Geometric methods for invariant manifolds in dynamical systems I.

Fixed points and periodic orbits

Maciej Capiński

AGH University of Science and Technology, Kraków

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M. Gidea, T. Kapela, C. Simó, P. Roldán, D. Wilczak, P. Zgliczyński

Plan of the lecture

- Motivation
- Examples of methodology that we shall use
- Brouwer theorem
- Interval Newton method
- Covering relations
- Example of application

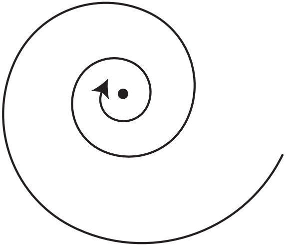
Motivation

ODEs

$\dot{x} = f(x)$

PDEs

- Fixed point $f(p) = 0$



$f : \mathbb{R}^2 \to \mathbb{R}^2$

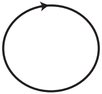
Motivation

ODEs

$$\dot{x} = f(x)$$

PDEs

- Periodic orbit



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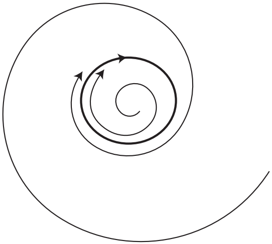

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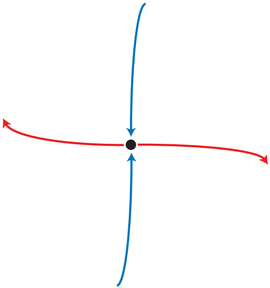
$\dot{x} = f(x)$

PDEs

- Stable, unstable manifolds



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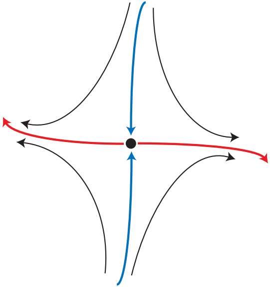

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Motivation

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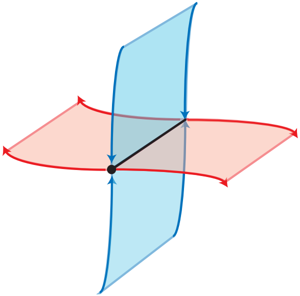
- Normally hyperbolic manifolds



$$f : \mathbb{R}^2 \times \mathbb{S}^1 \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$$

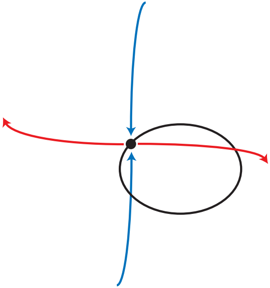
Motivation

ODEs

$\dot{x} = f(x)$

PDEs

- Normally hyperbolic manifolds



$f : \mathbb{R}^3 \to \mathbb{R}^3$

Motivation

ODEs

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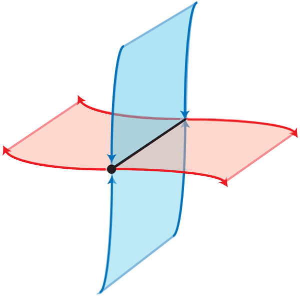
PDEs

- Normally hyperbolic manifolds



$$f : \mathbb{R}^n \to \mathbb{R}^n$$

Motivation

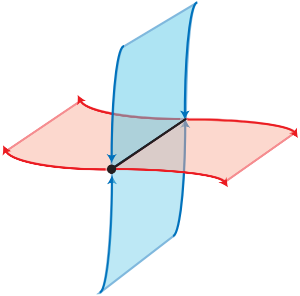
ODEs

$$\dot{x} = f(x)$$

PDEs

$$u_t = Lu + N$$

- Normally hyperbolic manifolds



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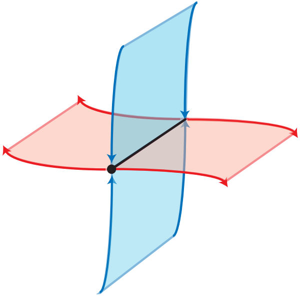


