Aubry Mather Theory from a Topological Viewpoint

II. Topological approach to Aubry-Mather theory

Marian Gidea[1,2]

[1]Northeastern Illinois University, Chicago

[2]Institute for Advanced Study, Princeton

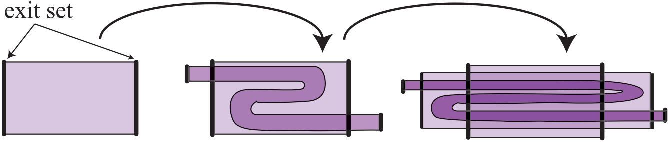
WORKSHOP ON INTERACTIONS BETWEEN DYNAMICAL SYSTEMS AND PARTIAL DIFFERENTIAL EQUATIONS (JISD2012)

May 28 - June 1, 2012

Outline

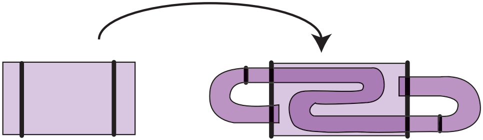
Correctly aligned windows (2-dimensional)

- **Window:** $W = c([0,1] \times [0,1])$, where c is a C^0-coordinate system; exit set $W^{\mathrm{ex}} = c(\partial[0,1] \times [0,1])$; entry set $W^{\mathrm{en}} = c([0,1] \times \partial[0,1])$
- **Definition:** W_1 correctly aligned with W_2 under f, if there exist $0 \leq a < b \leq 1$ such that, via coordinates:
  (i) $f([a,b] \times [0,1]) \subseteq \mathbb{R} \times (0,1)$
  (ii) $f(\{a\} \times [0,1]) \subseteq \{x < 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x > 1\}$
- Weak alignment if instead of (ii)
  (ii)' $f(\{a\} \times [0,1]) \subseteq \{x \leq 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x \geq 1\}$
- **Theorem:** Correct alignment is robust. Weak alignment is not.
- **Theorem:** Given $(W_i)_{i\in\mathbb{Z}}$, $(f_i)_{i\in\mathbb{Z}}$, such that W_i correctly aligned with W_{i+1} under f_i, then $\exists\,(p_i)_{i\in\mathbb{Z}}$, $p_i \in W_i$ and $f_i(p_i) = p_{i+1}$
  If $(W_i)_{i=1,\dots,d}$ form a loop, then $\exists$ a closed orbit.



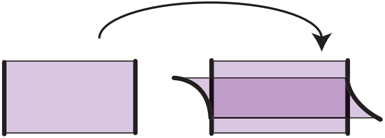
Correctly aligned windows (2-dimensional)

- **Window:** $W = c([0,1] \times [0,1])$, where c is a C^0-coordinate system; exit set $W^{\mathrm{ex}} = c(\partial[0,1] \times [0,1])$; entry set $W^{\mathrm{en}} = c([0,1] \times \partial[0,1])$
- **Definition:** W_1 correctly aligned with W_2 under f, if there exist $0 \leq a < b \leq 1$ such that, via coordinates:
  (i) $f([a,b] \times [0,1]) \subseteq \mathbb{R} \times (0,1)$
  (ii) $f(\{a\} \times [0,1]) \subseteq \{x < 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x > 1\}$
- Weak alignment if instead of (ii)
  (ii)' $f(\{a\} \times [0,1]) \subseteq \{x \leq 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x \geq 1\}$
- **Theorem:** Correct alignment is robust. Weak alignment is not.
- **Theorem:** Given $(W_i)_{i\in\mathbb{Z}}$, $(f_i)_{i\in\mathbb{Z}}$, such that W_i correctly aligned with W_{i+1} under f_i, then $\exists\,(p_i)_{i\in\mathbb{Z}}$, $p_i \in W_i$ and $f_i(p_i) = p_{i+1}$
  If $(W_i)_{i=1,\ldots,d}$ form a loop, then $\exists$ a closed orbit.



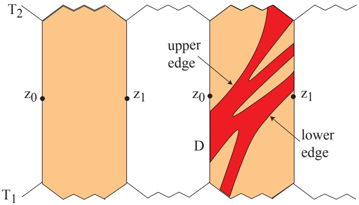
Correctly aligned windows (2-dimensional)

- **Window:** $W = c([0,1] \times [0,1])$, where c is a C^0-coordinate system; exit set $W^{\rm ex} = c(\partial[0,1] \times [0,1])$; entry set $W^{\rm en} = c([0,1] \times \partial[0,1])$
- **Definition:** W_1 correctly aligned with W_2 under f, if there exist $0 \leq a < b \leq 1$ such that, via coordinates:
  (i) $f([a,b] \times [0,1]) \subseteq \mathbb{R} \times (0,1)$
  (ii) $f(\{a\} \times [0,1]) \subseteq \{x < 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x > 1\}$
- Weak alignment if instead of (ii)
  (ii)' $f(\{a\} \times [0,1]) \subseteq \{x \leq 0\}$ and $f(\{b\} \times [0,1]) \subseteq \{x \geq 1\}$
- **Theorem:** Correct alignment is robust. Weak alignment is not.
- **Theorem:** Given $(W_i)_{i\in\mathbb{Z}}$, $(f_i)_{i\in\mathbb{Z}}$, such that W_i correctly aligned with W_{i+1} under f_i, then $\exists\, (p_i)_{i\in\mathbb{Z}}$, $p_i \in W_i$ and $f_i(p_i) = p_{i+1}$
  If $(W_i)_{i=1,\dots,d}$ form a loop, then $\exists$ a closed orbit.



