Geometric methods for invariant manifolds in dynamical systems III.

Normally hyperbolic invariant manifolds

Maciej Capiński

AGH University of Science and Technology, Kraków

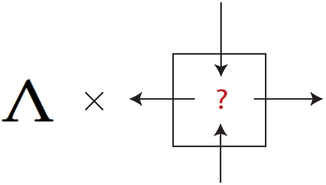
Plan of the lecture

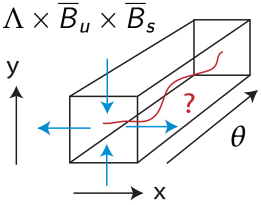
- Statement of the problem
- Normally hyperbolic invariant manifold theorem
- Existence of the center-stable manifold
- Existence of the center-unstable manifold
- Intersections of manifolds - normally hyperbolic manifold
- Main results
- Examples
  - Rotating Hénon map
  - Driven logistic map
  - Center manifold in the restricted three body problem

Statement of the problem

$$f : \Lambda \times \overline{B}_u \times \overline{B}_s \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s$$

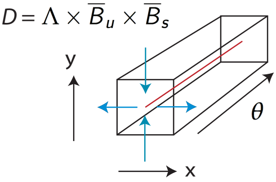
Λ is compact manifold without a boundary $\qquad (\Lambda = \mathbb{S}^1)$



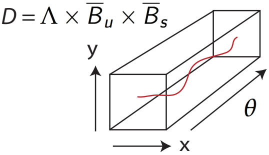


Do we have an invariant manifold in $\Lambda \times \overline{B}_u \times \overline{B}_s$?

Normally hyperbolic invariant manifold theorem



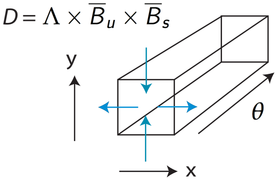
$f : D \to \Lambda \times \mathbb{R}^2$



$f_\varepsilon = f + \varepsilon g$

- we start with the region D and devise conditions which ensure the existence of the manifold
- the conditions are verifiable with rigorous numerics

Normally hyperbolic invariant manifold theorem



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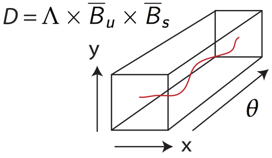

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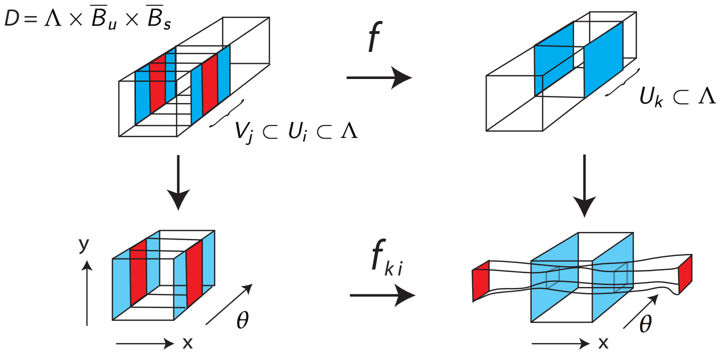
Local maps

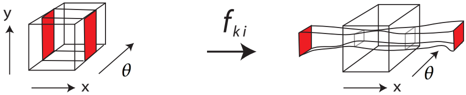
Topological conditions (covering relations)



$\{V_j\}$ and $\{U_i\}$ are coverings of Λ

$$f_{k\,i}\left(V_j \times \overline{B}_u \times \overline{B}_s\right) \subset U_k \times \mathbb{R}^u \times \mathbb{R}^s$$

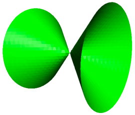
Cones



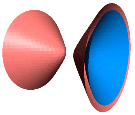
In local coordinates we define

$$Q(\theta, x, y) = \|x\|^2 - \|y\|^2 - \|\theta\|^2$$

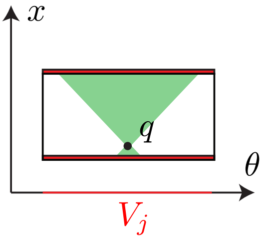
Horizontal cone $Q \geq 0$:



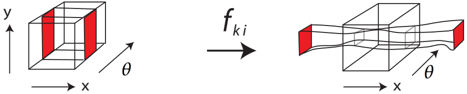
$Q = a$ and $Q = b$ for $0 < a < b$:



For each point $q \in D$ we have a local set which contains the cone starting from q.



