Announcement of the seventh
JORNADES D'INTRODUCCIÓ ALS SISTEMES DINÀMICS I A LES EDP'S (JISD2008)

Barcelona, February 18-22, 2008

The seventh edition of the JORNADES D'INTRODUCCIÓ ALS SISTEMES DINÀMICS I A LES EDP'S (JISD2008) will be held in Barcelona from February 18th to February 22nd 2008 at the **Universitat Politècnica de Catalunya (UPC)**.

The JISD2008 will have two courses in Partial Differential Equations and Dynamical Systems.

The courses belong to the Master in Applied Mathematics and Mathematical Engineering inside the Graduate studies at UPC, and are organized by Prof. Xavier Cabré and Prof. Tere M. Seara.

These courses are supported by the grant Ayuda de movilidad asociada a los Masters oficiales (UPC),

REGISTRATION FORM *seventh* JISD'2008

*You can see the courses' **schedule** here*

Contents

February 18-22. **Courses will be held in the room 102 of the FME building (Facultat de Matemàtiques i Estadística), at C/ Pau Gargallo, n. 5 Barcelona, 08028.**

Course	Abstract
Asymptotic methods for non-linear diffusion equations *Juan Luis Vázquez* (Universidad Autónoma de Madrid) (Syllabus)	The aim of the course if to present some aspects of the theory of nonlinear diffusion with emphasis on the asymptotic methods. The use of scaling groups, selfsimilarity and entropy functionals will be stressed. **References:** 1. Vázquez, Juan Luis. "The porous medium equation. Mathematical theory". Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. MR2286292 2. Vázquez, Juan Luis. "Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type". Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006. MR2282669 3. Gilding, Brian H.; Kersner, Robert. "Travelling waves in nonlinear diffusion-convection reaction". Progress in Nonlinear Differential Equations and their Applications, 60. Birkhäuser Verlag, Basel, 2004. MR2081104
Generic Bifurcations in Dynamical Systems *Marco Antonio Teixeira* (Universidade Estatual de Campinas) (Syllabus) Prof.Teixeira's notes (PDF) **NEW**	The main aim is to present some aspects of generic bifurcation of dynamical systems represented by vector fields defined in a manifold. Discussion on the Thom-Smale program will be developed. **References:** 1. Garcia, R. and Teixeira M.A. Vector fields in manifolds with boundary and reversibility- an expository account : Qualitative Theory of Dynamical Systems, V.4, (2004), 311-327. 2. J. Sotomayor , Introduction al estúdio de lãs bifurcaiones de los sistemas dinâmicos,, VII ELAM, Caracas Venezuela, 1984. 3. J. Sotomayor, Generic one-parameter familes of vector fields on 2-dimensional manifolds, Public. IHES, Vol. 43, 1974. 4. Teixeira, M.A., Generic bifurcation on manifolds with boundary, JDE, V. 25, 1, 1977.

(*) For further details, please contact Prof. Xavier Cabré (xavier.cabre upc.edu),

March-08 - RMC

JISD2008 SCHEDULE

18	15.00 - 17.00	Juan Luis Vázquez / Lecture 1
	17.30 - 19.30	Marco Antonio Teixeira / Lecture 1
19	15.00 - 17.00	Juan Luis Vázquez / Lecture 2
	17.30 - 19.30	Marco Antonio Teixeira / Lecture 2
20	15.00 - 17.00	Juan Luis Vázquez / Lecture 3
	17.30 - 19.30	Marco Antonio Teixeira / Lecture 3
21	15.00 - 17.00	Juan Luis Vázquez / Lecture 4
	17.30 - 19.30	Marco Antonio Teixeira / Lecture 4
22	15.00 - 17.00	Juan Luis Vázquez / Lecture 5
	17.30 - 19.30	Marco Antonio Teixeira / Lecture 5

For more information, please contact xavier.cabre upc.edu

January 08 -RMC

JORNADES D'INTRODUCCIÓ ALS SISTEMES DINÀMICS I A LES EDP'S (JISD2008)