$$f : \mathbb{R}^n \to \mathbb{R}^n$$

PDEs

$$u_t = Lu + N$$

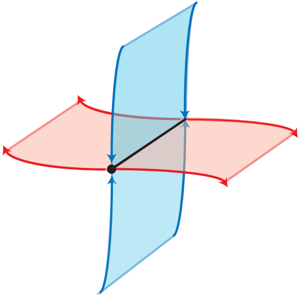
$$u(t,x) = \sum_k a_k(t) e^{ik\pi x}$$

Motivation

ODEs

$$\dot{x} = f(x)$$

- Normally hyperbolic manifolds



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$$u_t = Lu + N$$

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Take $\mathbf{a} = (a_i, \ldots, a_{i+n})$

$$\dot{\mathbf{a}} = f(\mathbf{a}) + R$$

If $\dot{a}_k < 0$ for $k \notin \{i, \ldots, i+n\}$

Motivation

ODEs

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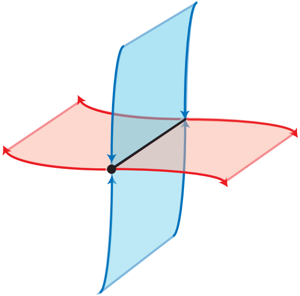

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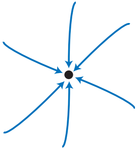
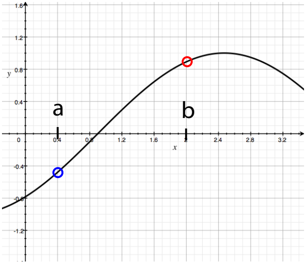


Kinds of tools that we shall use

Bolzano theorem

$$f : \mathbb{R} \to \mathbb{R} \qquad\qquad f(x) \stackrel{?}{=} 0$$



$$f(a) < 0 \qquad f(b) > 0$$

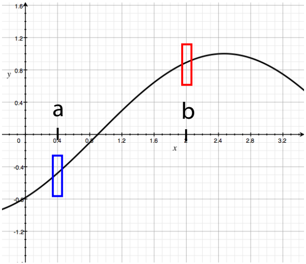
- There exists an x^* in (a, b) such that

$$f(x^*) = 0$$

Kinds of tools that we shall use

Bolzano theorem - no need to be too accurate

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) \overset{?}{=} 0$$



$$f(a) < 0 \qquad f(b) > 0$$

- There exists an x^* in (a, b) such that

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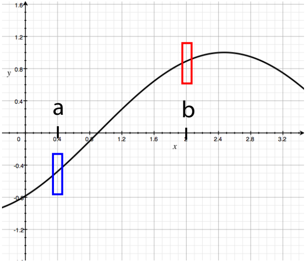
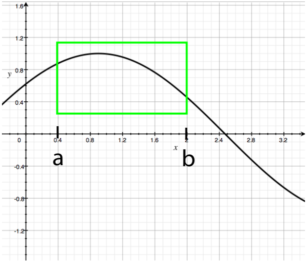
Kinds of tools that we shall use

Bolzano theorem - some more information

$f : \mathbb{R} \to \mathbb{R}$



$f' : \mathbb{R} \to \mathbb{R}$



$f(a) < 0 \qquad f(b) > 0 \qquad f'(x) > 0, \quad x \in [a, b]$

- There exists a **unique** x^* in (a, b) such that

$$f(x^*) = 0$$

Kinds of tools that we shall use

Interval arithmetic

computations on intervals:

$$[1,2]+[3,4]=[4,6]$$
$$[1,2]-[3,4]=[-3,-1]$$
$$[1,2]*[3,4]=[3,12]$$
$$[1,2]/[3,4]=[\frac{1}{4},\frac{2}{3}]$$
$$[1,2]^{[3,4]}=[1^3,2^4]$$

...

extends to higher dimensions

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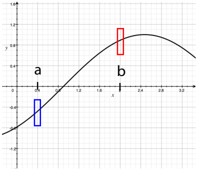

$f:\mathbb{R}^n \to \mathbb{R}^n$

What can be computed:

- $[f(U)]$
- $[Df(U)]$
- higher order derivatives
- linear algebra; eg. $[A^{-1}]$

CAPD library

Kinds of tools that we shall use

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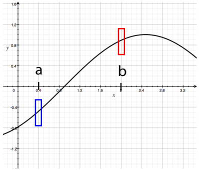

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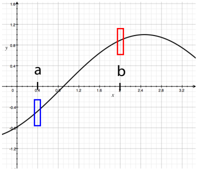

