so that

$$C(n, L)\bar{\rho}^{1+\gamma} \leq \frac{\eta}{2}\bar{\rho},$$

since the u_k converge uniformly to u^* on $B_{1/2}$ we obtain that for all k large

$$\|u_k - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{\rho}})} \leq \eta\bar{\rho},$$

a contradiction. □

Remark. To obtain Theorem 4.3 from the main Lemma, just rescale by setting

$$\tilde{u}(x) = \frac{u(\bar{\eta}x/L)}{\bar{\eta}/L}$$

where $\bar{\eta}$ is as in the Main Lemma. Then, in Theorem 4.3, choose

$$\delta_0 = \min\left\{\bar{\eta}, \bar{\delta}(\bar{\eta})\right\}^2$$

where $\bar{\delta}(\bar{\eta})$ is as in Lemma 4.9.

4.3 Strategy for the improvement of flatness

We outline the main differences in the two cases degenerate/nondegenerate.

Let us start with the **nondegenerate case.** As in the one phase case, the key lemma is:

Lemma 4.10 (Basic improvement). *Let the solution u satisfy*

$$U_\beta(x_n - \varepsilon) \le u(x) \le U_\beta(x_n + \varepsilon) \quad \text{in } B_1, \tag{27}$$

with $0 < \beta \le L$ and

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^2 \beta.$$

If $0 < r \le r_0$ for r_0 universal, and $0 < \varepsilon \le \varepsilon_0$ for some ε_0 depending on r, then

$$U_{\beta'}(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \le u(x) \le U_{\beta'}(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r, \tag{28}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le \tilde{C}\varepsilon$, and $|\beta - \beta'| \le \tilde{C}\beta\varepsilon$ for a universal constant $\tilde{C}$.

Proof. The proof of Lemma 4.10 follows the same three steps of the corresponding proof in the one phase case. Steps 1 and 2 are only outlined here. The proof is completed in Section 5.

Step 1: *compactness.* Fix $r \le r_0$, to be chose suitably. By contradiction assume that, for some sequences $\varepsilon_k \to 0$ and u_k, solutions of our f.b.p. in B_1 with r.h.s. f_k such that $\|f_k\|_{L^\infty(B_1)} \le \varepsilon_k^2 \beta_k$ and

$$U_{\beta_k}(x_n - \varepsilon_k) \le u_k(x) \le U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1,\ 0 \in F(u_k), \tag{29}$$

with $0 \le \beta_k \le L$, $\alpha_k^2 = 1 + \beta_k^2$, but the conclusion of the lemma does not hold.

Then one proves via a **Harnack type** inequality (Lemma 5.1), that the sequence of normalized functions

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\[2ex] \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

converges uniformly (up to a subsequence) to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$. Also $\alpha_k^2 = 1 + \beta_k^2$ converges to $\tilde{\alpha}^2 = 1 + \tilde{\beta}^2$.

Step 2: *limit function.* The limit function $\tilde{u}$ is a viscosity solution of the transmission problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\[2ex] \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{30}$$

One proves that $\tilde{u}$ is regular in the closure of both half-balls (see Lemmas 5.4, 5.5). Hence we can write that, since $\tilde{u}(0) = 0$, for all $r \le 1/4$ (say),

$$|\tilde{u}(x) - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \le Cr^2, \quad x \in B_r, \tag{31}$$

with

$$\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \le C.$$

Step 3: *contradiction.* From (31), since $\tilde{u}_k$ converges, uniformly to $\tilde{u}$ in $B_{1/2}$ we have

$$|\tilde{u}_k(x) - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq C'r^2, \quad x \in B_r. \tag{32}$$

Set

$$\beta'_k = \beta_k(1 + \varepsilon_k \tilde{q}) \qquad \nu_k = \frac{1}{\sqrt{1+\varepsilon_k^2|\nu'|^2}}(e_n + \varepsilon_k(\nu', 0)).$$

Then,

$$\alpha'_k = \sqrt{1 + \beta_k'^2} = \alpha_k(1 + \varepsilon_k \tilde{p}) + O(\varepsilon_k^2), \quad \nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2 \tau, \quad |\tau| \leq C,$$

where to obtain the first equality we used that $\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0$ and hence

$$\frac{\beta_k^2}{\alpha_k^2}\tilde{q} = \tilde{p} + o(1).$$

With these choices we can now show that (for k large and $r \leq r_0$)

$$\widetilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq \tilde{u}_k(x) \leq \widetilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

where again we are using the notation:

$$\widetilde{U}_{\beta'_k}(x) = \begin{cases} \dfrac{U_{\beta'_k}(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(U_{\beta'_k}) \cup F(U_{\beta'_k}) \\[2ex] \dfrac{U_{\beta'_k}(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(U_{\beta'_k}). \end{cases}$$

This will clearly imply that

$$U_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq u_k(x) \leq U_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

leading to a contradiction.

In view of (32) we need to show that in B_r

$$\widetilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2$$

and

$$\widetilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

This can be shown after some elementary calculations as long as $r \leq r_0$, r_0 universal, and $\varepsilon \leq \varepsilon_0(r)$.

We now examine the **degenerate case.** This time the key lemma is:

Lemma 4. 11. *Let the solution u satisfy*

$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1,\ 0 \in F(u) \tag{33}$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4,$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2. \tag{34}$$

There exists a universal r_1, such that if $0 < r \leq r_1$ and $0 < \varepsilon \leq \varepsilon_1$ for some ε_1 depending on r, then

$$U_0(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r, \tag{35}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ for a universal constant C.

Proof. The proof follows the usual same 3-steps pattern. Steps 1 and 2 are only outlined here. The proof is completed in Section 6.

Step1: *compactness.* Fix $r \leq r_0$, to be chose suitably. By contradiction assume that, for some sequences $\varepsilon_k \to 0$ and u_k, solutions of our f.b.p. in B_1 with r.h.s. f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^4$ and

$$\|u_k^-\|_{L^\infty(B_1)} \leq \varepsilon_k^2,$$

$$U_0(x_n - \varepsilon_k) \leq u_k(x) \leq U_0(x_n + \varepsilon_k) \quad \text{in } B_1, 0 \in F(u_k)$$

but the conclusion of the lemma does not hold.

Then one proves via a **Harnack type** inequality (Lemma 6.1), that the sequence of normalized functions

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k} \quad x \in B_1^+(u_k) \cup F(u_k)$$

converges to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$.

Step2: *limit function.* The limit function $\tilde{u}$ is a viscosity solution of the Neumann problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{36}$$

This will be proved in Lemma 6.4. The regularity of $\tilde{u}$ has been already established in Lemma 3.1.

Step 3: *contradiction.* The contradiction argument proceeds exactly as in the one phase case. □

Notice that the improvement in flatness is obtained through the closedness of the positive phase to a *one plane solution*, as long as inequality (34) holds. This inequality expresses quantitatively the degeneracy of the negative phase and should be kept valid at each step of the final iteration of lemma 4.11. However, it could happen that this is not the case and in some step of the iteration, at some level ε_k of flatness, the norm $\|u^-\|_{L^\infty(B_1)}$ becomes of order ε_k^2. When these occurs, a suitable rescaling restores a nondegenerate situation and we are back to Lemma 4.10.

The situation is precisely described in the following lemma, in which we work in B_2 for simplicity.