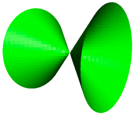
Cone conditions



$m > 1$. If $Q(x_1 - x_2) \geq 0$ then

$$Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$$

Horizontal cone $Q \geq 0$:



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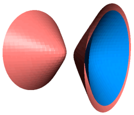

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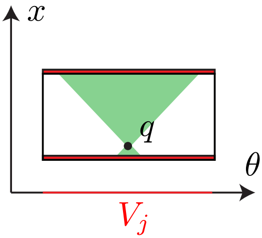
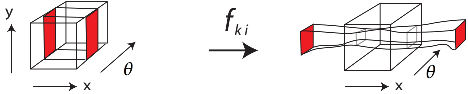
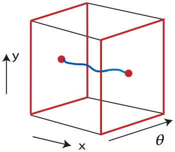


Horizontal discs

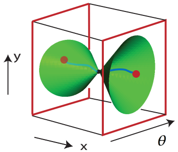


If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

A horizontal disc:
$b : \overline{B}_u \to V_j \times \overline{B}_u \times \overline{B}_s$



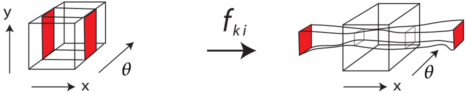
A horizontal disc which satisfies cone conditions:



Lemma

An image of a horizontal disc which satisfies cone conditions is a horizontal disc which satisfies cone conditions.

Horizontal discs



If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

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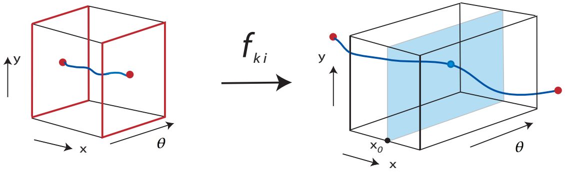
Proof.



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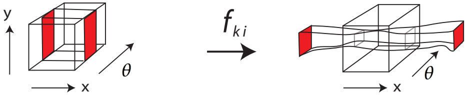

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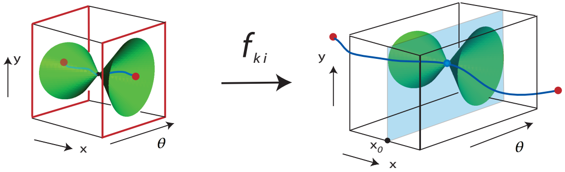

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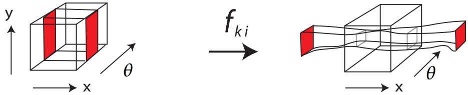

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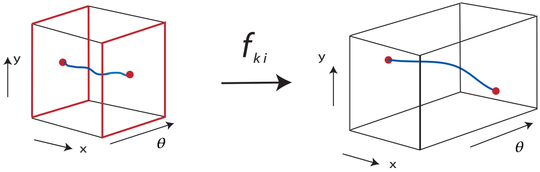

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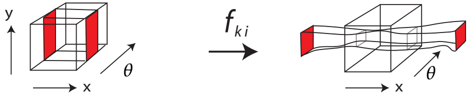
Forward iterations



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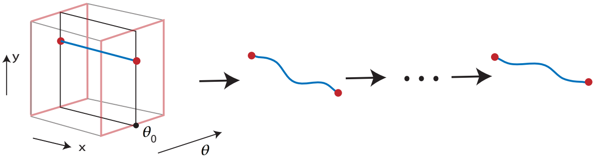
Lemma

For any $\theta_0 \in \Lambda$ we have a vertical disc of points in $\{\theta_0\} \times \overline{B}_u \times \overline{B}_s$ which stay inside of $\Lambda \times \overline{B}_u \times \overline{B}_s$.

Proof.



