Geometric methods for invariant manifolds in dynamical systems II.

Invariant manifolds associated with hyperbolic fixed points

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Plan of the lecture

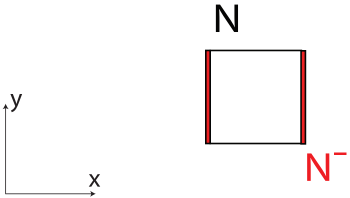
- Covering relations
- Cone conditions
- Horizontal discs
- Existence of stable manifold
- Existence of unstable manifold
- Intersections of manifolds

Covering relations

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = \overline{B_u} \times \overline{B_s}$$

$$N^- = \partial B_u \times \overline{B_s}$$



Theorem

If $N \overset{F}{\Rightarrow} N$ *then* $\exists q^* \in N$

$$F(q^*) = q^*$$

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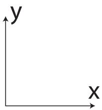

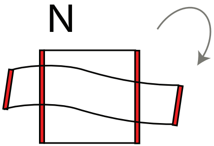

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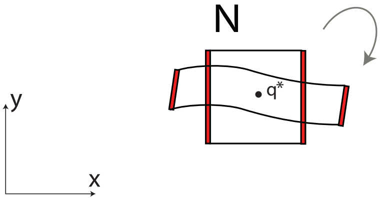

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Covering relations

Our goals

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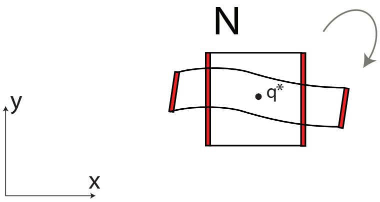

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If $N \overset{F}{\Rightarrow} N$ *then* $\exists q \in N$

$$F(q) = q$$

Questions:

- Uniqueness of a fixed point? $F(q^*) = q^*$
- unstable manifold?
- stable manifold?

Covering relations

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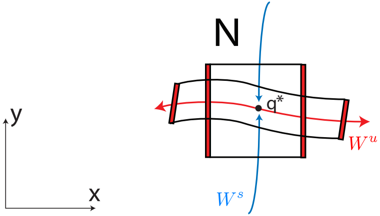

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- Uniqueness of a fixed point? $F(q^*) = q^*$
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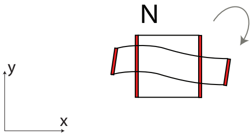
- stable manifold?

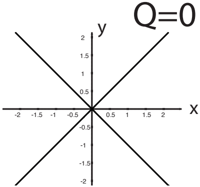
Cones



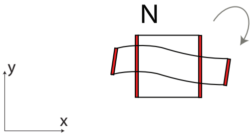
$$N \overset{F}{\Rightarrow} N$$

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$

$$Q(x, y) = \|x\|^2 - \|y\|^2$$



Cones



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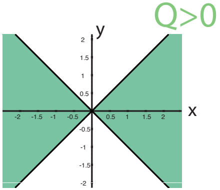
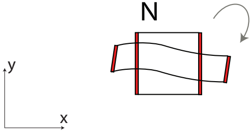


Cones

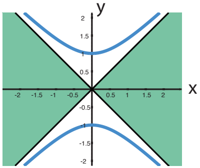


$$N \stackrel{F}{\Rightarrow} N$$

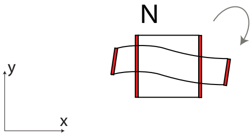
$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$

$$Q(x, y) = \|x\|^2 - \|y\|^2$$

$$Q = b < 0$$



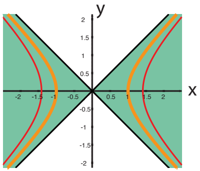
Cones



$$N \overset{F}{\Rightarrow} N$$

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$
$$Q(x, y) = \|x\|^2 - \|y\|^2$$

$$Q = a, Q = b, \quad a > b > 0$$



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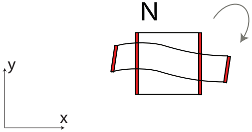

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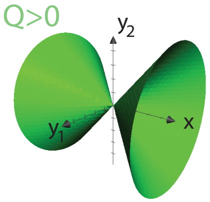
$$u = 1, \quad s = 2$$



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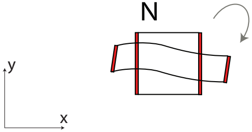

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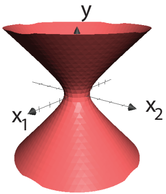
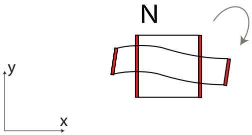
$$Q(x, y) = \|x\|^2 - \|y\|^2$$

$$u = 2, s = 1 \quad Q = a > 0$$



Cone conditions



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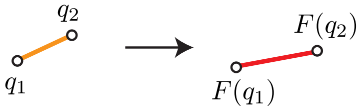
$$F : \mathbb{R}^n \to \mathbb{R}^n$$

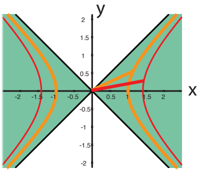
Definition (forward cone cond.)