Lemma 4.12. *Let u be a solution in B_2 satisfying*

$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1,\ 0 \in F(u) \tag{37}$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4,$$

and for $\tilde{C}$ universal,

$$\|u^-\|_{L^\infty(B_2)} \le \tilde{C}\varepsilon^2,\ \|u^-\|_{L^\infty(B_1)} > \varepsilon^2 \tag{38}$$

There exists (universal) ε_1 such that, if $0 < \varepsilon \le \varepsilon_1$, the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2} u\left(\varepsilon^{1/2} x\right)$$

satisfies, in B_1 :

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \le u_\varepsilon(x) \le U_{\beta'}(x_n + C'\varepsilon^{1/2})$$

with $\beta' \sim \varepsilon^2$ and C' depending on $\tilde{C}$.

Proof. Set

$$v = \frac{u^-}{\varepsilon^2}.$$

Then ve have:

$$F(v) \subset \{-\varepsilon < x_n < \varepsilon\}$$

$$v \ge 0 \text{ in } B_2 \cap \{x_n \le -\varepsilon\},\ v \equiv 0 \text{ in } B_2 \cap \{x_n > \varepsilon\}$$

and moreover

$$|\Delta v| \le \varepsilon^2 \qquad \text{in } B_2 \cap \{x_n \le -\varepsilon\},$$

$$0 \le v \le \tilde{C} \qquad \text{on } \partial B_2,$$

$$v(x^*) \ge 1$$

for some point x^* in B_1.

To get a control of v by above, we use comparison with the solution h of the problem

$$\Delta h = -\varepsilon^2 \text{ in } D = B_2 \cap \{x_n < \varepsilon\},\ \ h = v \text{ on } \partial D.$$

We have $v \le h$ in D and therefore also in B_2 since $v = 0$ for $x_n \ge \varepsilon$. By Lipschitz continuity we have, for k universal,

$$v(x) \le h(x) \le k(x_n - \varepsilon)^- \qquad \text{in } B_1. \tag{39}$$

In particular we deduce that

$$\text{dist}(x^*, \{x_n = -\varepsilon\}) \ge c > 0.$$

To get a control of v by below, we compare v in $B_1 \cap \{x_n < -\varepsilon\}$with the harmonic function w

$$w = 0 \text{ on } D = B_1 \cap \{x_n = -\varepsilon\},\ w = v \quad \text{on } \partial B_1 \cap \{x_n \le -\varepsilon\}.$$

By maximum principle, we have

$$w(x) + \varepsilon^2\left(|x|^2 - 3\right) \le v(x) \qquad \text{in } B_1 \cap \{x_n < -\varepsilon\}.$$

Also, from (39),

$$w(x) - \varepsilon k\left(|x|^2 - 3\right) \ge v(x) \qquad \text{on } \partial(B_1 \cap \{x_n < -\varepsilon\})$$

and hence in all $B_1 \cap \{x_n < -\varepsilon\}$. Therefore

$$|w - v| \leq c\varepsilon \qquad \text{in } B_1 \cap \{x_n < -\varepsilon\} \tag{40}$$

and in particular

$$w(x^*) \geq \frac{1}{2}.$$

Expanding w around $(0, -\varepsilon)$ and setting $\nabla w(0, -\varepsilon) = a$, we get

$$|w(x) - a|x_n + \varepsilon|| \leq C|x|^2 + C\varepsilon \tag{41}$$

in $B_1 \cap \{x_n < -\varepsilon\}$. Notice that $a \geq c > 0$, by Hopf Principle.

By (41) and (40), if we restrict to $B_{\varepsilon^{1/2}} \cap \{x_n < -\varepsilon\}$

$$|v(x) - a|x_n + \varepsilon|| \leq C\varepsilon.$$

Going back to u^- we can write

$$\left|u^-(x) - b\varepsilon^2|x_n + \varepsilon|\right| \leq C\varepsilon^3 \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n < -\varepsilon\}$$

and

$$u^-(x) \leq b\varepsilon^2 (x_n - \varepsilon)^- \quad \text{in } B_1$$

where b is universal.

Combining the last two inequalities with assumption (37), in $B_{\varepsilon^{1/2}}$ we have

$$(x_n - \varepsilon)^+ - b\varepsilon^2 (x_n - C\varepsilon)^- \leq u(x) \leq (x_n + \varepsilon)^+ - b\varepsilon^2 (x_n + C\varepsilon)^-$$

with $C > 0$, universal.

In terms of $u_\varepsilon(x) = \varepsilon^{-1/2} u\left(\varepsilon^{1/2} x\right)$, setting $\beta' = b\varepsilon^2$, this reads

$$(x_n - \varepsilon^{1/2})^+ - \beta'\left(x_n - C\varepsilon^{1/2}\right)^- \leq u_\varepsilon(x) \leq (x_n + \varepsilon^{1/2})^+ - \beta'\left(x_n + C\varepsilon^{1/2}\right)^-.$$

Setting $(\alpha')^2 = 1 + (\beta')^2 = 1 + b^2\varepsilon^4$, with small adjustements, we can write

$$\alpha'(x_n - C'\varepsilon^{1/2})^+ - \beta'(x_n - C'\varepsilon)^- \leq u_\varepsilon(x) \leq \alpha'(x_n + C'\varepsilon^{1/2})^+ - \beta'\left(x_n + C'\varepsilon^{1/2}\right)^-$$

with C' universal. □

Lesson 5

Outline

- *The nondegenerate case. Harnack inequality.*
- *A transmission problem.*
- *End of the proof of the improvement of flatness Lemma 4.10.*

5.1 The nondegenerate case. Harnack inequality.

In this case our solution is trapped between two translations of a *true* two-plane solution U_β, $\beta \neq 0$. The Harnack inequality takes the following form.

Theorem 5.1 (Harnack inequality). *Let u be a solution of our f.b.p. in B_2 with Lipschitz constant L. There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_2$ and u satisfies the following condition:*

$$U_\beta\left(x_n + a_0\right) \leq u\left(x\right) \leq U_\beta\left(x_n + b_0\right) \qquad \text{in } B_r\left(x_0\right) \subset B_2 \tag{42}$$

with

$$\|f\|_{L^\infty(B_2)} \leq \varepsilon^2 \beta, \qquad 0 < \beta \leq L$$

and

$$0 < b_0 - a_0 \leq \varepsilon r$$

for some $0 < \varepsilon \leq \tilde{\varepsilon}$, then

$$U_\beta\left(x_n + a_1\right) \leq u\left(x\right) \leq U_\beta\left(x_n + b_1\right) \qquad \text{in } B_{r/20}\left(x_0\right)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0 \quad \text{and} \quad b_1 - a_1 \leq (1-c)\,\varepsilon r$$

and $0 < c < 1$ universal.

As in the one phase case, a key consequence of the above Theorem is that for the renormalized function

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon}, & x \in B_2^+(u) \cup F(u) \\[2ex] \dfrac{u(x) - \beta x_n}{\beta \varepsilon}, & x \in B_2^-(u), \end{cases}$$

Corollary 2.2 still holds, with the same proof. Namely:

Corollary 5.2. *Let $r = 1$ in Theorem 5.1. Then*

$$\left|\tilde{u}_\varepsilon\left(x\right) - \tilde{u}_\varepsilon\left(x_0\right)\right| \leq C\left|x - x_0\right|^\gamma$$

for all $x \in B_1\left(x_0\right)$ such that $\left|x - x_0\right| \geq \varepsilon/\tilde{\varepsilon}$.

The analogous of Lemma 2.3 is the following.