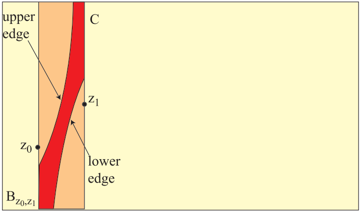
Diagonal sets in the closed annulus

- $A = \mathbb{T}^1 \times [0,1]$
- $B_{z_0,z_1} = \{w \in A \,|\, \pi_x(z_0) < \pi_x(w) < \pi_x(z_1)\}$
- $I_z = \{w \in A \,|\, \pi_x(w) = \pi_x(z)\}$, $I_z^+ = \{w \in I_z \,|\, \pi_y(w) \geq \pi_y(z)\}$, $I_z^- = \{w \in I_z \,|\, \pi_y(w) \leq \pi_y(z)\}$
- $D \subseteq \mathrm{cl}(B_{z_0,z_1})$ positive diagonal in B_{z_0,z_1}:
  - (i) D is simply connected and the closure of its interior;
  - (ii) $\partial D \cap \mathrm{cl}(B_{z_0,z_1}) \subseteq I_{z_0}^- \cup I_{z_1}^+ \cup \{y = 0\} \cup \{y = 1\}$;
  - (iii) $\partial D \cap I_{z_0}^- \neq \emptyset$ and $\partial D \cap I_{z_1}^+ \neq \emptyset$.
- $\partial D \cap B_{z_0,z_1}$ has two components connecting $I_{z_0}^- \cup \{y=0\}$ to $I_{z_1}^+ \cup \{y = 1\}$ — upper and lower edges



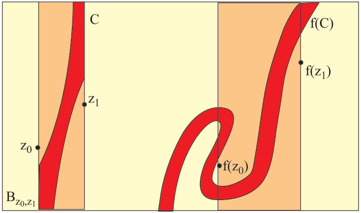
Twist maps and diagonal sets

- $f : A \to A$ orientation preserving, area preserving, boundary preserving, exact symplectic, monotone twist map of the annulus
- D positive diagonal in $B_{z_0,z_1} \Rightarrow f(D)$ has a positive diagonal component D' in $B_{f(z_0),f(z_1)}$
- If D has the upper/lower edge in $f^k(I^+_{w_0}), f^k(I^-_{w_1})$ and $\mathrm{bd}(D \cap B_{z_0,z_1}) \subseteq f^k(I^+_{w_0} \cup I^-_{w_1})$ for some w_0, w_1 and some $k > 0 \Rightarrow D'$ has upper/lower edge in $f^{k+1}(I^+_{w_0}), f^{k+1}(I^-_{w_1})$



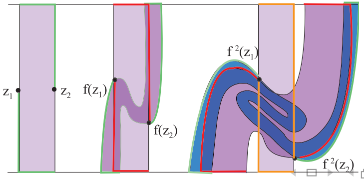
Twist maps and diagonal sets

- $f : A \to A$ orientation preserving, area preserving, boundary preserving, exact symplectic, monotone twist map of the annulus
- D positive diagonal in $B_{z_0,z_1} \Rightarrow f(D)$ has a positive diagonal component D' in $B_{f(z_0),f(z_1)}$
- If D has the upper/lower edge in $f^k(I^+_{w_0}), f^k(I^-_{w_1})$ and $\mathrm{bd}(D \cap B_{z_0,z_1}) \subseteq f^k(I^+_{w_0} \cup I^-_{w_1})$ for some w_0, w_1 and some $k > 0 \Rightarrow D'$ has upper/lower edge in $f^{k+1}(I^+_{w_0}), f^{k+1}(I^-_{w_1})$



Diagonal sets as windows

- Windows:
  - $W_1 = B_{z_1,z_2} = \{z : \pi_x(z_1) < \pi_x(z) < \pi_x(z_2)\}$;
    exit set $W_1^{\mathrm{ex}} = B_{z_1,z_2} \cap [(I^-_{z_1} \cup \{y = 0\}) \cup (I^+_{z_2} \cup \{y = 1\})]$
  - $W_2 = B_{f(z_1),f(z_2)} = \{z : \pi_x(f(z_1)) < \pi_x(z) < \pi_x(f(z_2))\}$;
    exit set $W_2^- = B_{f(z_1),f(z_2)} \cap [(I^-_{f(z_1)} \cup \{y = 0\}) \cup (I^+_{f(z_2)} \cup \{y = 1\})]$
- Then W_1 is correctly aligned with W_2 under f
- If W_1 (weakly) correctly aligned with W_2, ..., W_{k-1} (weakly) correctly aligned with W_k, then $f^{k-1}(W_1) \cap W_k$ determines a diagonal set in W_k
- Trivial remark: if z_1, z_2 are monotone (p, q)-points then there exists a loop of q weakly correctly aligned windows



Numerical implementation.

The diagonal method can be implemented in efficient numerical algorithms – Project M. Capinski and M.G.