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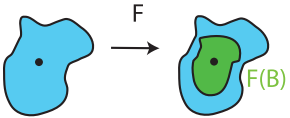
Brouwer theorem

$$\dot{x} = f(x) \qquad\qquad f(x^*) \overset{?}{=} 0$$

Theorem (Brouwer theorem)

If $F : B \rightarrow B$ is continuous, then there exists a $q \in B$

$$F(q) = q$$



Interval Newton method

$$F : \mathbb{R}^n \to \mathbb{R}^n, \qquad C^1$$

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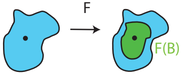

$$B = \Pi_{i=1}^n [a_i, b_i]$$

$$x_0 \in B$$

Theorem (interval Newton)

If

$$x_0 - [DF(B)]^{-1} F(x_0) \subset B$$

Then $\exists! x^* \in B$ such that

$$F(x^*) = 0$$

(intuition) Newton-Raphson:

$$F : \mathbb{R} \to \mathbb{R}$$

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

Proof. chalk.

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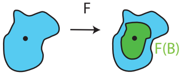

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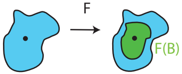

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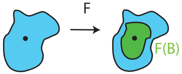

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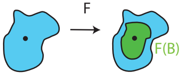

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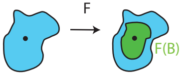

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Interval Newton method

Example - Hénon map

Theorem (interval Newton)

$x_0 - [DF(B)]^{-1}F(x_0) \subset B.$

Then $\exists! x^* \in B, \quad F(x^*) = 0.$

$$h(x,y) = (1 - ax^2 + y, bx)$$

$a = 1.4, b = 0.3$

Fixed point:

$$h(x,y) = (x,y)$$

$$F(x,y) = (1 - ax^2 + y - x, bx - y) = 0$$

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IVector x0(2);
x0[0] = sqrt(609)/28-0.25;
x0[1] = b*x0[0];

IVector B(2);
B[0] = x0[0] + interval(-1,1)*power(10,-14);
B[1] = x0[1] + interval(-1,1)*power(10,-14);

IVector N = x0 - gauss( F[B] , F(x0) );

if( subsetInterior(N,B) ) return 1;
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IVector N = x0 - gauss( F[B] , F(x0) );

if( subsetInterior(N,B) ) return 1;
```

Interval Newton method

Example - Hénon map

Theorem (interval Newton)

$x_0 - [DF(B)]^{-1}F(x_0) \subset B.$

Then $\exists! x^* \in B, \quad F(x^*) = 0.$

$$F(x, y) = (1 - ax^2 + y - x, bx - y) = 0$$

```
IMap F = "par:a,b;
          var:x,y;
          fun:1-a*x^2+y-x,b*x-y;";

IVector x0(2);
x0[0] = sqrt(609)/28-0.25;
x0[1] = b*x0[0];

IVector B(2);
B[0] = x0[0] + interval(-1,1)*power(10,-14);
B[1] = x0[1] + interval(-1,1)*power(10,-14);

IVector N = x0 - gauss( F[B] , F(x0) );