Lemma 5.3. *Let u be a viscosity solution of our f.b.p in B_2. .Assume that*

$$U_\beta\left(x\right) \leq u\left(x\right) \qquad \textit{in } B_1 \tag{43}$$

with

$$\|f\|_{L^{\infty}(B_1)} \leq \varepsilon^2 \beta \qquad 0 < \beta \leq L.$$

Let $\bar{x} = \frac{1}{5} e_n$*. There exists a universal* $\tilde{\varepsilon}$ *such that if* $\varepsilon \leq \tilde{\varepsilon}$*, and*

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon) \tag{44}$$

then,

$$u(x) \geq U_\beta(\bar{x}_n + c\varepsilon) \quad \text{ in } \bar{B}_{1/2}$$

for some universal $1 < c < 1$*. Similarly, if*

$$u(x) \leq U_\beta(x) \quad \text{ in } B_1$$

and

$$u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$$

then,

$$u(x) \leq U_\beta(\bar{x}_n - c\varepsilon) \quad \text{ in } \bar{B}_{1/2}$$

for some universal $1 < c < 1$

Proof. We prove only the first part. The second one is completely analogous.

Again, we first show that the interior gain (44) propagates into a neighborhood of $\bar{x}$. Clearly we have $u \geq U_\beta$ in B_1. Note that, since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u \geq U_\beta$ in B_1, then

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Also

$$B_{1/2} \subset\subset B_{3/4}(\bar{x}) \subset\subset B_1.$$

By Harnack inequality in $B_{1/10}(\bar{x})$, we get

$$u(x) - U_\beta(x) \geq c(u(\bar{x}) - U_\beta(\bar{x})) - C\|f\|_{L^\infty(B_{1/10})} \qquad \text{in } \bar{B}_{1/20}(\bar{x}).$$

Thus, from our assumptions ($\alpha > \beta$)

$$u(x) - U_\beta(x) = u(x) - \alpha x_n \geq c\alpha\varepsilon - C\alpha\varepsilon^2 \geq c_0\alpha\varepsilon \qquad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{45}$$

To propagate this gain up to $F(u)$ we construct a family of subsolutions. Let

$$A = B_{3/4}(\bar{x}) \setminus B_{1/20}(\bar{x}).$$

Define

$$w(x) = c\left[|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma}\right] \qquad \text{in } A$$

with the constant c chosen such that $w = 1$ on $\partial B_{1/20}(\bar{x})$ and γ (large) so that

$$\Delta w \geq k(n) > 0. \tag{46}$$

Note that $w \leq 1$ in A. Extend

$$w \equiv 1 \text{ in } B_{1/20}(\bar{x}).$$

Set, for $t \geq 0$, $x \in B_{3/4}(\bar{x})$, $\psi = 1 - w$ and

$$v_t(x) = U_\beta(x_n - c_0\varepsilon\psi + t\varepsilon).$$

Observe that

$$v_0 \leq U_\beta \leq u \qquad \text{in } B_{3/4}(x^*)$$

Let $\bar{t}$ the largest $t > 0$ such that

$$v_t \leq u \quad \text{in } B_{3/4}(x^*).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. In fact, in $B_{1/2}$,

$$u \geq v_{\bar{t}} = U_\beta (x_n - c_0\varepsilon\psi + \bar{t}\varepsilon) \geq U_\beta (x_n + c\varepsilon)$$

since $w \geq C_1 > 0$ in $B_{1/2}$.

By contradiction, suppose $\bar{t} < c_0$. At some point $\bar{x} \in B_{3/4}(x^*)$ we have

$$u(\bar{x}) = v_{\bar{t}}(\bar{x}).$$

We claim that $\bar{x}$ cannot belong to A and therefore $\bar{x} \in \bar{B}_{1/20}(x^*)$.

• $\bar{x} \notin \bar{B}_{3/4}(x^*)$. Indeed, on $\partial B_{3/4}(x^*)$ $w = 0$ whence

$$v_{\bar{t}} = U_\beta (x_n - c_0\varepsilon + \bar{t}\varepsilon) < U_\beta (x_n) \leq u.$$

• $\bar{x} \notin A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$. Indeed, in $A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$ is a strict supersolution for ε small, since (universal),

$$\Delta v_{\bar{t}} \geq c_0\varepsilon\beta\Delta w \geq c_0\varepsilon\beta k(n) > \varepsilon^2\beta \geq f.$$

• $\bar{x} \notin A \cap F(v_{\bar{t}})$. In fact,

$$\begin{aligned} \left(v_{\bar{t}}^+\right)_\nu^2 - \left(v_{\bar{t}}^-\right)_\nu^2 &= \alpha^2 \left|e_n - c_0\varepsilon\nabla\psi\right|^2 - \beta^2 \left|e_n - c_0\varepsilon\nabla\psi\right|^2 \\ &= 1 + c_0^2\varepsilon^2 \left|\nabla\psi\right|^2 - 2c_0\varepsilon\psi_n > 1 \end{aligned}$$

since $\psi_n < 0$ (as in the one phase case). This is a strict subsolution condition on $F(v_{\bar{t}})$ hence $v_{\bar{t}}$ cannot touch u by below.

Thus $\bar{x} \in \bar{B}_{1/20}(x^*)$. But then, since $\psi = 0$,

$$u(\bar{x}) = v_{\bar{t}}(\bar{x}) = U_\beta (\bar{x}_n + \bar{t}\varepsilon) < \alpha\bar{x}_n + c_0\alpha\varepsilon$$

contradicting (45). □

Proof of Theorem 5.1. We may assume $r = 2$, $x_0 = 0$.We distinguish 3 cases.

(a) $a_0 > 1/5$. Then it follows from (42) that $B_{1/5} \subset \{u > 0\}$ and

$$0 \leq v(x) \equiv \frac{u(x) - \alpha x_n}{\alpha\varepsilon} \leq 1$$

with

$$|\Delta v| \leq \varepsilon \qquad \text{in } B_{1/10}.$$

From Harnack inequality (ε small)

$$\text{osc}_{B_{1/20}}(v) \leq \theta\text{osc}_{1/10}(v) = \theta$$

with $\theta < 1$, universal. The conclusion follows easely.

(b) $a_0 < -1/5$. It follows from (42) that $B_{1/5} \subset \{u < 0\}$ and

$$0 \leq v(x) \equiv \frac{u(x) - \beta x_n}{\beta\varepsilon} \leq 1$$

with

$$|\Delta v| \le \varepsilon \qquad \text{in } B_{1/10}.$$

Again we conclude by Harnack inequality.