- $(u_n)_{n\in\mathbb{Z}}$ left sequence – $\pi_x(f^{-1}(u_{i+1})) \leq \pi_x(u_i)$ and $\pi_x(f(u_i)) \leq \pi_x(u_{i+1})$, $\forall i$
- $(v_n)_{n\in\mathbb{Z}}$ left sequence – $\pi_x(v_i) \leq \pi_x(f^{-1}(v_{i+1}))$ and $\pi_x(v_{i+1})) \leq \pi_x(f(v_i)$, $\forall i$, $\forall i$
- If $(u_n)_n$ left seq., $(v_n)_n$ right seq., and $\pi_x(u_i) \leq \pi_x(v_i)$, $\forall i$, then B_{u_i,v_i} is correctly aligned with $B_{u_{i+1},v_{i+1}}$ (robustly) under $f \Rightarrow$ $\exists z$, $\pi_x(u_i) \leq \pi_x(f^i(z)) \leq \pi_x(v_i)$, $\forall i$
- [Jungreis,1991] – nonexistence of invariant circles

Ends of a manifold

- $C \simeq S^1 \times \mathbb{R}$ — cylinder
- The cylinder has two ends $\{S, N\}$ †
- $A \cup \{S\} \cup \{N\} = S^2$ — end compactification ‡
- $A \simeq S^2 \setminus \{S, N\}$.

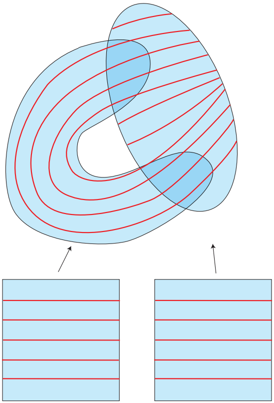
---

† End of a manifold M: function e to each compact set $K \subseteq M$ an unbounded non-empty component $e(K)$ of $M \setminus K$ s.t. $K_1 \subseteq K_2 \Rightarrow e(K_2) \subseteq e(K_1)$. $E(M) =$ the set of ends of M – the unbounded components of $M \setminus K$.

‡ End compactification of M: $\bar{M} = M \cup E(M)$ endowed with a topology satisfying: (i) M is an open subspace of $\bar{M}$, (ii) the fundamental open neighborhoods of $e \in E(M)$ are of the form $e(K) \cup \{e' \in E(M) \,|\, e'(K) = e(K)\}$ for all compacts $K \subseteq M$

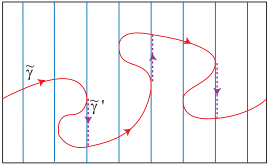
Foliations

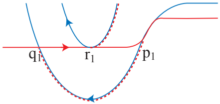
- M — manifold of dimension n
- Foliation atlas: foliation charts $\{\phi_i : U_i \subseteq M \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ s.t. $\phi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$
- Plaques: connected components of $\phi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$
- Leaves $\mathcal{F}$: connected, immersed q-dimensional manifolds formed with plaques $x, y \in F \in \mathcal{F}$: $\exists U_1, \ldots, U_k, x_0 = x, \ldots, x_i, \ldots, x_k = y$ s.t. x_{i-1}, x_i on the same plaque in U_i

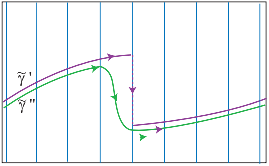


Foliation of the cylinder

- $\mathcal{V}$ – 1-dimensional foliation of C s.t. $\forall V \in \mathcal{V}$ embedded curve connecting S and N
- **Proposition:** There exists an essential circle T in C that meets each leaf $V \in \mathcal{V}$ transversally (exactly once)
- Proof. Start with essential circle γ
  Step 1. Modify $\gamma \rightsquigarrow$ essential circle γ' =finitely many curve segments of γ transverse $\mathcal{V}$ and finitely many curve segments included in $\mathcal{V}$
  Step 2. Modify the leaf-parts of $\gamma' \rightsquigarrow$ essential circle $\tilde{\gamma}''$ that is transverse to $\mathcal{V}$.
- **Coordinate system:** $c(x, y) = (\tau(x), c_x(y))$, where $x \in S^1$, $\tau(\mathbb{R}) = T$, $c_x(\mathbb{R}) = V_x$
- Foliation of the covering space $V_{x+n} = V_x + (n, 0)$

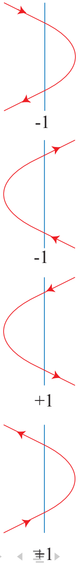






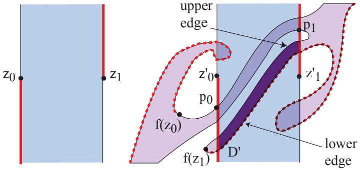
Twist and tilt conditions via foliation

- **Monotone twist:** f monotone twist condition if $f(S) = S$, $f(N) = N$, and $\forall V, V' \in \mathcal{V}$, $f(V)$ is transverse to V'
- If $f(V)$ intersecting V' in a finite set of points $\{z_0 = c_V(t_0), z_1 = c_V(t_1), \ldots, z_k = c_V(t_k)\}$ defined index $i_{[c_V(t_i), c_V(t_{i+1})]}$ as in Figure.
- **Tilt:** f right tilt condition if $f(S) = S$, $f(N) = N$, and for each $\forall V, V' \in \mathcal{V}$, with $f(V)$ intersecting V' in a finite set of points $\{z_0, z_1, \ldots, z_k\}$: $\sum_{i=0}^{j} i_{[c_V(t_i), c_V(t_{i+1})]} \leq 0$ for each $j = 1, \ldots, k-1$
- Note: $\sum_{i=0}^{k-1} i_{[c_V(t_i), c_V(t_{i+1})]} = 0$