$m > 1$. If $Q(q_1 - q_2) \geqslant 0$ then

$$Q(F(q_1) - F(q_2)) > mQ(q_1 - q_2)$$



$Q = a$, $Q = b$, $\quad a > b > 0$



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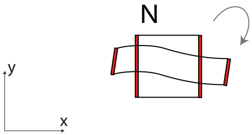

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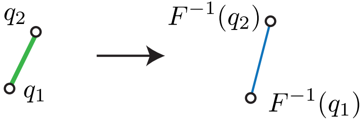
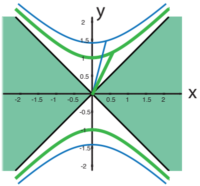
Definition (backward cone cond.)

$m > 1$. If $Q(q_1 - q_2) \leqslant 0$ then

$$Q(F^{-1}(q_1) - F^{-1}(q_2)) < mQ(q_1 - q_2)$$



$Q = a$, $Q = b$, $a < b < 0$



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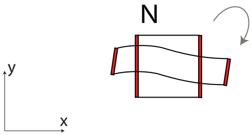

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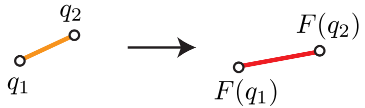

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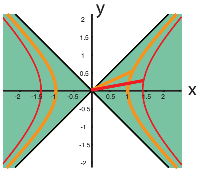
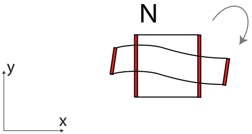


Our aim: W^s

Assumptions:

- Covering:

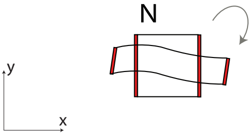


$$N \overset{F}{\Longrightarrow} N$$

Our aim: W^s

Assumptions:

- Covering:



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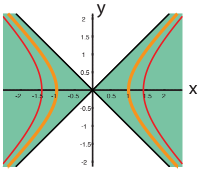

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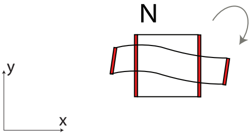

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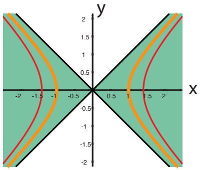
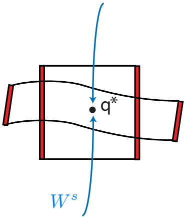
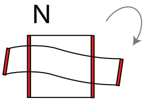
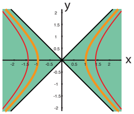


Claim: $\exists\, W^s$



Horizontal discs





$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = \overline{B_u} \times \overline{B_s}$$

$$N \stackrel{F}{\Rightarrow} N$$

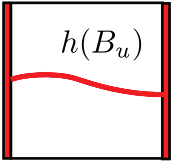
Definition (horizontal disc)

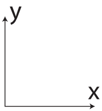
$$h : B_u \to N$$

- $\pi_x h(x) = x$

cone conditions:

- $x_1 \neq x_2$ then $Q(h(x_1) - h(x_2)) > 0$





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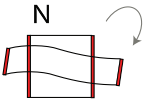

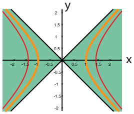

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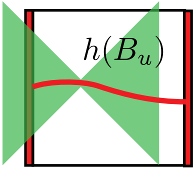

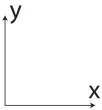

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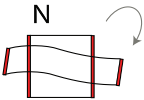
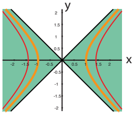
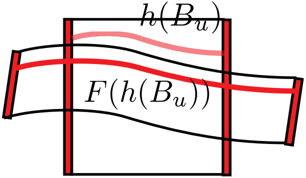




$$N \overset{F}{\Rightarrow} N$$

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = B_u \times B_s$$

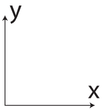


Lemma

There exists $h^* : B_u \to N$

$$F(h(B_u)) \cap N = h^*(B_u)$$

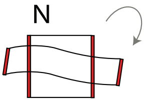
Proof. chalk.