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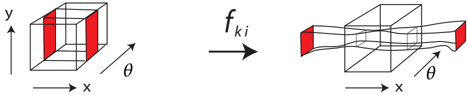

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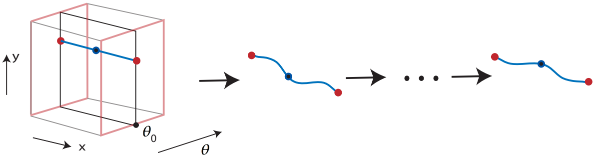

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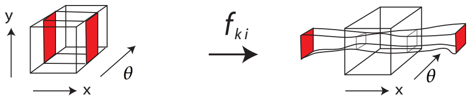

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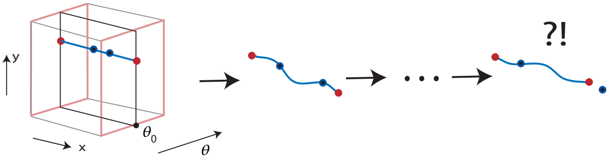

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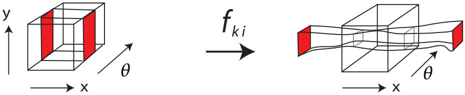

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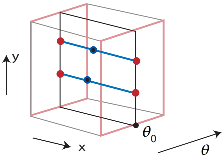

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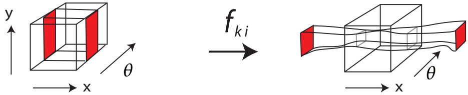

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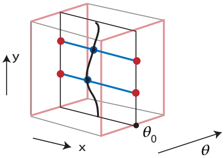

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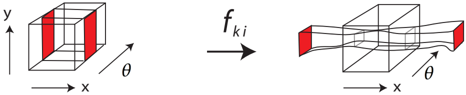

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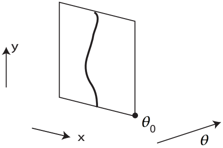

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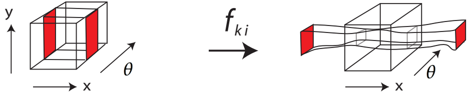
Main Result



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Theorem (Main result)

If both the forward map f and its inverse f^{-1} satisfy the the topological and cone conditions then there exists a continuous map

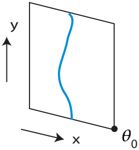
$$\chi : \Lambda \to \Lambda \times \overline{B}_u \times \overline{B}_s$$

such that

$$\chi(\Lambda) = \mathrm{inv}(f, \Lambda \times \overline{B}_u \times \overline{B}_s).$$

Proof

a vertical disc of forward invariant points:



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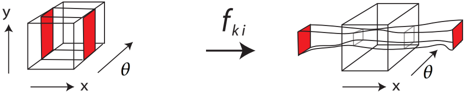

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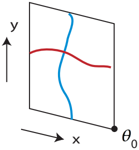
a horizontal disc of backward invariant points:



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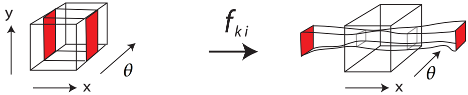

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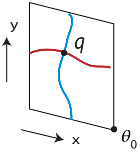
gives $\chi(\theta_0) := q$



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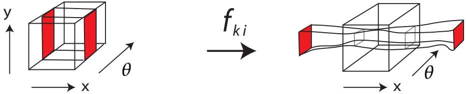

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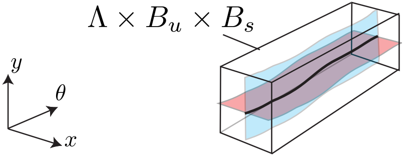

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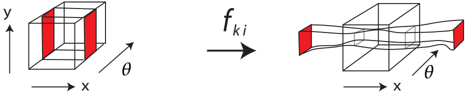

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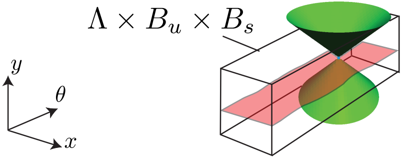

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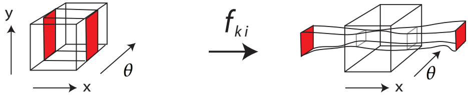

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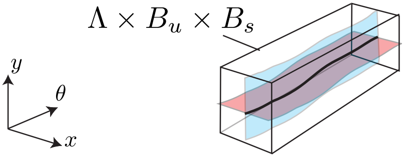