Diagonal sets in the cylinder

- Notation: $z \in A$, V_z = the unique leaf in $\mathcal{V}$ passing through z
- $V^+_{z_i} = \{w \in V_z \,|\, w \text{ above } z_i \text{ in } V_{z_0}\}$,
  $V^-_{z_i} = \{w \in V_z \,|\, w \text{ below } z_i \text{ in } V_{z_0}\}$
- $B_{z_0,z_1} = \{V_w \,|\, V_w \text{ is between } V_{z_0} \text{ and } V_{z_1}\}$
- D positive diagonal in B_{z_0,z_1}
  - (i) D is simply connected and the closure of its interior;
  - (ii) $\partial D \cap \mathrm{cl}(B_{z_0,z_1}) \subseteq V^-_{z_0} \cup V^+_{z_1}$;
  - (iii) $\partial D \cap V^-_{z_0} \neq \emptyset$ and $\partial D \cap V^+_{z_1} \neq \emptyset$.
- **Definition:** Infinite window
  $W_{z_0,z_1} = \mathrm{cl}(B_{z_0,z_1})$, together with
  $W^{\mathrm{ex}}_{z_0,z_1} = V^-_{z_0} \cup V^+_{z_1}$
  $W^{\mathrm{en}}_{z_0,z_1} = V^+_{z_0} \cup V^-_{z_1}$



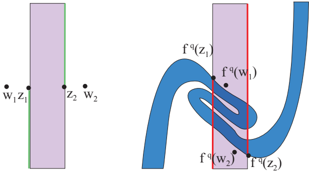
Existence of Aubry-Mather sets

- $f : A \to A$ satisfies the circle intersection property if for every lift $\tau : \mathbb{R} \to \mathbb{R}^2$ of an essential circle in the annulus A we have $f(\tau(\mathbb{R})) \cap \tau(\mathbb{R}) \neq \emptyset$
- In the sequel: assume $f : C \to C$ is orientation preserving, satisfies the circle intersection property, maps each end of the cylinder to itself, twists each end infinitely, preserves a measure absolutely continuous with respect to the Lebesgue measure, and satisfies a twist condition or a tilt condition relative to the foliation $\mathcal{V}$
- For every (p, q), there exists a monotone (p, q)-periodic orbit
- For each $\omega \in \mathbb{R}$, there exists an Aubry-Mather set Σ_ω (non-unique) of rotation number ω
    - Choose $p_n/q_n \to \omega \rightsquigarrow \exists z_n$ monotone (p_n, q_n)-periodic point with $\pi_x(z_n) \in [0, 1]$ — subsequence of z_n converges to z_ω monotone and $\rho(z_\omega) = \omega$

Existence of monotone (p, q)-orbits

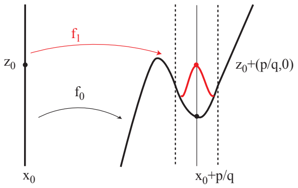
Idea: Existence of non-monotone orbits implies existence of monotone orbits

- Step 1: Given z_1, z_2, w_1, w_2
  - z_1, z_2 are (p, q)-points
  - $\forall i = 0, \ldots, q$, $\pi_x(f^i(z_1)) < \pi_x(f^i(z_2))$, $\pi_x(f^i(w_1)) < \pi_x(f^i(w_2))$, $\pi_x(f^i(w_1)) < \pi_x(f^i(z_2))$, $\pi_x(f^i(z_1)) < \pi_x(f^i(w_2))$,
  - $\pi_x(w_j) - \pi_x(z_j)$ and $\pi_x(f^q(z_j)) - \pi_x(f^q(z_j))$ have the same sign, for $j = 1, 2$
  - $O(w_1)$ turns around $O(z_1)$ at some point, and $O(w_2)$ turns around $O(z_2)$ at some point
  - Then B_{z_1,z_2} is correctly aligned with $B_{z_1+(p,0),z_2+(p,0)}$ under f^q. Hence, $f^q - (p, 0)$ has a fixed point in the interior of B_{z_1,z_2} and so does every map sufficiently close to f.



Existence of monotone (p, q)-orbits

- Step 2: If f has a monotone (p, q)-orbit and a non-monotone (p, q)-orbit, then f has a second monotone (p, q)-orbit and so does every map sufficiently close to f.
- Step 3: Assume w_0 is a non-monotone (p, q)-periodic orbit. Choose $z_0 \notin EO(w_0)$. There exists a small homotopy f_t with $f_0 = f$ and $f_t = f$ outside a finite collection of narrow vertical strips about $z_0 + i/q$, $i \in \mathbb{Z}$, and off $EO(w_0)$, s.t. $f_1^i(z_0) = z_0 + i(p/q, 0)$ – change the y-coordinate of images of points. Hence f_1 has a (p, q)-monotone periodic orbit.