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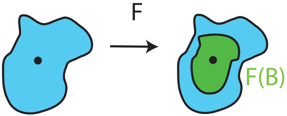
```

From ODEs to maps

Theorem (Brouwer theorem)

If $F : B \to B$ is continuous, then there exists a $q \in B$

$$F(q) = q$$



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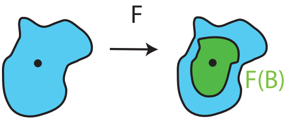

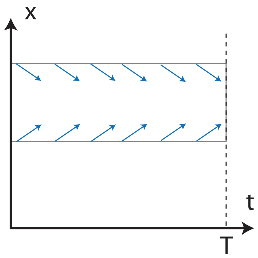

From ODEs to maps

Theorem (Brouwer theorem)

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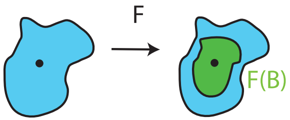
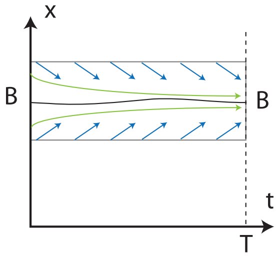
$$F(q) = q$$

$$\dot{x} = f(x, t) \qquad P(q) = q$$





From ODEs to maps

Poincaré map

$$\dot{x} = f(x)$$
$$x(0) = x_0$$

$$f : \mathbb{R}^n \to \mathbb{R}^n$$
$$V \subset \mathbb{R}^n$$

Flow

$$\phi(t, x_0) = x(t)$$

Time T-shift map

$$P(x) = \phi(T, x)$$

Poincaré map:

$$P : V \to V$$

$$P(x) = \phi(\tau(x), x)$$

From ODEs to maps

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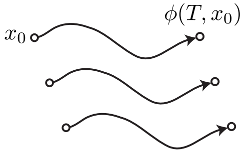

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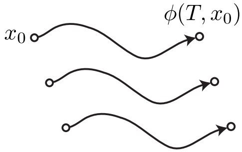

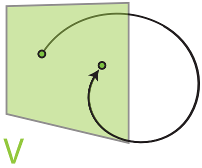

Periodic orbits of ODEs

Interval Newton method

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$$P : V \to V$$
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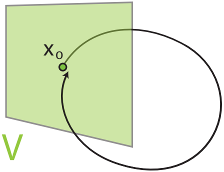

$$P(x^*) = x^*$$

Theorem

If

$$x_0 - [DF(B)]^{-1} F(x_0) \subset B$$

Then $\exists! x^* \in B$ *such that*

$$F(x^*) = 0$$

$$F(x) = P(x) - x$$

$$x_0 - [DP(B) - Id]^{-1}(P(x_0) - x_0) \subset B$$

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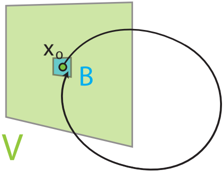

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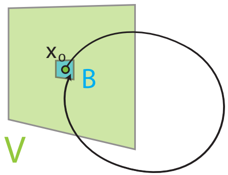

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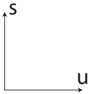
$$x_0 - [DP(B) - Id]^{-1} (P(x_0) - x_0) \subset B$$

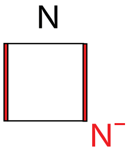
Topological covering

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = \overline{B_u} \times \overline{B_s}$$

$$N^- = \partial B_u \times \overline{B_s}$$
$$N^+ = B_u \times \partial B_s$$





Definition (covering)

$$N \overset{F}{\Rightarrow} N$$

- $\pi_u F(N^-) \cap \overline{B_u} = \emptyset$
- $\pi_s F(N) \subset B_s$
- $\exists q_0 \in N$ s.t. $F(q_0) \in \mathrm{int} N^{(*)}$

Theorem

There exists a point $p \in N$ such that

$$F(p) = p$$

Proof. chalk.