(c) $|a_0| \le 1/5$. From (42), we get

$$U_\beta\left(x_n + a_0\right) \le u\left(x\right) \le U_\beta\left(x_n + a_0 + \varepsilon\right) \qquad \text{in } \ B_1. \tag{47}$$

Let $x^* = \frac{1}{5}e_n$. Then either $u\left(x^*\right) \ge U_\beta\left(x_n^* + a_0 + \frac{\varepsilon}{2}\right)$ or $u\left(x^*\right) \le U_\beta\left(x_n^* + a_0 + \frac{\varepsilon}{2}\right)$. Assume the first case occurs (the other one is similar). Then, setting

$$v\left(x\right) = u\left(x - a_0 e_n\right),$$

(47) reads

$$U_\beta\left(x_n\right) \le v\left(x\right) \le U_\beta\left(x_n + \varepsilon\right) \qquad \text{in } \ B_{4/5}$$

with

$$v\left(x^*\right) \ge U_\beta\left(x_n^* + \frac{\varepsilon}{2}\right).$$

By Lemma 5.3,

$$v\left(x\right) \ge U_\beta\left(x_n + c\varepsilon\right) \qquad \text{in } B_{2/5}$$

or

$$u\left(x\right) \ge U_\beta\left(x_n + a_0 + c\varepsilon\right) \qquad \text{in } B_{2/5}$$

which is the desired improvement. □

5.2 A transmission problem

We consider solutions of the transmission problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_1 \cap \{x_n \ne 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{48}$$

Our main goal is to prove that viscosity solutions (see the definition below) are indeed classical. It is well known that the Dirichlet problem associated to (48) admits a unique classical solution. Precisely, we have:

Lemma 5.4. Let $h \in C\left(\partial B_1\right)$. *There exists a unique classical solution* $\tilde{v} \in C^\infty\left(\bar{B}_1^\pm\right)$ *to (48) such that* $\tilde{v} = h$ *on* ∂B_1. *In particular, there exists a universal constant* $\tilde{C}$ *such that*

$$\left|\tilde{v}\left(x\right) - \tilde{v}\left(y\right) - \left(\nabla_{x'}\tilde{v}\left(y\right) \cdot \left(x' - y'\right)\right) + \tilde{p}\left(y\right) x_n^+ - \tilde{q}\left(y\right) x_n^-\right| \le \tilde{C} \left\|\tilde{v}\right\|_{L^\infty(B_1)} r^2 \tag{49}$$

in $B_r\left(y\right)$, *for every* $r \le 1/4$, $y = \left(y', 0\right) \in B_{1/2}$, *with*

$$\tilde{\alpha}^2 \tilde{p}\left(y\right) - \tilde{\beta}^2 \tilde{q}\left(y\right) = 0.$$

Viscosity solutions are defined in the following way.

Definition 5.1. A function $u \in C\left(B_1\right)$ is a viscosity solution to (48) if:

(i) $\Delta u = 0$ (any sense is fine)

(ii) Consider functions of the form

$$\varphi(x) = A + px_n^+ - qx_n^- + BQ(y - x)$$

where A, B, p, q, are constants, $B > 0$, $y = (y', 0)$ and Q is the harmonic polynomial

$$Q(x) = \frac{1}{2}\left[(n-1)x_n^2 + |x'|^2\right].$$

Then, if

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0 \quad \text{(strict subsolution condition)},$$

φ cannot touch u strictly by below at a point $x_0 = (x_0', 0) \in B_1$, while if

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q < 0 \quad \text{(strict supersolution condition)},$$

φ cannot touch u strictly by above at a point $x_0 = (x_0', 0) \in B_1$.

Remark 5.1. Condition (ii) in the above definition is given in terms of sub/super solutions represented in both the upper and lower halfball, by quadratic harmonic polynomials.

Conditions (i) and (ii) are equivalent to ask that any *classical strict sub/super solution cannot touch u by below/above at a point in* B_1.

We want to show that a viscosity solution is indeed classical. Precisely, we have.

Theorem 5.5. *Let $\tilde{u}$ be a viscosity solution to (48) in B_1 such that $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$. Let $\tilde{v}$ be the classical solution to (48) in $B_{1/2}$ with $\tilde{v} = \tilde{u}$ on $\partial B_{1/2}$. Then $\tilde{v} = \tilde{u}$ in $B_{1/2}$.*

Proof. We only prove that $\tilde{v} \leq \tilde{u}$. We use once more a *sliding technique.* For $\varepsilon > 0$ fixed, $t \in \mathbb{R}$, define in $\bar{B}_{1/2}$,

$$v_{t,\varepsilon}(x) = \tilde{v}(x) + \varepsilon|x_n| + \varepsilon x_n^2 - \varepsilon - t.$$

Notice that $v_{t,\varepsilon}$ is a classical strict *subsolution in* $B_{1/2}$, since $\Delta v_{t,\varepsilon} = 2\varepsilon > 0$ outside $x_n = 0$ and

$$\tilde{\alpha}^2\left(\tilde{v}_n^+ + \varepsilon\right) - \tilde{\beta}^2\left(\tilde{v}_n^- - \varepsilon\right) = \left(\tilde{\alpha}^2 + \tilde{\beta}^2\right)\varepsilon > 0$$

on $x_n = 0$.

For $t > 0$ and large

$$v_{t,\varepsilon} < \tilde{u} \quad \text{in } B_{1/2}. \tag{50}$$

Let $\hat{t}$ be the smallest t such that (50) holds and let $\hat{x}$ such that

$$v_{\hat{t},\varepsilon}(\hat{x}) = \tilde{u}(\hat{x}).$$

Since $v_{\hat{t},\varepsilon}$ a classical strict *subsolution in* $B_{1/2}$, it cannot touch $\tilde{u}$ by below at a point in $B_{1/2}$. Hence it must be[3] $\hat{x} \in \partial B_{1/2}$, where $\tilde{v} = \tilde{u}$. This forces

$$\hat{t} = \varepsilon|\hat{x}_n| + \varepsilon\hat{x}_n^2 - \varepsilon < 0,$$

---

[3] An alternative way to proceed, that is using quadratic polynomials, is to suppose that $\hat{t} \geq 0$ and construct

$$\varphi(x) = A + px_n^+ - qx_n^- + BQ(y - x)$$

as in Definition 5.1, with

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0 \quad \text{(strict subsolution condition)},$$

that touches $v_{\hat{t},\varepsilon}$ (hence $\tilde{u}$) strictly by below.

See ([DFS]) for the details.

so that

$$\tilde{v}(x) + \varepsilon |x_n| + \varepsilon x_n^2 - \varepsilon < \tilde{u}(x) \qquad \text{in } B_{1/2}.$$

Letting $\varepsilon \to 0$ we get $\tilde{v} \leq \tilde{u}$. □

5.3 End of the proof of the improvement of flatness Lemma 4.10.

We are now ready to complete the proof of Lemma 4.10.

Step 1. (compactness). Recall that, for $r \leq r_0$, chosen in *step 3*, we assume that there exist sequences $\varepsilon_k \to 0$, β_k, α_k with

$$0 \leq \beta_k \leq L, \alpha_k^2 = 1 + \beta_k^2,$$

and u_k, solutions of our f.b.p. in B_1 with r.h.s. f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2 \beta_k$ and

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1,\, 0 \in F(u_k), \tag{51}$$

but not satisfying the conclusion of Lemma 4.10.

Consider the sequence of normalized functions

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\[2ex] \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Then,

$$|\tilde{u}_k| \leq 1 \quad \text{in } B_1$$

and from Corollary 5.2,

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C|x-y|^\gamma$$

for C and $\gamma \in (0,1)$ universal and

$$|x - y| \geq \varepsilon_k / \tilde{\varepsilon}.$$

As in the one phase case, since $F(u_k)$ converges in Hausdorff distance to $B_1 \cap \{x_n = 0\}$, we infer that the graph of $\tilde{u}_k$ converges (up to a subsequence) in Hausdorff distance to a graph of a Hölder continuous $\tilde{u}$ in $B_{1/2} \cap \{x_n \geq 0\}$. Also, up to a subsequence

$$\beta_k \to \tilde{\beta}$$

and

$$\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}.$$

Step 2: limit function. We now show that $\tilde{u}$ solves the following linearized problem (transmission problem):

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\[2ex] \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{52}$$

Since

$$|\Delta u_k| \leq \varepsilon_k^2 \beta_k \quad \text{in } B_1^+(u_k) \cup B_1^-(u_k),$$

one easily deduces that $\tilde{u}$ is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that $\tilde{u}$ satisfies the transmission condition on $\{x_n = 0\}$ in the viscosity sense. Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x-y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in, B > 0$$

and for which the subsolution transmission condition holds, namely:

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

We must show that $\tilde{\phi}$ cannot touch u strictly by below at a point $x_0 = (x_0', 0) \in B_{1/2}$ (the analogous statement by above follows with a similar argument.)

Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point. To reach a contradiction we construct a sequence of classical subsolutions ψ_k touching by below u_k.

Let

$$\Gamma(x) = \frac{1}{n-2}[(|x'|^2 + |x_n - 1|^2)^{\frac{2-n}{2}} - 1]. \tag{53}$$

This is a fundamental solution with pole at $(0', 1)$. Note that

$$\Gamma(x) = x_n + Q(x) + O\left(|x|^3\right) \qquad x \in B_1.$$

Let $z_k = y + e_n\left(\frac{1}{B\varepsilon_k} - A\varepsilon_k\right)$ and $B_k = B_{1/B\varepsilon_k}(z_k)$. We have

$$\partial B_k = \left\{x : |x' - y'|^2 + \left(x_n + A\varepsilon_k - \frac{1}{B\varepsilon_k}\right)^2\right\} = \frac{1}{B^2\varepsilon_k^2}.$$

Thus, the harmonic function

$$\begin{aligned} \Gamma_k(x) &= \frac{1}{(n-2)\, B\varepsilon_k}[(B^2\varepsilon_k^2|x' - y'|^2 + \left(B\varepsilon_k x_n + AB\varepsilon_k^2 - 1\right)^2)^{\frac{2-n}{2}} - 1] \\ &= \frac{1}{B\varepsilon_k}\Gamma(B\varepsilon_k(x-y) + AB\varepsilon_k^2 e_n) \end{aligned}$$

vanishes on ∂B_k. Moreover $|\nabla \Gamma_k| = 1$ on ∂B_k and

$$\Gamma_k(x) = A\varepsilon_k + x_n + B\varepsilon_k Q(x) + O\left(\varepsilon_k^2\right) \qquad x \in B_1.$$

Introduce now the *signed distance function* from B_k given by

$$d_k(x) = d(x, B_k) - d(x, \mathbb{R}^n \backslash B_k)$$

and define

$$\phi_k(x) = a_k \Gamma_k^+(x) - b_k \Gamma_k^-(x) + \alpha_k (d_k^+(x))^2 \varepsilon_k^{3/2} + \beta_k (d_k^-(x))^2 \varepsilon_k^{3/2}$$

where

$$a_k = \alpha_k(1 + \varepsilon_k p), \quad b_k = \beta_k(1 + \varepsilon_k q).$$

Note that also ϕ_k vanishes on ∂B_k.

Finally, let

$$\tilde{\phi}_k(x) = \begin{cases} \dfrac{\phi_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(\phi_k) \cup F(\phi_k) \\ \\ \dfrac{\phi_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(\phi_k). \end{cases}$$

It follows that in $B_1^+(\phi_k) \cup F(\phi_k)$

$$\tilde{\phi}_k(x) = A + BQ(x-y) + px_n + A\varepsilon_k p + Bp\varepsilon_k Q(x-y) + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k)$$

and analogously in $B_1^-(\phi_k)$

$$\tilde{\phi}_k(x) = A + BQ(x-y) + qx_n + A\varepsilon_k p + Bq\varepsilon_k Q(x-y) + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2}$.

Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly by below at x_0, we conclude that there exist a sequence of constant $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + c_k e_n)$$

touches u_k by below at x_k. We thus get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is

$$\begin{cases} \Delta\psi_k > \varepsilon_k^2 \beta_k \geq \|f_k\|_\infty, & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ \\ (\psi_k^+)_\nu^2 - (\psi_k^-)_\nu^2 > 1, & \text{on } F(\psi_k). \end{cases}$$

It is easily checked that away from the free boundary

$$\Delta\psi_k \geq \beta_k \varepsilon_k^{3/2} \Delta d_k^2(x + \varepsilon_k c_k e_n)$$

and the first condition is satisfied for k large enough.

Finally, since on ∂B_k, $|\nabla \Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to showing that

$$a_k^2 - b_k^2 > 1.$$

Using the definition of a_k, b_k we need to check that

$$(\alpha_k^2 p^2 - \beta_k^2 q^2)\varepsilon_k + 2(\alpha_k^2 p - \beta_k^2 q) > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Thus $\tilde{u}$ is a solution to the linearized problem.

Since we already proved *step 3*, the proof of Lemma 4-10 is complete.□

Lesson 6

Outline

- *The degenerate case. Harnack inequality.*
- *End of the proof of the improvement of flatness Lemma.*
- *Proof of the main Lemma 4.6.*
- *A Liouville Theorem and the proof of Theorem 4.4.*

6.1 The degenerate case. Harnack inequality

In this case, the negative part is very small compared to the positive one, which, in turn is closed to a one plan solution $U_0(x_n) = x_n^+$. Harnack inequality takes the following form:

Theorem 6.1 (Harnack inequality). *Let u be a solution of our f.b.p. in B_2 with Lipschitz constant L. There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_2$ and u satisfies the following condition*

$$(x_n + a_0)^+ \leq u^+(x) \leq (x_n + b_0)^+ \qquad \text{in } B_r(x_0) \subset B_2 \tag{54}$$

with

$$\|f\|_{L^\infty(B_2)} \leq \varepsilon^4, \quad \|f\|_{L^\infty(B_2)} \leq \varepsilon^2$$

and

$$0 < b_0 - a_0 \leq \varepsilon r$$

for some $0 < \varepsilon \leq \tilde{\varepsilon}$, then

$$(x_n + a_1)^+ \leq u^+(x) \leq (x_n + b_1)^+ \qquad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0 \quad \text{and} \quad b_1 - a_1 \leq (1 - c)\varepsilon r$$

and $0 < c < 1$ universal.

As in the previous cases, a key consequence of the above Theorem is that for the renormalized function

$$\tilde{u}_\varepsilon(x) = \frac{u^+(x) - x_n}{\varepsilon}, \quad x \in B_1(x_0)$$

Corollary 2.2 still holds, with the same proof. Namely:

Corollary 6.2. *Let $r = 1$ in Theorem 5.1. Then*

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma$$

for all $x \in B_1(x_0)$ such that $|x - x_0| \geq \varepsilon/\tilde{\varepsilon}$.

The analogous of Lemma 2.3 is the following.

Lemma 6.3. *Let u be a viscosity solution of our f.b.p in B_2. .Assume that*

$$x_n^+ \leq u^+(x) \qquad \textit{in } B_1 \tag{55}$$

with

$$\|f\|_{L^\infty(B_2)} \leq \varepsilon^4, \quad \|f\|_{L^\infty(B_2)} \leq \varepsilon^2.$$

Let $\bar{x} = \frac{1}{5}e_n$. *There exists a universal* $\tilde{\varepsilon}$ *such that if* $\varepsilon \leq \tilde{\varepsilon}$, *and*

$$u^+(\bar{x}) \geq (\bar{x}_n + \varepsilon)^+ \tag{56}$$

then,

$$u^+(x) \geq (\bar{x}_n + c\varepsilon)^+ \quad \text{in } \bar{B}_{1/2}$$

for some universal $1 < c < 1$. *Similarly, if*

$$u^+(x) \leq x_n^+ \quad \textit{in } B_1$$

and

$$u(\bar{x}) \leq (\bar{x}_n - \varepsilon)^+$$

then,

$$u(x) \leq (\bar{x}_n - c\varepsilon)^+ \quad \text{in } \bar{B}_{1/2}$$

for some universal $1 < c < 1$

Proof. We prove only the first part. The second one is completely analogous.