Existence of monotone (p, q)-orbits

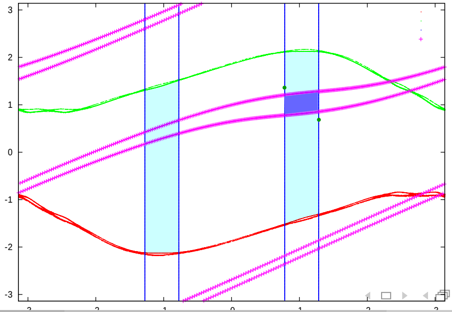
- Step 4:
  - Lemma: if $f_n \to f$ in C^0 as $n \to \infty$, with f_n, f twist maps (tilt maps), if z_n has a monotone orbit for f_n, and if $z_n \to z$, then z has a monotone orbit for f, and $\rho(z) = \lim_{n\to\infty} \rho(z_n)$
    $\Rightarrow$ the set $t \in [0, 1]$ for which f_t has a monotone (p, q)-orbit is an closed set.
  - By Step 2, the set $t \in [0, 1]$ for which f_t has a monotone (p, q) is an open set.
  - By connectedness $f_0 = f$ must have a (p, q)-monotone periodic orbit

Hall's shadowing lemma

- **Theorem (Hall):** Given $\{\Sigma_{\omega_s}\}_{s\in\mathbb{Z}}$. For every $\{n_s\}_{s=1,\ldots,m} \subseteq \mathbb{N}$, there exists an orbit $\{f^j(\zeta)\}$ that follows each Σ_{ω_s} for a number of n_s iterates, i.e.,

$$\pi_x(f^j(w_s^1)) < \pi_x(f^j(\zeta)) < \pi_x(f^j(w_s^2))$$

for n_s-consecutive j's, and for some $w_s^1, w_s^2 \in \Sigma_{\omega_s}$.
In particular, $h_{\mathrm{top}}(f) > 0$.

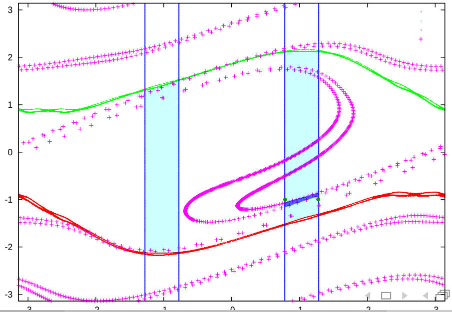


Hall's shadowing lemma

- **Theorem (Hall):** Given $\{\Sigma_{\omega_s}\}_{s\in\mathbb{Z}}$. For every $\{n_s\}_{s=1,\dots,m}\subseteq\mathbb{N}$, there exists an orbit $\{f^j(\zeta)\}$ that follows each Σ_{ω_s} for a number of n_s iterates, i.e.,

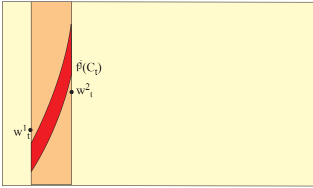
$$\pi_x(f^j(w_s^1))<\pi_x(f^j(\zeta))<\pi_x(f^j(w_s^2))$$

for n_s-consecutive j's, and for some $w_s^1, w_s^2\in\Sigma_{\omega_s}$.
In particular, $h_{\rm top}(f)>0$.



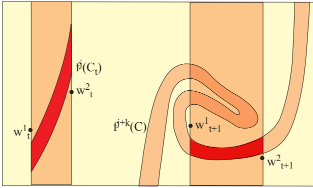
Proof of Hall's shadowing lemma

- Recursive argument:
  - start with adjacent points w_1^1, w_1^2 in Σ_{ω_1}
  - construct windows $B_{f^j(w_1^1)f^j(w_1^2)}$, $j = 1, n_1 - 1$, with $B_{f^j(w_1^1),f^j(w_1^2)}$ correctly aligned with $B_{f^{j+1}(w_1^1),f^{j+1}(w_1^2)}$ under f – each orbit that follows the windows satisfies the order relation
  - $\exists$ p_0 near $y = 0$ that gets near $y = 1$, and $\exists$ P_1 near $y = 1$ that gets near $y = 0$ $\Rightarrow$ for m_1 (sufficiently large) no. of iterates
  - then $B_{f^{n_1+k_1+m_1}(w_1^1),f^{n_1+k_1+m_1}(w_1^2)}$ is correctly aligned with $B_{f^{k_1+m_1}(p_0),f^{k_1+m_1}(p_1)}$, of width > 1
  - hence $B_{f^{n_1+k_1+m_1}(w_1^1),f^{n_1+k_1+m_1}(w_1^2)}$ is correctly aligned with $B_{w_2^1,w_2^2}$



Proof of Hall's shadowing lemma

- Recursive argument:
  - start with adjacent points w_1^1, w_1^2 in Σ_{ω_1}
  - construct windows $B_{f^j(w_1^1)f^j(w_1^2)}$, $j = 1, n_1 - 1$, with $B_{f^j(w_1^1),f^j(w_1^2)}$ correctly aligned with $B_{f^{j+1}(w_1^1),f^{j+1}(w_1^2)}$ under f – each orbit that follows the windows satisfies the order relation
  - $\exists$ p_0 near $y = 0$ that gets near $y = 1$, and $\exists$ P_1 near $y = 1$ that gets near $y = 0 \Rightarrow$ for m_1 (sufficiently large) no. of iterates
  - then $B_{f^{n_1+k_1+m_1}(w_1^1),f^{n_1+k_1+m_1}(w_1^2)}$ is correctly aligned with $B_{f^{k_1+m_1}(p_0),f^{k_1+m_1}(p_1)}$, of width > 1
  - hence $B_{f^{n_1+k_1+m_1}(w_1^1),f^{n_1+k_1+m_1}(w_1^2)}$ is correctly aligned with $B_{w_2^1,w_2^2}$