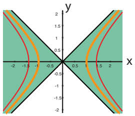


Stable manifold theorem

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = B_u \times B_s$$



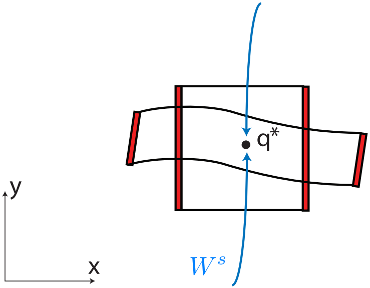


$$N \overset{F}{\Rightarrow} N$$

Lemma

There exists $h^* : B_u \to N$

$$F(h(B_u)) \cap N = h^*(B_u)$$



Theorem

There exists W^s *such that for any* $q \in W^s$

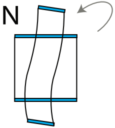
$$F^n(q) \in N \quad n \geqslant 0$$

Proof. chalk.

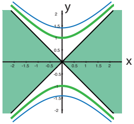
Unstable manifold theorem

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = B_u \times B_s$$



$$N \overset{F^{-1}}{\Rightarrow} N$$

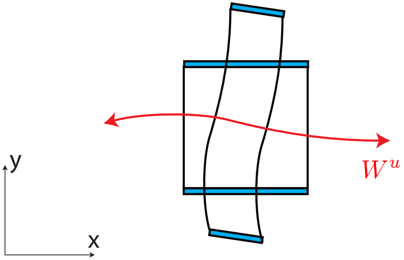


- F invertible.
- F^{-1} satisfies covering and cone conditions.

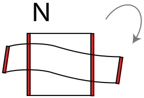
Theorem

There exists W^u *such that for any* $q \in W^u$

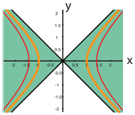
$$F^{-n}(q) \in N \quad n \geqslant 0$$

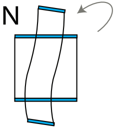


Summing up

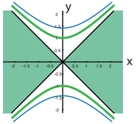


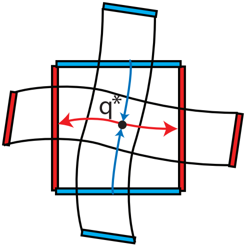
$$N \overset{F}{\Rightarrow} N$$



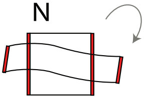


$$N \overset{F^{-1}}{\Rightarrow} N$$

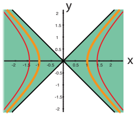


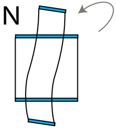


Summing up



$N \overset{F}{\Rightarrow} N$





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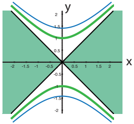
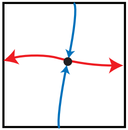


Theorem (Brouwer)

If $f : N \to N$ is C^0 then $\exists q \in N$

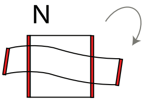
$$f(q) = q$$



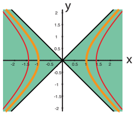
Exercise:

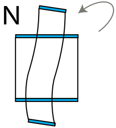
- horizontal and vertical discs intersect
- W^u is indeed unstable
- W^s is indeed stable

Tightening bounds

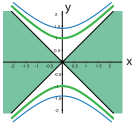


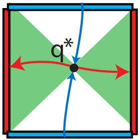
$N \overset{F}{\Rightarrow} N$





$N \overset{F^{-1}}{\Rightarrow} N$





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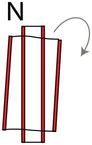

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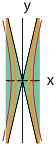

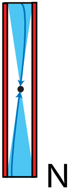

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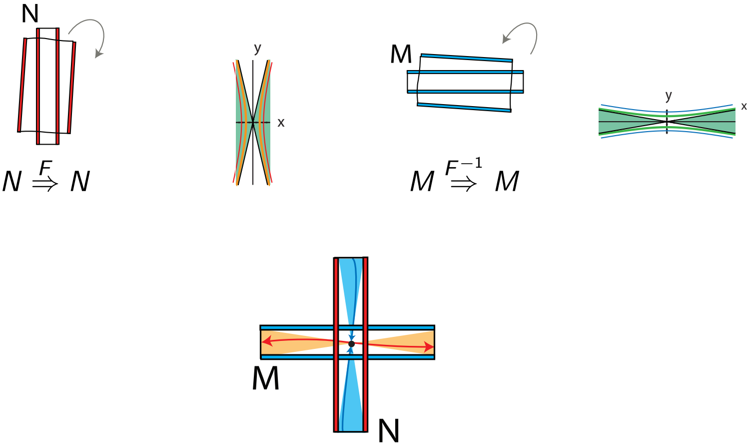
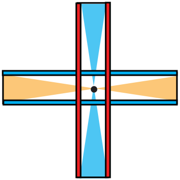


Transversal intersections of manifolds



Theorem (Smale-Birkhoff)

The system has a topological horseshoe (is chaotic).