Barcelona, February 18 - 22, 2008

Course	Syllabus
Asymptotic methods for non-linear diffusion equations ***Juan Luis Vázquez*** (Universidad Autónoma de Madrid) ***Schedule***	• Lect. 1. Nonlinear Diffusion: Introduction. Physical Problems. • Lect. 2. Special solutions. Finite propagation. Basic theory. • Lect. 3. Asymptotic behaviour by scaling methods. Renormalized flows. • Lect. 4. The use of entropies. Calculation of stabilization rates. • Lect. 5. Other equations and problems.

Generic Bifurcations in Dynamical Systems
 	
Marco Antonio Teixeira
(Universidade Estatual de Campinas)
Schedule
	• Lect. 1. Preliminaries:
	  ◦ introduction of the terminology and main definitions
	  ◦ some background and some examples.
	• Lect. 2. Notions of equivalence of discrete and continuous systems. Vector fields defined in a compact planar region. Structural stability and local classification. Peixoto's theorem.
	• Lect. 3. Local bifurcation; singularities of codimension k.
	• Lect. 4. Global bifurcation.
	• Lect. 5. Non-smooth systems.

February 08 - RMC

Perturbation Theory for Non-smooth Systems

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Article Outline

I. GLOSSARY

Non-smooth dynamical system: systems derived from ordinary differential equations when the non-uniqueness of solutions is allowed. In this article we deal with discontinuous vector fields in R^n where the discontinuities are concentrated in a codimension-one surface.
Bifurcation: in a $k-parameter$ families of systems, a bifurcation is a parameter value at which the phase portrait is not structurally stable.
Typical singularity: are points on the discontinuity set where the orbits of the system through them must be distinguished.

II. DEFINITION OF THE SUBJECT AND ITS IMPORTANCE

In this article we survey some qualitative and geometric aspects of non-smooth dynamical systems theory. Our goal is to provide an overview of the state of the art on the theory of contact between a vector field and a manifold, and on discontinuous vector fields and their perturbations. We also establish a bridge between 2-dimensional non-smooth systems and the geometric singular perturbation theory. Non-smooth dynamical systems is a subject that has been developing at a very fast pace in recent years due to various factors: its mathematical beauty, its strong relationship with others branches of science and the challenge in establishing reasonable and consistent definitions and conventions. It has become certainly one of the common frontiers between Mathematics and Physics/Engineering. We mention that certain phenomena in control systems, impact in mechanical systems and nonlinear oscillations are the main sources of motivation of our study concerning the dynamics of those systems that emerge from differential equations with discontinuous right-hand sides. We understand that non-smooth systems are driven by applications and they play an intrinsic role in a wide range of technological areas.

III. Introduction

The purpose of this article is to present some aspects of the geometric theory of a class of non-smooth systems. Our main concern is to bring the theory into the domain of geometry and topology in a comprehensive mathematical manner.

Since this is an impossible task, we do not attempt to touch upon all sides of this subject in one article. We focus in exploring the local behavior of systems around typical singularities. The first task is to describe a generic persistence of a local theory (structural stability and bifurcation) for discontinuous systems mainly in the $2-$ and $3-$dimensional cases. Afterwards we present some striking features and results of the regularization process of two-dimensional discontinuous systems in the framework developed by Sotomayor and Teixeira in [44] and establish a bridge between those systems and the fundamental role played by the Geometric Singular Perturbation Theory (GSPT). This transition was introduced in [10] and we reproduce here its main features in the 2-dimensional case. For an introductory reading on the methods of geometric singular perturbation theory we refer to [15], [18] and [30]. In Section 2 we introduce the setting of this article. In Section 3 we survey the state of the art of the contact between a vector field and a manifold. The results contained in this section are crucial for the development of our approach. In Section 4 we discuss the classification of typical singularities of non-smooth vector fields. The study of non-smooth systems, via GSPT, is presented in Section 5. In Section 6 some theoretical open problems are presented.