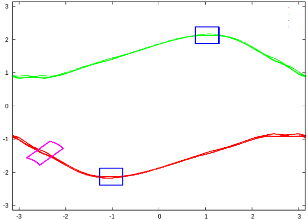


Two results

- **Proposition 1.** Given $\mathcal{Z}$ a BZI bounded by T_1 and T_2 (*not topologically transitive*). For any $z_1 \in T_1, z_2 \in T_2$, U_0, V_0 neighborhoods of z_1, z_2, there exists an orbit that goes from U_0 to V_0.
- **Proposition 2.** Given $\mathcal{Z}$ a BZI bounded by T_1, T_2, and $\{\Sigma_{\omega_s}\}_{i=1,\dots,m}$ vertically ordered. For any $z_1 \in T_1, z_2 \in T_2$, U_0 neighborhood of z_1, V_0 neighborhood of z_2, $\{n_s\}_{s=1,\dots,m}$ an increasing sequence in $\mathbb{N}$, there exists an orbit $f^n(z)$ that starts in an U_0, ends in V_0, and 'shadows' each Σ_{ω_s} for n_s iterates s.t.

$$\pi_x(f^j(w_s^1)) < \pi_x(f^j(\zeta)) < \pi_x(f^j(w_s^2))$$

for n_s-consecutive j's, and some $w_s^1, w_s^2 \in \Sigma_{\omega_s}$.

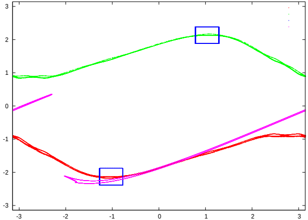


Two results

- **Proposition 1.** Given $\mathcal{Z}$ a BZI bounded by T_1 and T_2 (*not topologically transitive*). For any $z_1 \in T_1, z_2 \in T_2$, U_0, V_0 neighborhoods of z_1, z_2, there exists an orbit that goes from U_0 to V_0.
- **Proposition 2.** Given $\mathcal{Z}$ a BZI bounded by T_1, T_2, and $\{\Sigma_{\omega_s}\}_{i=1,\dots,m}$ vertically ordered. For any $z_1 \in T_1, z_2 \in T_2$, U_0 neighborhood of z_1, V_0 neighborhood of z_2, $\{n_s\}_{s=1,\dots,m}$ an increasing sequence in $\mathbb{N}$, there exists an orbit $f^n(z)$ that starts in an U_0, ends in V_0, and 'shadows' each Σ_{ω_s} for n_s iterates s.t.

$$\pi_x(f^j(w_s^1)) < \pi_x(f^j(\zeta)) < \pi_x(f^j(w_s^2))$$

for n_s-consecutive j's, and some $w_s^1, w_s^2 \in \Sigma_{\omega_s}$.

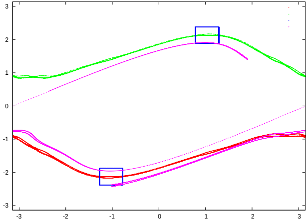


Two results

- **Proposition 1.** Given $\mathcal{Z}$ a BZI bounded by T_1 and T_2 (*not topologically transitive*). For any $z_1 \in T_1, z_2 \in T_2$, U_0, V_0 neighborhoods of z_1, z_2, there exists an orbit that goes from U_0 to V_0.
- **Proposition 2.** Given $\mathcal{Z}$ a BZI bounded by T_1, T_2, and $\{\Sigma_{\omega_s}\}_{i=1,\dots,m}$ vertically ordered. For any $z_1 \in T_1, z_2 \in T_2$, U_0 neighborhood of z_1, V_0 neighborhood of z_2, $\{n_s\}_{s=1,\dots,m}$ an increasing sequence in $\mathbb{N}$, there exists an orbit $f^n(z)$ that starts in an U_0, ends in V_0, and 'shadows' each Σ_{ω_s} for n_s iterates s.t.

$$\pi_x(f^j(w_s^1)) < \pi_x(f^j(\zeta)) < \pi_x(f^j(w_s^2))$$

for n_s-consecutive j's, and some $w_s^1, w_s^2 \in \Sigma_{\omega_s}$.



Proof of Proposition 1

Lemma [Kaloshin,2003] Suppose that T_1 and T_2 bound a BZI. Let Σ_ω be an Aubry-Mather set of rotation number ω inside the BZI. Let p be a recurrent point in Σ_ω and $W(p)$ be a neighborhood of p inside the BZI. The following hold true:

(i) For some positive number n^+ (resp. n^-) depending on $W(p)$ the set $\bigcup_{j=0}^{n^+} f^j(W(p))$ (resp. $\bigcup_{j=0}^{n^-} f^{-j}(W(p))$) separates the cylinder.