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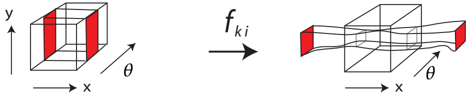
Verification of conditions



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We need

$$[Df(V_j)] \quad \longleftrightarrow \quad \begin{bmatrix} \left\| \frac{\partial f_1}{\partial \theta} \right\| \leq C & \varepsilon & \varepsilon \\ \varepsilon & \left\| \frac{\partial f_2}{\partial x} \right\| \geq \alpha & \varepsilon \\ \varepsilon & \varepsilon & \left\| \frac{\partial f_3}{\partial y} \right\| \leq \beta \end{bmatrix}$$

where

$$\beta < C < \alpha$$

with $\beta < 1 < \alpha$ and ε appropriately small.

Example of applications

Rotating Hénon map

$$\bar{x} = 1 + y - ax^2$$
$$\bar{y} = bx$$

For $a = 0.68$, $b = 0.1$ and $\varepsilon \leq \frac{1}{2}$

$$\Lambda \subset U_\varepsilon = \mathbb{T}^1 \times [x_0 - 1.1\varepsilon, x_0 + 1.1\varepsilon] \times [y_0 - 0.12\varepsilon, y_0 + 0.12\varepsilon],$$

where

$$x_0 = \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3,$$
$$y_0 = bx_0 \approx -0.204\,33.$$

Example of applications

Rotating Hénon map

$$\begin{aligned} \bar{\theta} &= \theta + \omega \qquad (\text{mod } 1), \\ \bar{x} &= 1 + y - ax^2 + \varepsilon \cos(2\pi\theta), \\ \bar{y} &= bx \end{aligned}$$

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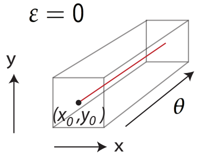
Example of applications

Rotating Hénon map



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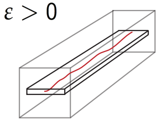
$$



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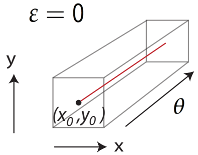
Rotating Hénon map



$$
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\bar{y} &= bx
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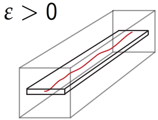
$$



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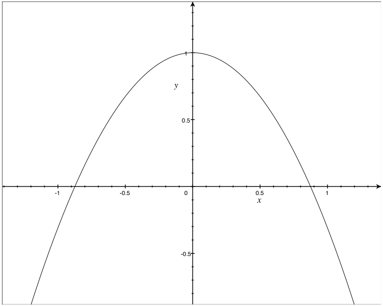
Driven logistic map



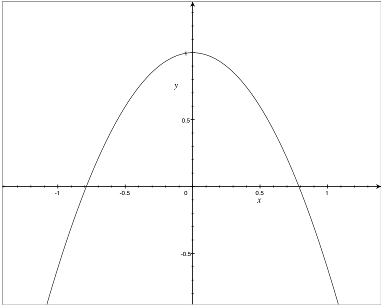
$\theta = 0$

$$T : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$$

$$T(x, \theta) = \left(\theta + \alpha, 1 - \left(a + \varepsilon \sin(2\pi\theta)\right) x^2\right)$$

$$a = 1.31, \quad \varepsilon = 0.3 \quad g = \frac{\sqrt{5} - 1}{2}, \quad \alpha = \frac{g}{N}.$$

Driven logistic map



$\theta = \frac{1}{4}$

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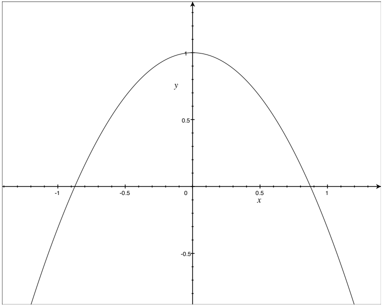

$\theta = \frac{1}{2}$

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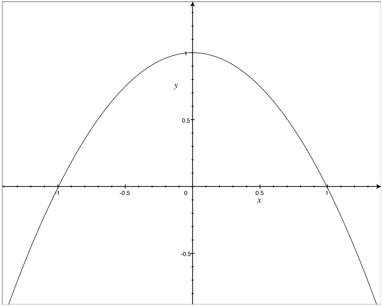