One aspect of the qualitative point of view is the problem of structural stability, the most comprehensive of many different notions of stability. This theme was studied in 1937 by Andronov-Pontryagin (see [4]). This problem is of obvious importance, since in practice one obtains a lot of qualitative information not only on a fixed system but also on its nearby systems.

We deal with non-smooth vector fields in R^{n+1} having a codimension-one submanifold M as its discontinuity set. The scheme in this work toward a systematic classification of typical singularities of non-smooth systems follows the ideas developed by Sotomayor-Teixeira in [43] where the problem of contact between a vector field and the boundary of a manifold was discussed. Our approach intends to be self-contained and is accompanied by an extensive bibliography. We will try to focus here on areas that are complimentary to some recent reviews made elsewhere.

The concept of structural stability in the space of non-smooth vector fields is based on the following definition:

Definition 1. *Two vector fields Z and $\tilde{Z}$ are C^0 equivalent if there is an $M-$invariant homeomorphism $h : R^{n+1} \to R^{n+1}$ that sends orbits of Z to orbits of $\tilde{Z}$.*

A general discussion is presented to study certain unstable non-smooth vector fields within a generic context. The framework in which we shall pursue these unstable systems is sometimes called generic bifurcation theory. In [4] the concept of kth-order structural stability is also presented; in a local approach such setting gives rise to the notion of a codimension-k singularity. In studies of classical dynamical systems, normal form theory has been well accepted as a powerful tool in studying the local theory (see [6]). Observe that, so far, bifurcation and normal form theories for non-smooth vector fields have not been extensively studied in a systematic way

Control Theory is a natural source of mathematical models of these systems (see, for instance, [5], [8], [20], [41] and [45]). Interesting problems concerning discontinuous systems can be formulated in systems with hysteresis ([41]), economics ([25], [23]) and biology ([7]). It is worth mentioning that in [3] a class of relay systems in R^n is discussed. They have the form:

$$X = A\,x + sgn(x_1)\,k$$

where $x = (x_1, x_2, \ldots, x_n)$, $A \in M_R(n, n)$ and $k = (k_1, k_2, ...k_n)$ is a constant vector in R^n. In [27] and [28] the generic singularities of reversible relay systems in $4D$ were classified. In [54] some properties of non-smooth dynamics are discussed in order to understand some phenomena that arise in chattering control. We mention the presence of chaotic behavior in some non-smooth systems (see for example [12]). It is worthwhile to cite [17], where the main problem in the classical calculus of variations was carried out to study discontinuous Hamiltonian vector fields. We refer to [14] for a comprehensive text involving non-smooth systems which includes many models and applications. In particular motivating models of several non-smooth dynamical systems arising in the occurrence of impacting motion in mechanical systems, switchings in electronic systems and hybrid dynamics in control systems are presented together with an extensive literature on impact oscillators which we do not attempt to survey here. For further reading on some mathematical aspects of this subject we recommend [11] and references therein. A setting of general aspects of non-smooth systems can be found also in [35] and references therein. Our discussion does not focus on continuous but rather on non-smooth dynamical systems and we are aware that the interest in this subject goes beyond the approach adopted here.

The author wishes to thank R. Garcia, T.M. Seara and J. Sotomayor for many helpful conversations.

IV. Preliminaries

Now we introduce some of the terminology, basic concepts and some results that will be used in the sequel.

Definition 2. *Two vector fields Z and $\tilde{Z}$ on R^n with $Z(0) = \tilde{Z}(0)$ are germ-equivalent if they coincide on some neighborhood V of 0.*

The equivalent classes for this equivalence are called germs of vector fields. In the same way as defined above, we may define germs of functions. For simplicity we are considering the germ notation and we will not distinguish a germ of a function and any one of its representatives. So, for example, the notation $h : R^n, 0 \to R$ means that the h is a germ of a function defined in a neighborhood of 0 in R^n. Refer to [16] for a brief and nice introduction of the concepts of *germ* and *k-jet* of functions.

