Geometric methods for invariant manifolds in dynamical systems IV.

Fibres and verification of conditions

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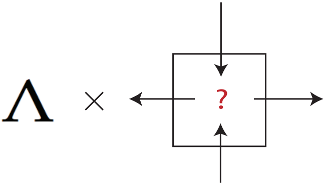
Plan of the lecture

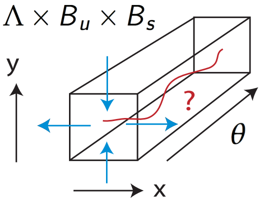
- Overview
- Invariant fibres on stable/unstable manifolds
- Verification of cone conditions
- Covering and cone conditions for vector fields
- Example

Overview

$$f : \Lambda \times B_u \times B_s \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s$$

Λ is compact manifold without a boundary $(\Lambda = \mathbb{S}^1)$

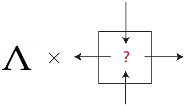


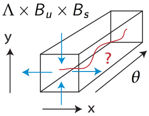


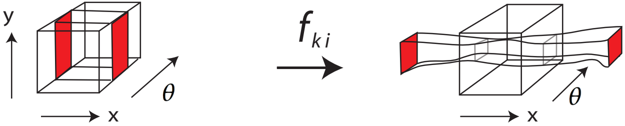
Do we have an invariant manifold in $\Lambda \times B_u \times B_s$?

Overview

Covering relations





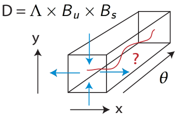


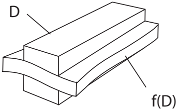
$\{V_j\}$ and $\{U_i\}$ are coverings of Λ

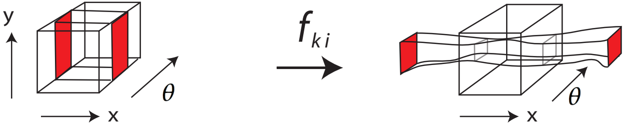
$$f_{k\,i}\left(V_j \times \overline{B}_u \times \overline{B}_s\right) \subset U_k \times \mathbb{R}^u \times \mathbb{R}^s$$

Overview

Covering relations







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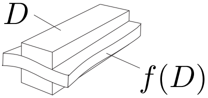
Overview

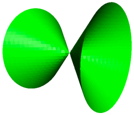
Cone conditions



$$Q_h(\theta, x, y) = \|x\|^2 - \|y\|^2 - \|\theta\|^2$$

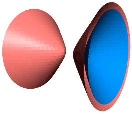
$$Q_v(\theta, x, y) = -\|x\|^2 + \|y\|^2 - \|\theta\|^2$$

Horizontal cone $Q_h \geq 0$:



If $Q_h(q_1 - q_2) = a > 0$ then

$Q_h(f(q_1) - f(q_2)) = b > ma$



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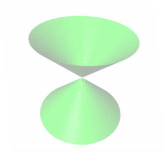
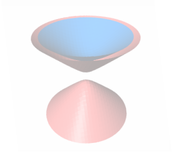
Vertical cone $Q_v \geq 0$:



If $Q_v(q_1 - q_2) = a$ then

$Q_v(f^{-1}(q_1) - f^{-1}(q_2)) = b > ma$



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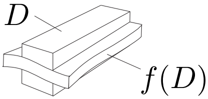

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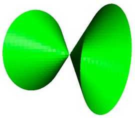

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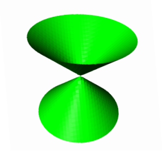

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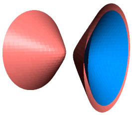

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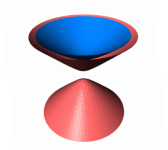
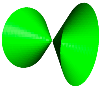
If $Q_v(q_1 - q_2) = a$ then

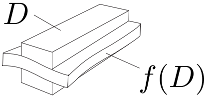
$Q_v(f^{-1}(q_1) - f^{-1}(q_2)) = b > ma$

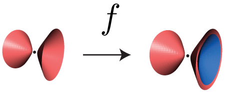


Overview

Normally hyperbolic invariant manifolds





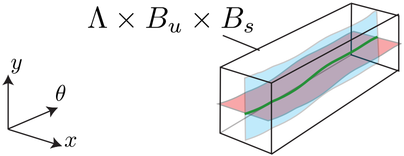


If $Q_h(q_1 - q_2) = a > 0$ then

$Q_h(f(q_1) - f(q_2)) = b > ma$

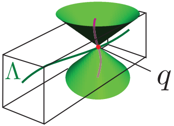
Theorem

If f and f^{-1} satisfy covering and cone conditions, then there exists a manifold $\Lambda \in D$.
Moreover, there exist manifolds W^u and W^s.