$\theta = \frac{3}{4}$

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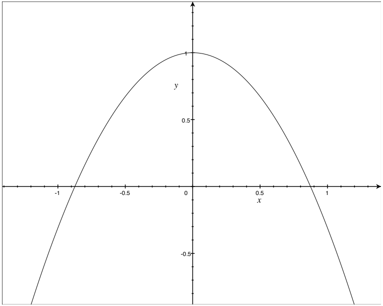

$\theta = 0$

$$T : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$$

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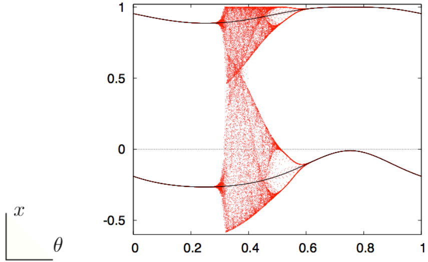

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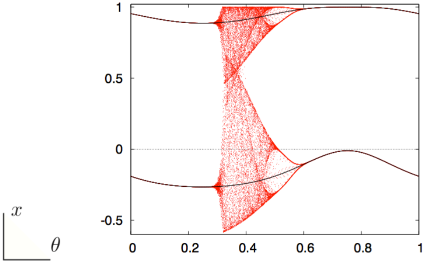

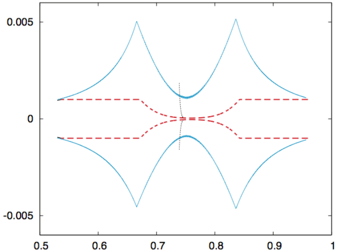

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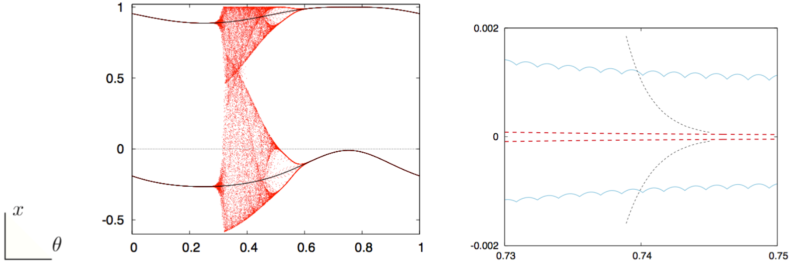

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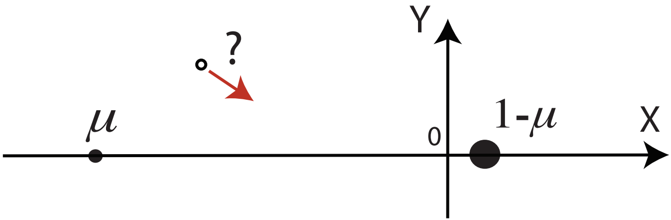
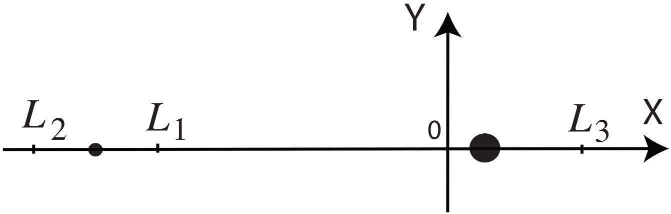
The planar restricted three body problem



$$H = \frac{1}{2}\left(P_X^2 + P_Y^2\right) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

The planar restricted three body problem



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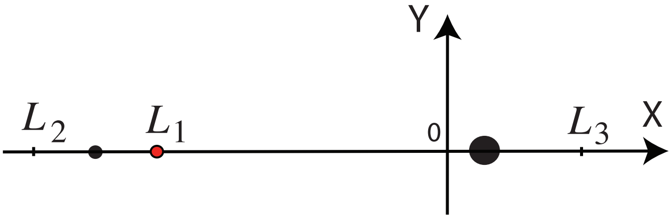