(*) stronger conditions needed: $\exists h : [0,1] \times N \to \mathbb{R}^n$, homotopy, such that (see [GZ] for details):
$h_0 = F$, $h_\lambda(N^-) \cap N = \emptyset$, $h_\lambda(N) \cap N^+ = \emptyset$ $h_1 = (A, 0)$, where A is a matrix s.t. $\overline{B_u} \subset AB_u$

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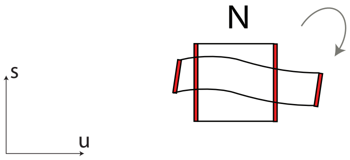

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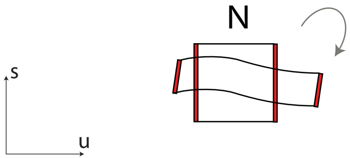

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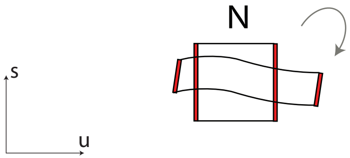

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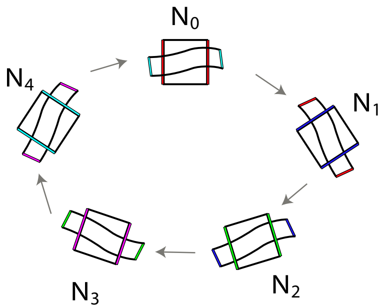
Correctly aligned windows



Theorem

$$N_0 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} \ldots \overset{F}{\Rightarrow} N_0$$

Then there exists a periodic orbit passing through the sets.

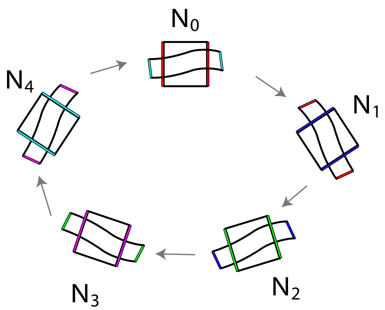
Proof.

$$\mathbf{N} = N_0 \times \ldots \times N_k$$

$$\mathbf{F}(x_0, \ldots, x_k) = (F(x_k), F(x_0), \ldots, F(x_{k-1}))$$

$$\mathbf{N} \overset{\mathbf{F}}{\Rightarrow} \mathbf{N}$$

Correctly aligned windows



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$$\mathbf{N} \overset{\mathbf{F}}{\Rightarrow} \mathbf{N}$$

Chaos

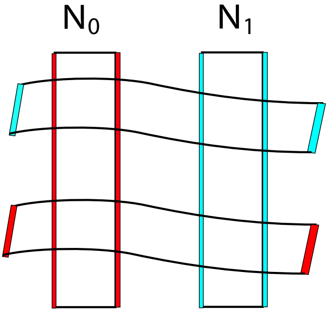
Two correctly aligned windows

$$N_0 \overset{F}{\Rightarrow} N_1$$

$$N_0 \overset{F}{\Rightarrow} N_0$$

$$N_1 \overset{F}{\Rightarrow} N_0$$

$$N_1 \overset{F}{\Rightarrow} N_1$$



- For any sequence of zeros and ones we have

0	0	1	0	1	1	...
$F^0(p) \in N_0$	$F^1(p) \in N_0$	$F^2(p) \in N_1$	$F^3(p) \in N_0$	$F^4(p) \in N_1$	$F^5(p) \in N_1$	...

Chaos

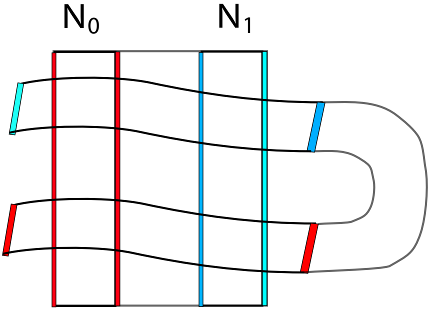
Horseshoe

$$N_0 \overset{F}{\Rightarrow} N_1$$

$$N_0 \overset{F}{\Rightarrow} N_0$$

$$N_1 \overset{F}{\Rightarrow} N_0$$

$$N_1 \overset{F}{\Rightarrow} N_1$$



- Orbits of any prescribed sequences of zeros and ones.
- Periodic orbits of any period. For example:

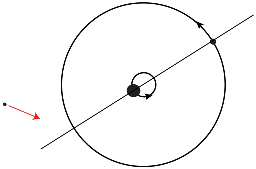
$$N_0 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_0$$

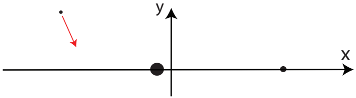
Example of application

The three body problem

$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$

$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$



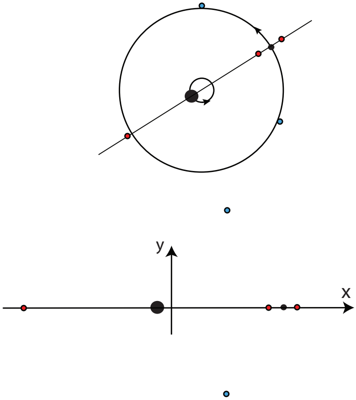


- Two planets rotate on circular orbits around center of mass
- Third, small, massless particle does not influence their motion