Again, we first show that the interior gain (56) propagates into a neighborhood of $\bar{x}$. Since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u^+ \geq U_0$ in B_1, then

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Also

$$B_{1/2} \subset\subset B_{3/4}(\bar{x}) \subset\subset B_1.$$

By Harnack inequality in $B_{1/10}(\bar{x})$, we get

$$u(x) - x_n \geq c(u(\bar{x}) - \bar{x}_n) - C\|f\|_{L^\infty\left(B_{1/10}\right)} \geq c_0\varepsilon \qquad \text{in } \bar{B}_{1/20}(\bar{x}). \tag{57}$$

Set, in $\bar{B}_{3/4}(\bar{x})$

$$v_t(x) = (x_n - c_0\varepsilon\psi(x) + t\varepsilon)^+ - C_1\varepsilon^2(x_n - c_0\varepsilon\psi(x) + t\varepsilon)^-$$

where $\psi = 1 - w$ is as in Lemma 5.3 and C_1 is to be chosen later.

We claim that

$$v_0 \leq u \quad \text{in } \bar{B}_{3/4}(\bar{x}).$$

Indeed, where $u \geq 0$, we have $u \geq x_n \geq v$. Where u is negative, we compare u^- with the solution of the problem

$$\Delta v = -\varepsilon^4 \text{ in } B_1 \cap \{x_n < 0\}, \quad v = u^- \text{ on } \partial(B_1 \cap \{x_n < 0\}).$$

Since $\{u < 0\} \subset \{x_n < 0\}$, it follows that

$$u^-(x) \leq Cx_n^-\varepsilon^2 \qquad \text{in } B_{8/9},\ C \text{ universal.} \tag{58}$$

Since for $C_1 > C$ and $x_n < 0$,

$$C_1(x_n - c_0\varepsilon\psi(x))^- < Cx_n^-$$

our claim follows.

Let t^* be the largest t such that

$$v_t \le u \qquad \text{in } \bar{B}_{3/4}(\bar{x}).$$

We want to show that $t^* \ge c_0$. Then, from

$$u^+(x) \ge (x_n - c_0\varepsilon\psi(x) + t^*\varepsilon)^+ \quad \text{in } \bar{B}_{3/4}(\bar{x})$$

we get

$$u^+(x) \ge (x_n + c\varepsilon)^+ \quad \text{in } \bar{B}_{1/2}$$

with $c < c_0 \min_{B_{1/2}} w$.

Suppose $t^* < c_0$. Then at some $\tilde{x} \in \overline{B}_{3/4}(\bar{x})$ we have

$$v_{t^*}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\overline{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $t^* < c_0$

$$v_{t^*}(x) = (x_n - \varepsilon c_0 + t^*\varepsilon)^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0 + t^*\varepsilon)^- < u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

In the set where $u \ge 0$ this can be seen using that $u \ge x_n^+$ while in the set where $u < 0$ again we can use the estimate (58).

We now show that $\tilde{x}$ cannot belong to the annulus $A = B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$. Indeed,

$$\Delta v_{t^*} \ge \varepsilon^3 c_0 k(n) > \varepsilon^4 \ge \|f\|_\infty, \quad \text{in } A^+(v_{t^*}) \cup A^-(v_{t^*})$$

for ε small enough.

Also,

$$(v_{t^*}^+)_\nu^2 - (v_{t^*}^-)_\nu^2 = (1 - \varepsilon^4 C_1^2)(1 + \varepsilon^2 c_0^2 |\nabla\psi|^2 - 2\varepsilon c_0 \psi_n) \quad \text{on } F(v_{t^*}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A$$

as long as ε is small enough (as in the non-degenerate case one can check that $\inf_{F(v_{\bar{t}})\cap A}(-\psi_n) > c > 0$, c universal.) Thus, v_{t^*} is a strict subsolution to in A which lies below u, hence by definition, $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \overline{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = (\tilde{x}_n + \bar{t}\varepsilon) < \tilde{x}_n + c_0\varepsilon$$

contradicting (57). □

6.2 End of the proof of the improvement of flatness Lemma 4.11

Corollary 6.2 implies that the sequence of normalized functions

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k} \quad x \in B_1^+(u_k) \cup F(u_k)$$

converges to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$.

Lemma 6.4. *$\tilde{u}$ is a viscosity solution of the Neumann problem*

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

Proof. As before, the interior condition follows easily thus we focus on the boundary condition. We keep the same notations in the proof of subsection 5.3.

Let $\tilde{\phi}$ be a classical strict subfunction of the form solution of the form

$$\tilde{\phi}(x) = A + px_n + BQ(x-y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in, B > 0$$

and

$$p > 0.$$

Then we must show that $\tilde{\phi}$ cannot touch u strictly by below at a point $x_0 = (x_0', 0) \in B_{1/2}$. Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point. Call

$$\phi_k(x) = a_k \Gamma_k^+(x) + (d_k^+(x))^2 \varepsilon_k^2, \quad a_k = (1 + \varepsilon_k p)$$

where d_k is the signed distance to $B_{\frac{1}{B\varepsilon_k}}(z_k)$.

Let

$$\tilde{\phi}_k(x) = \frac{\phi_k(x) - x_n}{\varepsilon_k}.$$

As in the previous case, it follows that in $B_1^+(\phi_k) \cup F(\phi_k)$,

$$\tilde{\phi}_k(x) = A + BQ\,(x-y) + px_n + A\varepsilon_k p + Bp\varepsilon_k Q\,(x-y) + \varepsilon_k d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2} \cap \{x_n \geq 0\}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly by below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + c_k e_n)$$

touches u_k by below at $x_k \in B_1^+(u_k) \cup F(u_k)$. We claim that x_k cannot belong to $B_1^+(u_k)$. Otherwise, in a small neighborhood $\mathcal{N}$ of x_k we would have that

$$\Delta \psi_k > \varepsilon_k^4 \geq \|f_k\|_\infty = \Delta u_k, \quad \psi_k < u_k \text{ in } \mathcal{N} \setminus \{x_k\}, \psi_k(x_k) = u_k(x_k)$$

a contradiction.