Stable fibers

$$Q_v(\theta, x, y) = -\|x\|^2 + \|y\|^2 - \|\theta\|^2$$



Definition

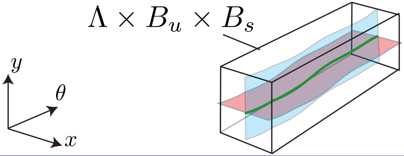
A vertical disc $v \subset W^s$ is a stable fiver of $q \in \Lambda$ if for $n \geq 0$

$$\|f^n(q) - f^n(v(y))\| \stackrel{n\to+\infty}{\to} 0$$

$$Q_v(f^n(q) - f^n(v(y))) > 0$$

Theorem

If f and f^{-1} satisfy covering and cone conditions, then there exists a manifold $\Lambda \in \Lambda \times \overline{B_u} \times \overline{B_s}$. Moreover, there exist manifolds W^u and W^s.



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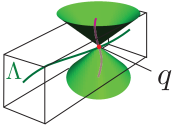


Rate condition:

$$Q_v(f^n(q) - f^n(v(y))) > 0$$

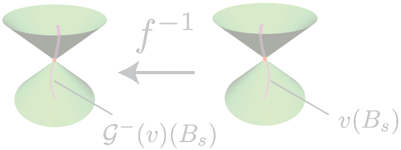
$$[Df] \in \begin{pmatrix} \mathbf{C} & \epsilon_1 \\ \epsilon_2 & \mathbf{B} \end{pmatrix} \qquad \frac{\|\mathbf{B}\| + \|\epsilon_2\|}{m(\mathbf{C}) - \|\epsilon_1\|} < 1$$

Theorem

For any $q \in \Lambda$ a stable fibre exists and is unique.

Lemma

If f^{-1} satisfies covering and cone cond. then $f^{-1}(v)$ is a vertical disc.



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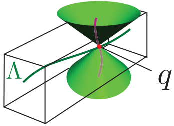

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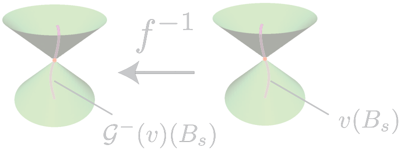

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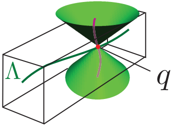

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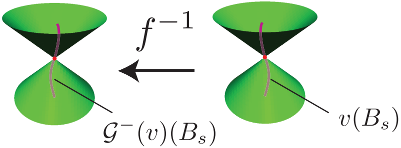

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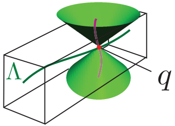

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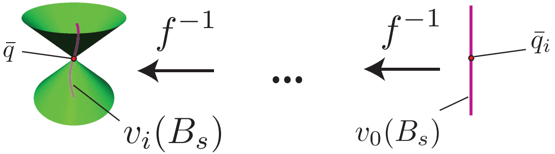
For any $\bar{q} \in \Lambda$ a stable fibre exists and is unique.

Proof. $\bar{q}_i = f^i(\bar{q})$

$$v_i = \mathcal{G}^{-i}(v_0)$$



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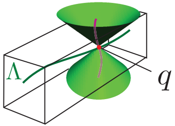

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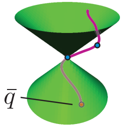


suppose q_1, q_2 such that

$$\|\pi_{x,\theta}(q_1 - q_2)\| > \|\pi_y(q_1 - q_2)\|$$

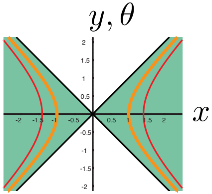
This contradicts rate conditions (chalk). ∎

Verifying cone conditions

$$Q(x, y, \theta) = \|x\|^2 - \|y\|^2 - \|\theta\|^2$$
$$= \|x\|^2 - \|(y, \theta)\|^2$$

without loss of generality

$$Q(x, y) = \|x\|^2 - \|y\|^2$$



Definition (cone condition)

$m > 1$. If $Q(q_1 - q_2) \geq 0$ then $Q(f(q_1) - f(q_2)) > mQ(q_1 - q_2)$.