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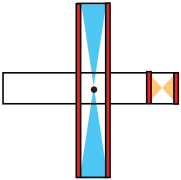

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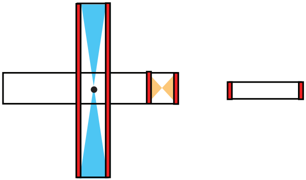

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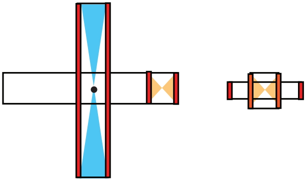

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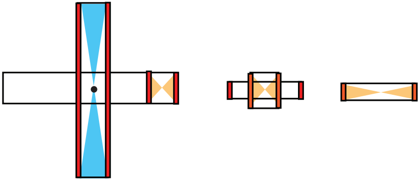

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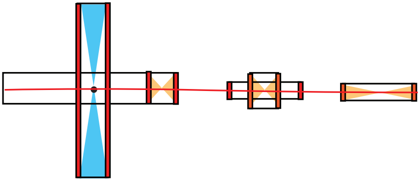

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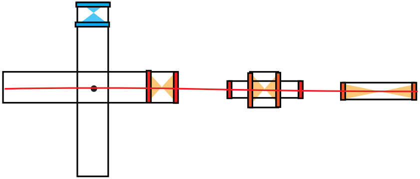

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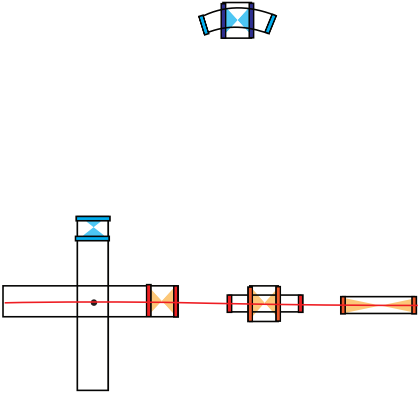

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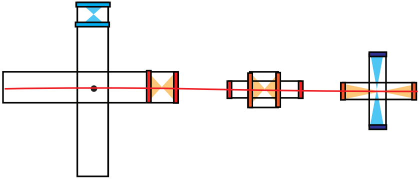

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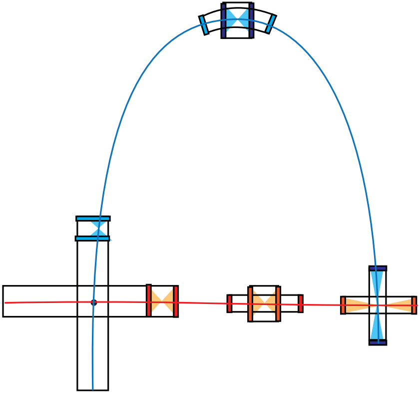

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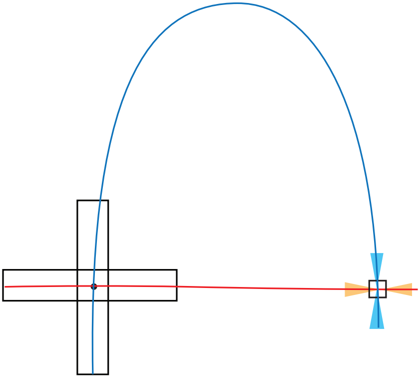

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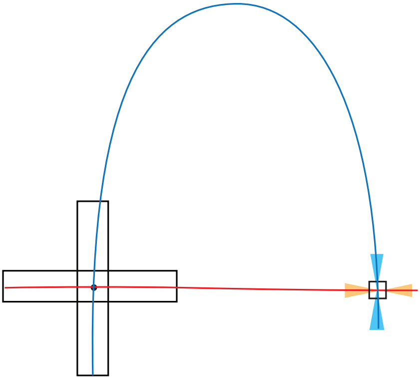
Next lectures



- Not only fixed points
- How to verify the covering condition
- How to verify cone conditions
- Examples

Thank you for your attention

References

- Covering relations:

[GZ] M. Gidea, P.Zgliczyński, Covering relations for multidimensional dynamical systems I, , J. of Diff. Equations, 202(2004) 32–58

- Stable and unstable manifolds of hyperbolic fixed points:

[Z] P.Zgliczyński, Covering relations, cone conditions and stable manifold theorem , J. of Diff. Equations 246 (2009) 1774–1819