Thus, since $u_k\,(x_k) = \psi_k(x_k) = 0$, we have $x_k \in F(u_k) \cap \partial \mathcal{B}_k$ where

$$\mathcal{B}_k = B_{\frac{1}{B\varepsilon_k}}(z_k - e_n c_k).$$

Let $\mathcal{N}_\rho$ be a small neighborhood of x_k of size ρ. Since

$$\|u_k^-\|_\infty \leq \varepsilon_k^2, \quad u_k^+ \geq (x_n - \varepsilon_k)^+$$

as in the proof of Harnack inequality, using the fact that $x_k \in F(u_k) \cap \partial \mathcal{B}_k$ we can conclude by the comparison principle that

$$u_k^-(x) \le c\varepsilon_k^2 d(x, \mathcal{B}_k)^-, \quad \text{in } \mathcal{N}_{\frac{3}{4}\rho}.$$

Let

$$\Psi_k(x) = \begin{cases} \psi_k & \text{in } \mathcal{B}_k \\ c\varepsilon_k^2 \left[-3d(x, \mathcal{B}_k) + d^2(x, \mathcal{B}_k)\right] & \text{outside } \mathcal{B}_k. \end{cases}$$

Then Ψ_k touches by below u_k at $x_k \in F(u_k) \cap F(\Psi_k)$. Since $p > 0$, for k large enough, we have

$$(\Psi_k^+)_\nu^2 - (\Psi_k^-)_\nu^2 = a_k^2 - c\varepsilon_k^4 = (1 + \varepsilon_k p)^2 - c\varepsilon_k^4 > 1$$

which makes Ψ_k a subsolution. But this is a contradiction. □

6.3 Proof of the main Lemma 4.6

To prove the main Lemma 4.6 we iterate Lemma 4.10 or Lemma 4.11, after proper rescaling. Let r_0, r_1 as in those lemmas and fix a universal $\bar{r}$ such that

$$\bar{r} \le \min\left\{r_0, r_1, \frac{1}{16}\right\}.$$

Also fix a universal $\tilde{\varepsilon}$ such that

$$\tilde{\varepsilon} \le \min\left\{\varepsilon_0(\bar{r}), \frac{\varepsilon_1(\bar{r})}{2}, \frac{1}{2\tilde{C}}, \frac{\varepsilon_2}{2}\right\}$$

where $\varepsilon_0(\bar{r})$, $\varepsilon_1(\bar{r})$, $\varepsilon_2(\bar{r})$ and $\tilde{C}$ are as in Lemmas 4.10, 4.11 and 4.12.

Now let

$$\bar{\eta} = \tilde{\varepsilon}^3.$$

Suppose our assumptions hold in the ball B_2.

Case 1. $\beta \ge \tilde{\varepsilon}$ (non degenerate case).

In view of Lemma 4.5 and our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the assumptions of Lemma 5.1:

$$U_\beta(x_n - \tilde{\varepsilon}) \le u(x) \le U_\beta(x_n + \tilde{\varepsilon}) \qquad \text{in } B_1$$

with $0 < \beta \le L$ and

$$\|f\|_{L^\infty(B_1)} \le \tilde{\varepsilon}^3 \le \tilde{\varepsilon}^2 \beta.$$

From Lemma 4.10, we get

$$U_{\beta_1}(x \cdot \nu_1 - \bar{r}\frac{\tilde{\varepsilon}}{2}) \le u(x) \le U_{\beta_1}(x \cdot \nu_1 + \bar{r}\frac{\tilde{\varepsilon}}{2}) \quad \text{in } B_{\bar{r}},$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le \tilde{C}\tilde{\varepsilon}$, $0 < \beta_1 \le L$, $|\beta_1 - \beta| \le \tilde{C}\beta\tilde{\varepsilon}$ and $\alpha_1 = \sqrt{1 + \beta_1^2}$. In particular, by our choice of $\tilde{\varepsilon}$ we have

$$\beta_1 \ge \beta\left(1 - \tilde{C}\tilde{\varepsilon}\right) \ge \frac{\tilde{\varepsilon}}{2}.$$

We can therefore rescale and iterate the argument above. Assume at the step k, for $k = 1, 2...$, we have $(\beta = \beta_0, e_n = \nu_0)$

$$U_{\beta_k}(x \cdot \nu_k - \bar{r}^k \tilde{\varepsilon}_k) \le u(x) \le U_{\beta_k}(x \cdot \nu_k + \bar{r}^k \tilde{\varepsilon}_k) \quad \text{in } B_{\bar{r}^k},$$

with $\varepsilon_k = 2^{-k}\tilde{\varepsilon}$, $|\nu_k| = 1$, $|\nu_k - \nu_{k-1}| \leq \tilde{C}\tilde{\varepsilon}_{k-1}$,

$$|\beta_k - \beta_{k-1}| \leq \tilde{C}\beta_{k-1}\tilde{\varepsilon}_{k-1}, \quad \varepsilon_k \leq \beta_k \leq L$$

and $\alpha_k = \sqrt{1+\beta_k^2}$. Set

$$\rho_k = \bar{r}^k$$

and

$$u_k(x) = \frac{1}{\rho_k} u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x) \qquad x \in B_1.$$

We have

$$\|f_k\|_{L^\infty(B_1)} \leq \rho_k \tilde{\varepsilon}^3 \leq \tilde{\varepsilon}_k^2 \beta_k$$

and

$$U_{\beta_k}(x \cdot \nu_k - \tilde{\varepsilon}_k) \leq u_k(x) \leq U_{\beta_k}(x \cdot \nu_k + \tilde{\varepsilon}_k) \quad \text{in } B_1.$$

Then, applying Lemma 4.10, we get

$$U_{\beta_{k+1}}(x \cdot \nu_{k+1} - \bar{r}\tilde{\varepsilon}_{k+1}) \leq u_{k+1}(x) \leq U_{\beta_{k+1}}(x \cdot \nu_{k+1} + \bar{r}\tilde{\varepsilon}_{k+1}) \quad \text{in } B_{\bar{r}},$$

or, rescaling back,

$$U_{\beta_{k+1}}(x \cdot \nu_{k+1} - \bar{r}^{k+1}\tilde{\varepsilon}_{k+1}) \leq u(x) \leq U_{\beta_{k+1}}(x \cdot \nu_{k+1} + \bar{r}^{k+1}\tilde{\varepsilon}_{k+1}) \quad \text{in } B_{\bar{r}^{k+1}},$$

with $|\nu_{k+1}| = 1$, $|\nu_{k+1} - \nu_k| \leq \tilde{C}\tilde{\varepsilon}_k$, $|\beta_k - \beta_{k+1}| \leq \tilde{C}\beta_k\tilde{\varepsilon}_k$, and $\varepsilon_k \leq \beta_k \leq L$.

We conclude that $F(u)$ is $C^{1,\alpha}$ at the origin.

Case 2. $\beta < \tilde{\varepsilon}$ (degenerate case).

In view of Lemma 4.5 and our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the assumptions of Lemma 4.11

$$U_0(x_n - \tilde{\varepsilon}) \leq u^+(x) \leq U_0(x_n + \tilde{\varepsilon}) \quad \text{in } B_1.$$

Since (see (23)

$$\|u - U_\beta\|_{L^\infty(B_1)} \leq \bar{\eta} = \tilde{\varepsilon}^3$$

we infer

$$\left\|u^-\right\|_{L^\infty(B_1)} \leq \beta + \tilde{\varepsilon}^3 \leq 2\tilde{\varepsilon}.$$

Call $\varepsilon' = \sqrt{2\tilde{\varepsilon}}$. Then

$$U_0(x_n - \varepsilon') \leq u^+(x) \leq U_0(x_n + \varepsilon') \quad \text{in } B_1$$

and

$$\|f\|_{L^\infty(B_1)} \leq (\varepsilon')^4, \|u^-\|_{L^\infty(B_1)} \leq (\varepsilon')^2.$$

From Lemma 4.11, we get

$$U_0(x \cdot \nu_1 - \bar{r}\frac{\varepsilon'}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + \bar{r}\frac{\varepsilon'}{2}) \quad \text{in } B_{\bar{r}}.$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon'$ for a universal constant C.