Normal form at L_1

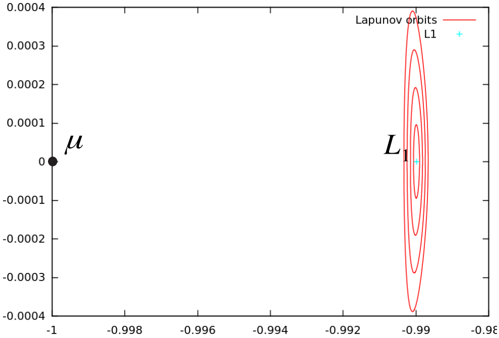
$$(x_1, y_1, x_2, y_2) = \phi(X, Y, P_X, P_Y)$$

$$I = \frac{x_2^2 + y_2^2}{2}$$

$$H = \lambda x_1 y_1 + \omega I + \sum_{N \geq i > 2} H_i(x_1 y_1, I) + h.o.t.$$

The planar restricted three body problem

Lapunov orbits around L_1 - Sun-Earth system

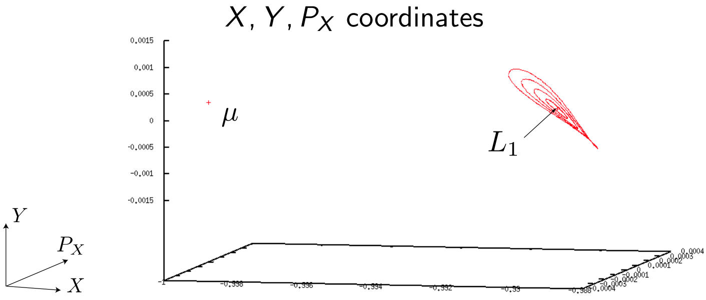


$$\mu = 3.040423398444176 \times 10^{-6}$$

$$H = \lambda x_1 y_1 + \omega I + \sum_{N \geq i > 2} H_i(x_1 y_1, I) + h.o.t.$$

The planar restricted three body problem

The approximate center manifold at L_1



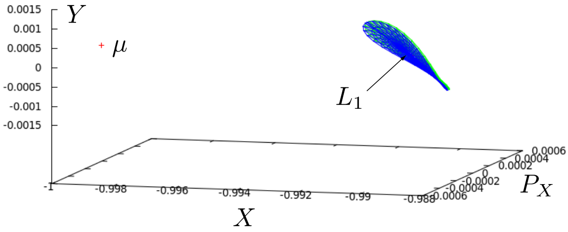
Set $x_1 = y_1 = 0$

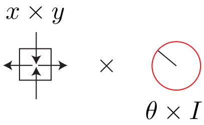
$$\phi^{-1}(0, 0, [0, I_1], S^1)$$

$$H = \lambda x_1 y_1 + \omega I + \sum_{N \geq i > 2} H_i(x_1 y_1, I) + h.o.t.$$

The planar restricted three body problem

Rigorous enclosure of the center manifold at L_1

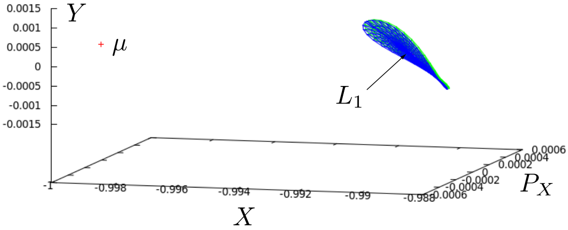




$$\phi^{-1}([-\delta,\delta],[-\delta,\delta],[0,I_1],S^1)$$

$$H = \lambda x_1 y_1 + \omega I + \sum_{N \geq i > 2} H_i(x_1 y_1, I) + h.o.t.$$

Next Lecture



- Verification of covering and cone conditions
- Conditions for vector fields
- Foliations
- Some more examples

Thank you for your attention

References

- Existence of normally hyperbolic manifolds:

[CZ] M.J.Capiński, P.Zgliczyński, Cone Conditions and Covering Relations for Topologically Normally Hyperbolic Invariant Manifolds. Discrete and Continuous Dynamical Systems A. Vol. 30, No 3, July 2011, pp. 641670

[CS] M.J.Capiński, C. Somó, Computer Assisted Proof for Normally Hyperbolic Invariant Manifolds. Nonlinearity 25 (2012) 1–30.

- 3 body problem example:

[CS] M.J.Capiński, P.Roldan, Existence of a Center Manifold in a Practical Domain Around L1 in the Restricted Three Body Problem. SIAM J. Appl. Dyn. Syst. 11, pp. 285-318