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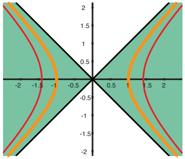

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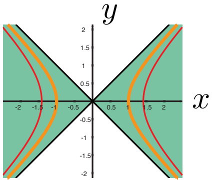

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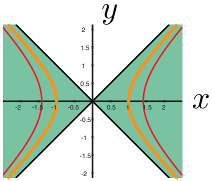

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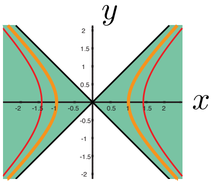

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$$\|\epsilon_i\| < \varepsilon$$

Lemma

$m(\mathbf{A})^2 - \varepsilon(\|\mathbf{B}\| + \|\mathbf{A}\| + \varepsilon) > m$

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then f satisfies cone conditions.

Proof. chalk.

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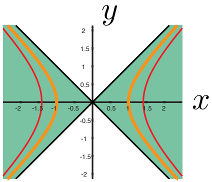

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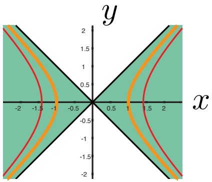

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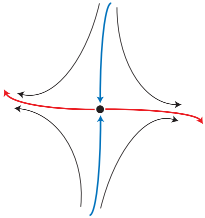
Cone conditions for vector fields

$$x' = F(q)$$



$$Q(x,y) = \|x\|^2 - \|y\|^2$$

$$Q \quad \sim \quad C = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$$

Lemma

$$\frac{d}{dt} Q(\phi_t(q_1) - \phi_t(q_2))|_{t=0} \in 2(q_1 - q_2)^T C[DF](q_1 - q_2)$$

Proof. chalk.

Conditions implying

$$v^T C v \geq 0 \quad \Longrightarrow \quad v^T (C[DF]) v > \delta v^T C v$$

are analogous to ones on previous slide. Then

$$Q(\phi_t(q_1) - \phi_t(q_2)) > (1 + 2\delta t) Q(q_1 - q_2)$$

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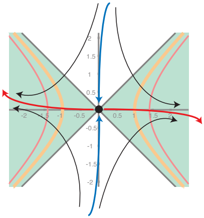

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Cone conditions for vector fields

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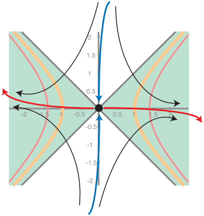

$$Q(x,y) = \|x\|^2 - \|y\|^2$$

$$Q \quad \sim \quad C = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$$

Lemma

$$\frac{d}{dt} Q(\phi_t(q_1) - \phi_t(q_2))|_{t=0} \in 2(q_1 - q_2)^T C[DF](q_1 - q_2)$$

Proof. chalk.

Conditions implying

$$v^T C v \geq 0 \quad \Longrightarrow \quad v^T (C[DF]) v > \delta v^T C v$$

are analogous to ones on previous slide. Then

$$Q(\phi_t(q_1) - \phi_t(q_2)) > (1 + 2\delta t) Q(q_1 - q_2)$$

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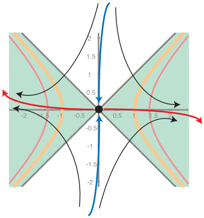

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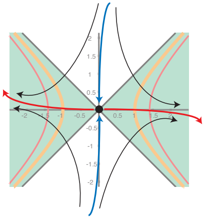

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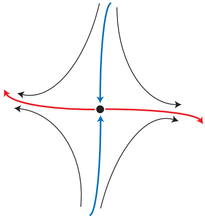
Covering conditions for vector fields

$$q' = F(q)$$



$$[DF] \subset \begin{pmatrix} \mathbf{A} & \epsilon_1 \\ \epsilon_2 & \mathbf{B} \end{pmatrix}$$

Lemma

$\frac{d}{dt} \|\pi_x(\phi_t(q) - \phi_t(0))\|^2 |_{t=0} = 2x^T(\mathbf{A}x + \epsilon_1 y)$

Proof. Chalk.

Assume $x^T(\mathbf{A}x + \epsilon_1 y) > a > 0$. Taking $q \in N^-$

$$\|\pi_x \phi_t(q)\| \geq \sqrt{\|\pi_x(\phi_t(q) - \phi_t(0))\|^2} - \|\pi_x \phi_t(0)\|$$
$$> \sqrt{1 + at} - tc$$

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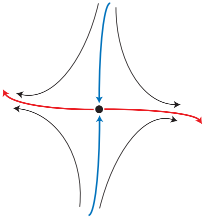

Lemma

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Proof. Chalk.