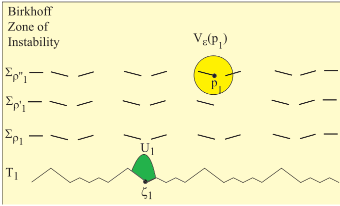
(ii) The set $W^{+\infty} := \bigcup_{j=0}^{\infty} f^j(W(p))$ (resp. the set $W^{-\infty} := \bigcup_{j=0}^{\infty} f^{-j}(W(p))$), is connected and open.

(iii) The closure of $W^{+\infty}$ (resp. $W^{-\infty}$) contains both boundary tori T_1 and T_2.

(iv) The set $W^{\infty} := \bigcup_{j=-\infty}^{\infty} f^j(W(p))$ is invariant, and both $W^{+\infty}$ and $W^{-\infty}$ are open and dense in W^{∞}.

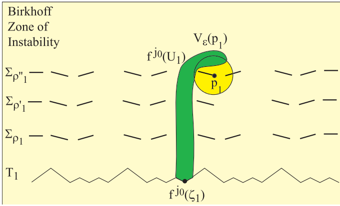
Proof of Proposition 1

- choose $\Sigma_{\rho_1} < \Sigma_{\rho_1'} < \Sigma_{\rho_1''}$ — A-M sets
- $W_\varepsilon(p_1)$ neighborhood of $p_1 \in \Sigma_{\rho_1''}$
- $\mathrm{cl}[\cup_{j=0}^{\infty} f^{-j}(W_\varepsilon(p_1))] \supseteq T_1$ [Kaloshin,2003]
- $\exists U_m \subseteq U_{m-1} \subseteq \ldots \subseteq U_0$ s.t.
  - $f^{j_m}(U_m)$ intersects $W_\varepsilon(p_1) + (h_m, 0)$
  - $f^{j_m}(U_m)$ crosses the gaps $[a^m_{\rho_1}, b^m_{\rho_1}]$ of Σ_{ρ_1} and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ of $\Sigma_{\rho_1'}$
  - $[a^m_{\rho_1}, b^m_{\rho_1}]$ and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ — shifted apart
- $f^{j_1}(U_1) \cup f^{j_m}(U_m)$ forms an 'arch' over some part of Σ_{ρ_1} $\rightsquigarrow$ a neighborhood $\mathcal{U}$ of a point in Σ_{ρ_1}
- Similarly — an 'arch' over a part of some A-M set Σ_{ρ_2} near T_2 $\rightsquigarrow$ a neighborhood $\mathcal{V}$ of a point in Σ_{ρ_2}
- Mather connecting property: orbit from $\mathcal{U}$ to $\mathcal{V}$ $\rightsquigarrow$ orbit from U_0 to V_0



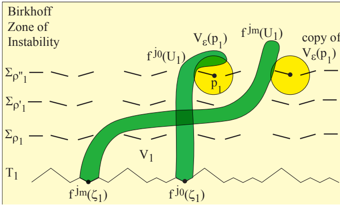
Proof of Proposition 1

- choose $\Sigma_{\rho_1} < \Sigma_{\rho_1'} < \Sigma_{\rho_1''}$ — A-M sets
- $W_\varepsilon(p_1)$ neighborhood of $p_1 \in \Sigma_{\rho_1''}$
- $\mathrm{cl}[\cup_{j=0}^{\infty} f^{-j}(W_\varepsilon(p_1))] \supseteq T_1$ [Kaloshin,2003]
- $\exists U_m \subseteq U_{m-1} \subseteq \ldots \subseteq U_0$ s.t.
  - $f^{j_m}(U_m)$ intersects $W_\varepsilon(p_1) + (h_m, 0)$
  - $f^{j_m}(U_m)$ crosses the gaps $[a^m_{\rho_1}, b^m_{\rho_1}]$ of Σ_{ρ_1} and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ of $\Sigma_{\rho_1'}$
  - $[a^m_{\rho_1}, b^m_{\rho_1}]$ and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ — shifted apart
- $f^{j_1}(U_1) \cup f^{j_m}(U_m)$ forms an 'arch' over some part of Σ_{ρ_1} $\rightsquigarrow$ a neighborhood $\mathcal{U}$ of a point in Σ_{ρ_1}
- Similarly — an 'arch' over a part of some A-M set Σ_{ρ_2} near T_2 $\rightsquigarrow$ a neighborhood $\mathcal{V}$ of a point in Σ_{ρ_2}
- Mather connecting property: orbit from $\mathcal{U}$ to $\mathcal{V}$ $\rightsquigarrow$ orbit from U_0 to V_0