The three body problem

Fixed points

$$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$$
$$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$$



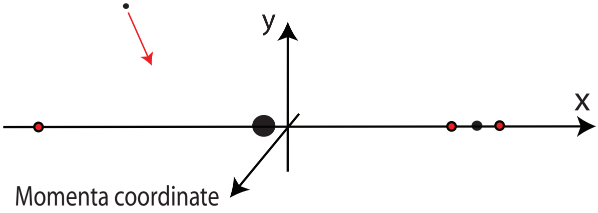
- We can position a satellite in 5 points and it will remain motionless

The three body problem

The problem is three dimensional

$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$

$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$

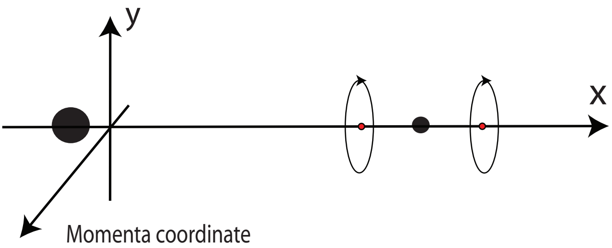


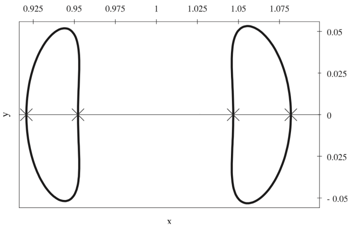
- Coordinates $x, y, \dot{x}, \dot{y}$
- (Conservation of energy reduces the dimension by one)

The three body problem

Periodic orbits

$$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$$
$$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$$

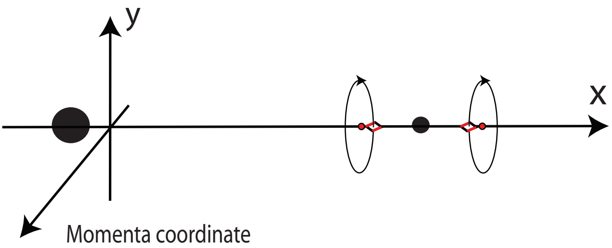


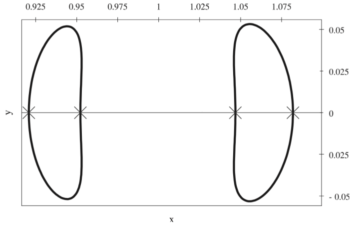


The three body problem

Where is the map and windows?

$$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$$
$$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$$



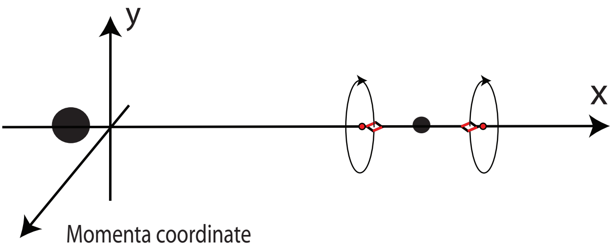


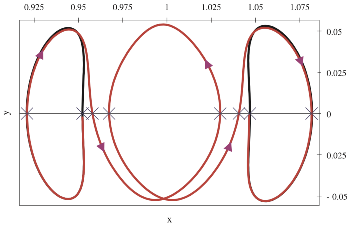
The three body problem

Transition from one window to another

$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{F}{\Rightarrow} N_0$

$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{F}{\Rightarrow} N_1$

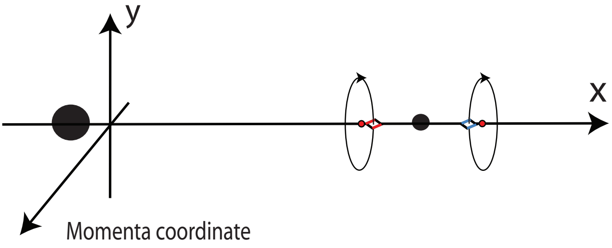




The three body problem

Chaos in celestial mechanics

$$N_0 \overset{F}{\Rightarrow} N_0 \qquad N_1 \overset{f}{\Rightarrow} N_0$$
$$N_0 \overset{F}{\Rightarrow} N_1 \qquad N_1 \overset{f}{\Rightarrow} N_1$$



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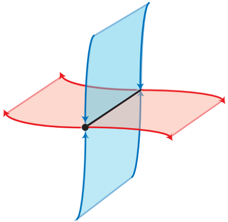
[WZ] D. Wilczak, P. Zgliczyński, Comm. Math. Phys. 2003, 2005

Closing remarks

We can:

- compute fixed points
- compute periodic orbits
- prove chaotic dynamics



next lectures

Thank you for your attention

References

- Interval Newton method:

[N] A. Neumeier, Interval methods for systems of equations. Cambridge University Press, 1990.

- Covering relations:

[GZ] M. Gidea, P.Zgliczyński, Covering relations for multidimensional dynamical systems I, J. of Diff. Equations, 202(2004) 32–58

- 3 body problem example:

[WZ] D. Wilczak, P.Zgliczyński, Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - A Computer Assisted Proof, Comm. Math. Phys. 234 (2003) 1, 37-75.