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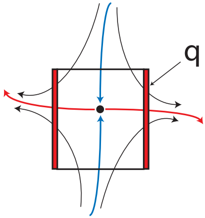

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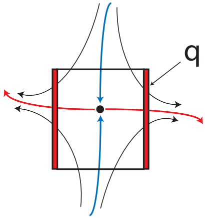

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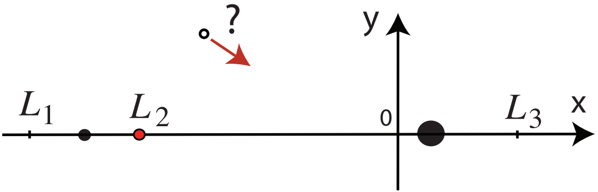
Example

Restricted three body problem



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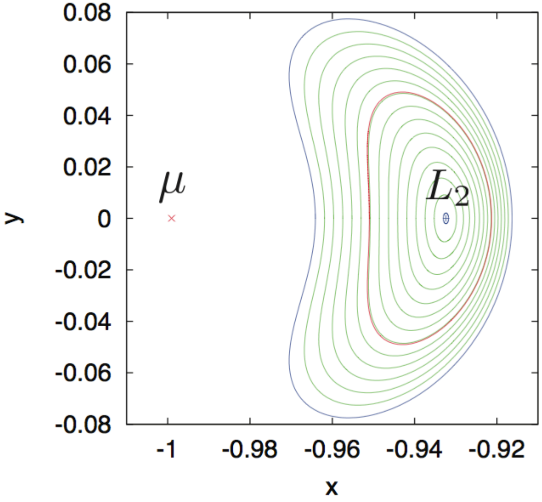

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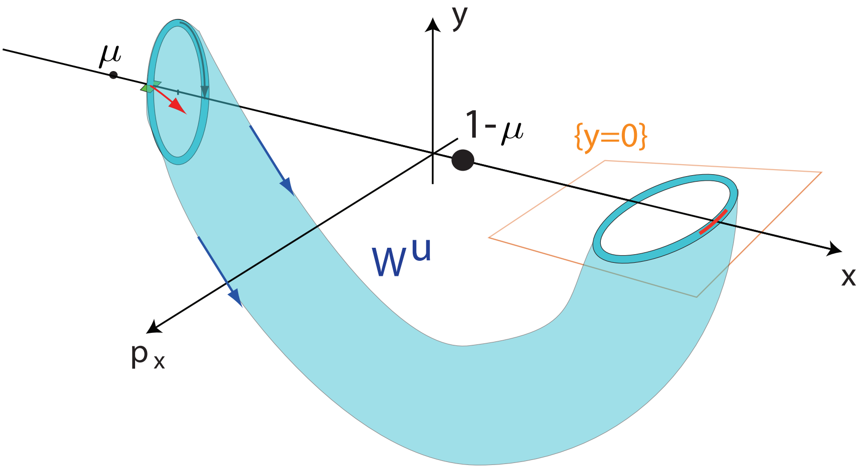

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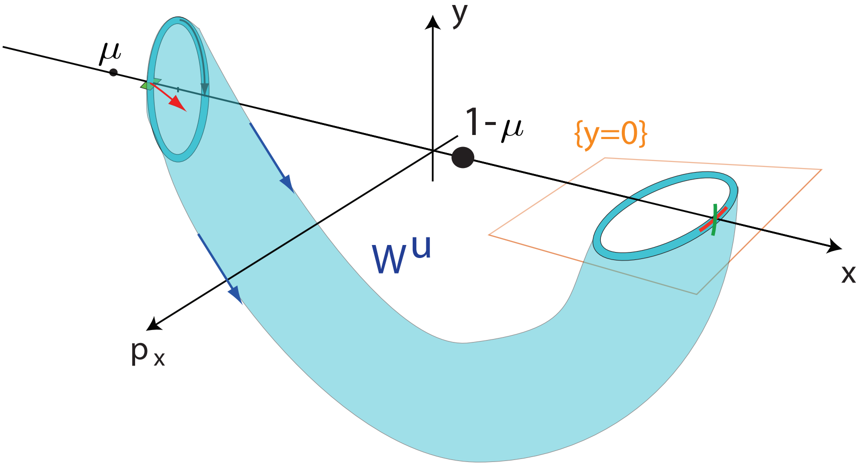