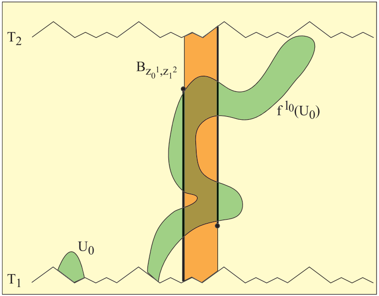
Proof of Proposition 1

- choose $\Sigma_{\rho_1} < \Sigma_{\rho_1'} < \Sigma_{\rho_1''}$ — A-M sets
- $W_\varepsilon(p_1)$ neighborhood of $p_1 \in \Sigma_{\rho_1''}$
- $\mathrm{cl}[\cup_{j=0}^{\infty} f^{-j}(W_\varepsilon(p_1))] \supseteq T_1$ [Kaloshin,2003]
- $\exists U_m \subseteq U_{m-1} \subseteq \ldots \subseteq U_0$ s.t.
  - $f^{j_m}(U_m)$ intersects $W_\varepsilon(p_1) + (h_m, 0)$
  - $f^{j_m}(U_m)$ crosses the gaps $[a^m_{\rho_1}, b^m_{\rho_1}]$ of Σ_{ρ_1} and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ of $\Sigma_{\rho_1'}$
  - $[a^m_{\rho_1}, b^m_{\rho_1}]$ and $[a^m_{\rho_1'}, b^m_{\rho_1'}]$ — shifted apart
- $f^{j_1}(U_1) \cup f^{j_m}(U_m)$ forms an 'arch' over some part of Σ_{ρ_1} $\rightsquigarrow$ a neighborhood $\mathcal{U}$ of a point in Σ_{ρ_1}
- Similarly — an 'arch' over a part of some A-M set Σ_{ρ_2} near T_2 $\rightsquigarrow$ a neighborhood $\mathcal{V}$ of a point in Σ_{ρ_2}
- Mather connecting property: orbit from $\mathcal{U}$ to $\mathcal{V}$ $\rightsquigarrow$ orbit from U_0 to V_0



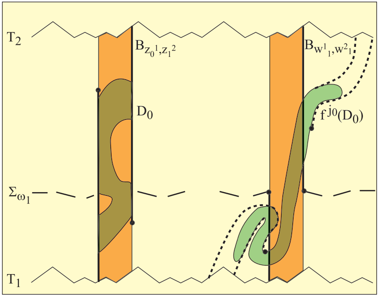
Proof of Proposition 2

- $\exists l_0 > 0$, $\exists B_{z_0^1, z_0^2}$ such that $F^{l_0}(U_0) \cap \mathrm{cl}(B_{z_0^1, z_0^2})$ has a component that is a positive diagonal D_0 in $B_{z_0^1, z_0^2}$
- $\exists j_0 > 0$, $\exists B_{w_1^1, w_1^2}$ with $w_1^1, w_1^2 \in \Sigma_{\omega_1}$ such that $F^{j_0}(D_0) \cap \mathrm{cl}(B_{w_1^1, w_1^2})$ has a component D_1 that is a positive diagonal in $B_{w_1^1, w_1^2}$
- upper and lower edges of D_1 contained in $F^{j_0}(\mathrm{bd}(U_0))$



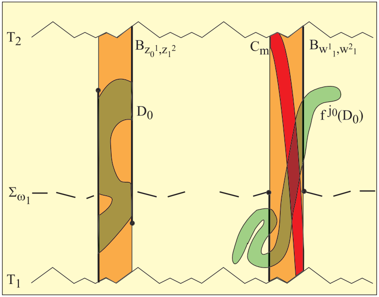
Proof of Proposition 2

- $\exists l_0 > 0$, $\exists B_{z_0^1, z_0^2}$ such that $F^{l_0}(U_0) \cap \mathrm{cl}(B_{z_0^1, z_0^2})$ has a component that is a positive diagonal D_0 in $B_{z_0^1, z_0^2}$
- $\exists j_0 > 0$, $\exists B_{w_1^1, w_1^2}$ with $w_1^1, w_1^2 \in \Sigma_{\omega_1}$ such that $F^{j_0}(D_0) \cap \mathrm{cl}(B_{w_1^1, w_1^2})$ has a component D_1 that is a positive diagonal in $B_{w_1^1, w_1^2}$
- upper and lower edges of D_1 contained in $F^{j_0}(\mathrm{bd}(U_0))$



Proof of Proposition 2

- [Hall,1989] $\rightsquigarrow \exists C_1 \supseteq C_2 \supseteq \ldots \supseteq C_m$ negative diagonals of $B_{w_1^1, w_1^2}$ s.t.
  - $f^{j_s+n_s}(C_s)$ is a positive diagonal in $B_{w_s^1, w_s^2}$, where $[w_s^1, w_s^2]$ is a gap in Σ_{ω_s}
- C_m intersects $F^{j_0}(\mathrm{bd}(U_0))$
- Similar argument about T_2
- There exists an orbit that goes from $\mathrm{bd}(U_0)$ to $\mathrm{bd}(V_0)$ and 'shadows' each Σ_{ω_s}



References

[1] M. Gidea and C. Robinson. Diffusion along transition chains of invariant tori and Aubry-Mather sets. Ergod. Th. & Dynam. Sys. (2012), to appear.

[2] G.R. Hall. A topological version of a theorem of Mather on shadowing in monotone twist maps. Dynamical Systems and Ergodic Theory (Warsaw, 1986) (Banach Center Publications, 23). PWN, Q11 Warsaw, 1989, 125–134.

[3] I. Jungreis. A method for proving that monotone twist maps have no invariant circles. Ergodic Theory Dynam. Systems 11(1) (1991), 79–84.

[4] K. Meyer, G.R. Hall and D. Offin. Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, 2nd Edition, Springer, 2009.

[5] P. Zgliczyński and M. Gidea. Covering relations for multidimensional dynamical systems. J. Differential Equations 202(1) (2004), 32-58.