We now rescale as in the previous case, setting

$$\rho_k = \bar{r}^k, \, \varepsilon_k = 2^{-k}\varepsilon'$$

and

$$u_k(x) = \frac{1}{\rho_k} u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x) \qquad x \in B_1.$$

Note that

$$\|f_k\|_{L^\infty(B_1)} \leq \rho_k (\varepsilon')^4 \leq \frac{1}{16} (\varepsilon')^4 = \varepsilon_k^4.$$

We can iterate Lemma 4.11 and obtain

$$U_0(x \cdot \nu_k - \varepsilon_k) \leq u_k^+(x) \leq U_0(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1.$$

with $|\nu_k - \nu_{k-1}| \leq C\varepsilon_{k-1}$, as long as

$$\|u_k^-\|_{L^\infty(B_1)} \leq \varepsilon_k^2.$$

Let $k^* > 1$ be the first integer for which this fails:

$$\|u_{k^*}^-\|_{L^\infty(B_1)} > \varepsilon_{k^*}^2$$

and

$$\|u_{k^*-1}^-\|_{L^\infty(B_1)} \leq \varepsilon_{k^*-1}^2.$$

We also have

$$U_0(x \cdot \nu_{k^*-1} - \varepsilon_{k^*-1}) \leq u_{k^*-1}^+(x) \leq U_0(x \cdot \nu_{k^*-1} + \varepsilon_{k^*-1}) \quad \text{in } B_1.$$

By usual comparison argument we can write

$$u_{k^*-1}^+(x) \leq C\,|x_n - \varepsilon_{k^*-1}|\,\varepsilon_{k^*-1}^2 \qquad \text{in } B_{19/20}$$

for C universal. Rescaling, we have

$$\|u_{k^*}^-\|_{L^\infty(B_1)} \leq C_1 \varepsilon_{k^*}^2$$

where C_1 universal (C_1 depends on $\bar{r}$). Then u_{k^*} satisfies the assumptions of Lemma 4.12 and therefore the rescaling

$$v(x) = \varepsilon_{k^*}^{-1/2} u_{k^*}(\varepsilon_{k^*}^{1/2} x)$$

satisfies in B_1 :

$$U_{\beta'}(x \cdot \nu_{k^*} - C'\varepsilon_{k^*}^{1/2}) \leq v(x) \leq U_{\beta'}(x \cdot \nu_{k^*} + C'\varepsilon_{k^*}^{1/2})$$

with $\beta' \sim \varepsilon_{k^*}^2$. Call $\hat{\varepsilon} = C'\varepsilon_{k^*}^{1/2}$. Then v is a solution of our f.b.p. in B_1 with r.h.s.

$$g(x) = \varepsilon_{k^*}^{1/2} f_{k^*}(\varepsilon_{k^*}^{1/2} x)$$

and the flatness assumption

$$U_{\beta'}(x \cdot \nu_{k^*} - \hat{\varepsilon}) \leq v(x) \leq U_{\beta'}(x \cdot \nu_{k^*} + \hat{\varepsilon}).$$

Since $\beta' \sim \varepsilon_{k^*}^2$, we have

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon_{k^*}^{1/2} \varepsilon_{k^*}^4 \leq \hat{\varepsilon}^2 \beta'$$

as long as $\hat{\varepsilon} \leq \min\left\{\varepsilon_0(\bar{r}), \frac{1}{2\tilde{C}}\right\}$, which is true if $C'(2\tilde{\varepsilon})^{1/4} \leq \min\left\{\varepsilon_0(\bar{r}), \frac{1}{2\tilde{C}}\right\}$ or

$$\tilde{\varepsilon} \leq \frac{1}{2C'^4} \min\left\{\varepsilon_0(\bar{r}), \frac{1}{2\tilde{C}}\right\}^4.$$

Under these restrictions, v satisfies the assumptions of the nondegenerate case and we can proceed accordingly.

This concludes the proof of the main Lemma. □

6.4 A Liouville Theorem and the proof of Theorem 4.4

Although not strictly necessary, we use the following Liouville type result for global viscosity solutions to a two-phase homogeneous free boundary problem.

Lemma 6. 5. *Let U be a global Lipschitz viscosity solution to*

$$\begin{cases} \Delta U = 0, & \text{in } \{U > 0\} \cup \{U \leq 0\}^{\circ} \\ (U_\nu^+)^2 - (U_\nu^-)^2 = 1, & \text{on } F(U) = \partial \{U > 0\} \end{cases} \tag{59}$$

Assume that $F(U) = \{x_n = g(x'), x' \in^{n-1}\}$ with Lip$(g) \leq M$. Then g is linear and $U(x) = U_\beta(x)$ for some $\beta \geq 0$.

Proof. Assume for simplicity, $0 \in F(U)$. Also, balls in $\mathbb{R}^{n-1}$ are denoted by B'_ρ.

By the regularity theory in [C1], since U is a solution in B_2, the free boundary $F(U)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n and on M. Thus,

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in B'_1$$

with C depending only on n, M. Moreover, since U is a global solution, the rescaling

$$g_R(x') = \frac{1}{R} g(Rx'), \quad x' \in B'_2,$$

which preserves the same Lipschitz constant as g, satisfies the same inequality as above i.e.

$$|g_R(x') - g_R(0) - \nabla g_R(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in B'_1.$$

This reads,

$$|g(Rx') - g(0) - \nabla g(0) \cdot Rx'| \leq CR|x'|^{1+\alpha}, \quad x' \in B'_1.$$

Thus,

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \leq C\frac{1}{R^\alpha}|y'|^{1+\alpha}, \quad y' \in B'_R.$$

Proof of Theorem 4.3. Let $\bar{\eta}$ be the universal constant in the main Lemma 4.6. Consider the blow-up sequence

$$u_k(x) = \frac{u(\delta_k)}{\delta_k}$$

with $\delta_k \to 0$ as $k \to \infty$. Each u_k is a solution of our f.b.p. with right hand side

$$f_k(x) = \delta_k f(\delta_k x)$$

and

$$\|f_k(x)\| \leq \delta_k \|f\|_{L^\infty} \leq \bar{\eta}$$

for k large enough. Standard arguments using the uniform Lischitz continuity of the u_k's and the nondegeneracy of their positive part u_k^+, imply that (up to a subsequence)

$$u_k \to \tilde{u} \quad \text{uniformly on compacts}$$

and

$$\{u_k^+ = 0\} \to \{\tilde{u} = 0\} \quad \text{in the Hausdorff distance.}$$

The blow-up limit $\tilde{u}$ solves the global homogeneous two-phase free boundary problem

$$\begin{cases} \Delta \tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \leq 0\}^{\circ} \\ (\tilde{u}_\nu^+)^2 - (\tilde{u}_\nu^-)^2 = 1, & \text{on } F(\tilde{u}) = \partial \{\tilde{u} > 0\} \end{cases} \tag{60}$$

Since $F(u)$ is a Lipschitz graph in a neighborhood of 0, it follows from Lemma 6.5 that $\tilde{u}$ is a two-plane solutions, $\tilde{u} = U_\beta$ for some $\beta \geq 0$. Thus, for k large enough

$$\|u_k - U_\beta\|_{L^\infty} \leq \bar{\eta}$$

and

$$\{x_n \leq -\bar{\eta}\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \bar{\eta}\}.$$

Therefore, we can apply our flatness Theorem and conclude that $F(u_k)$ and hence $F(u)$ is smooth. □

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