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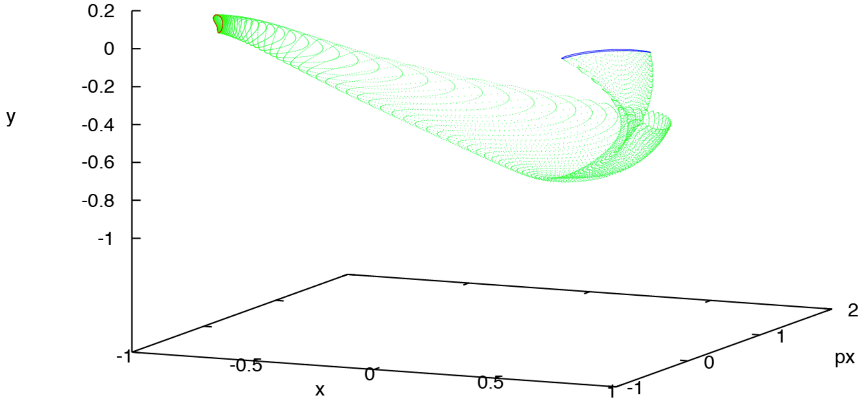

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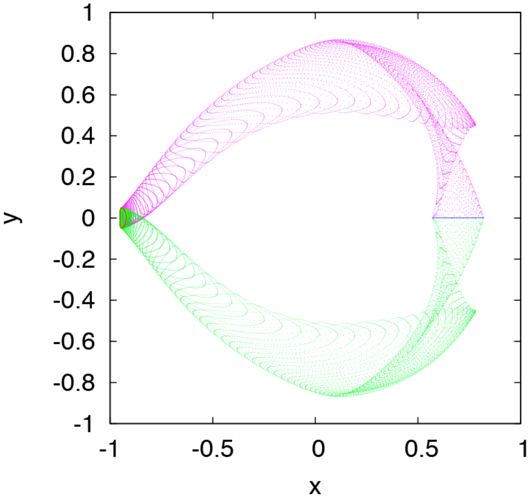

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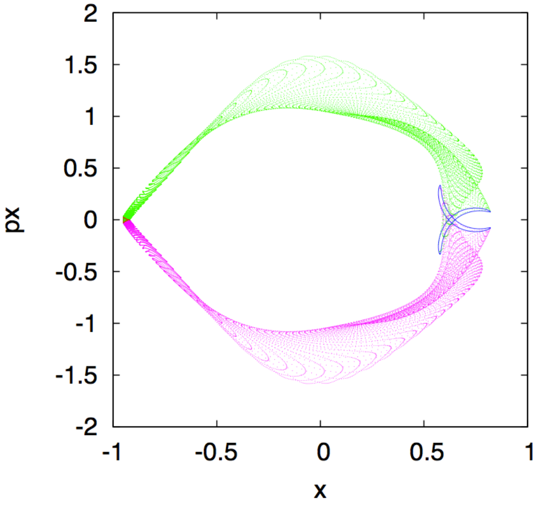
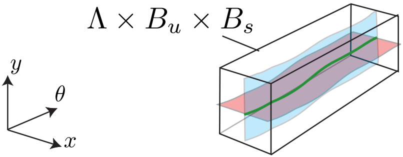
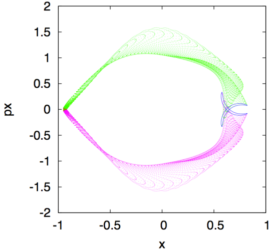


Closing remarks





- Invariant manifolds follow from geometric constructions
- Assumptions verifiable - suitable for computer assisted proofs
- All that is needed: $[f(q)]$, $[df(D)]$

Thank you for your attention.

References

- Verification of cone and covering conditions:

[Z] P.Zgliczyński, Covering relations, cone conditions and stable manifold theorem , J. of Diff. Equations 246 (2009) 1774–1819

[CS] M.J.Capiński, P.Roldan, Existence of a Center Manifold in a Practical Domain Around L1 in the Restricted Three Body Problem. SIAM J. Appl. Dyn. Syst. 11, pp. 285-318

- Fibres and higher order smoothness:

[CZ2] M.J.Capiński, P.Zgliczyński, Geometric Proof of the Normally Hyperbolic Invariant Manifold Theorem, preprint

- 3 body problem example:

[C] M.J.Capiński, Lyapunov Orbits at L2 and transversal Intersections of Invariant Manifolds in the Jupiter-Sun Planar Restricted Circular Three Body Problem. To appear in SIAM Journal on Applied Dynamical Systems.