IV.1. **Discontinuous Systems.** Let $M = h^{-1}(0)$, where h is (a germ of) a smooth function $h : R^{n+1}, 0 \longrightarrow R$ having $0 \in R$ as its regular value. We assume that $0 \in M$.

Designate by $\chi(n+1)$ the space of all germs of C^r vector fields on R^{n+1} at 0 endowed with the C^r–topology with $r > 1$ and large enough for our purposes. Call $\Omega(n+1)$ the space of all germs of vector fields Z in $R^{n+1}, 0$ such that

$$Z(q) = \begin{cases} X(q), & \text{for} \quad h(q) > 0, \\ Y(q), & \text{for} \quad h(q) < 0, \end{cases} \tag{1}$$

The above field is denoted by $Z = (X, Y)$. So we are considering $\Omega(n+1) = \chi(n+1) \times \chi(n+1)$ endowed with the product topology.

Definition 3. *We say that $Z \in \Omega(n+1)$ is structurally stable if there exists a neighborhood U of Z in $\Omega(n+1)$ such that every $\tilde{Z} \in U$ is C^0-equivalent with Z.*

To define the orbit solutions of Z on the switching surface M we take a pragmatic approach. In a well characterized open set O of M (described below) the solution of Z through a point $p \in O$ obeys the Filippov rules and on $M - O$ we accept it to be multivalued. Roughly speaking, as we are interested in studying the structural stability in $\Omega(n+1)$ it is convenient to take into account all the leaves of the foliation in R^{n+1} generated by the orbits of Z (and also the orbits of X and Y) passing through $p \in M$. (see Figure 1)

The trajectories of Z are the solutions of the autonomous differential system $\dot{q} = Z(q)$.

In what follows we illustrate our terminology by presenting a simplified model that is found in the classical electromagnetism theory (see for instance [26]):

$$\ddot{x} - \dddot{x} + \alpha sign x = 0.$$

with $\alpha > 0$.

So this system can be expressed by the following objects: $h(x, y, z) = x$ and $Z = (X, Y)$ with $X(x, y, z) = (y, z, z + \alpha)$ and $Y(x, y, z) = (y, z, z - \alpha)$.

For each $X \in \chi(n+1)$ we define the smooth function $Xh : R^{n+1} \to R$ given by $Xh = X \cdot \nabla h$ where $\cdot$ is the canonical scalar product in R^{n+1}.

We distinguish the following regions on the discontinuity set M :

(i) M_1 is the *sewing region* that is represented by $h = 0$ and $(Xh)(Yh) > 0$;

(ii) M_2 is the *escaping region* that is represented by $h = 0$, $(Xh) > 0$ and $(Yh) < 0$;

(iii) M_3 is the *sliding region* that is represented by $h = 0$, $(Xh) < 0$ and $(Yh) > 0$.

We set $\mathcal{O} = \bigcup_{i=1,2,3} M_i$.

Consider $Z = (X, Y) \in \Omega(n+1)$ and $p \in M_3$. In this case, following Filippov's convention, the solution $\gamma(t)$ of Z through p follows, for $t \geq 0$, the orbit of a vector field tangent to M. Such system is called *sliding vector field* associated to Z and it will be defined below.

Definition 4. *The sliding vector field associated to $Z = (X, Y)$ is the smooth vector field Z^s tangent to M and defined at $q \in M_3$ by $Z^s(q) = m - q$ with m being the point where the segment joining $q + X(q)$ and $q + Y(q)$ is tangent to M.*

It is clear that if $q \in M_3$ then $q \in M_2$ for $-Z$ and then we define the *escaping vector field* on M_2 associated to Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^M for both cases.

We recall that sometimes Z^M is defined in an open region U with boundary. In this case it can be C^r extended to a full neighborhood of $p \in \partial U$ in M.

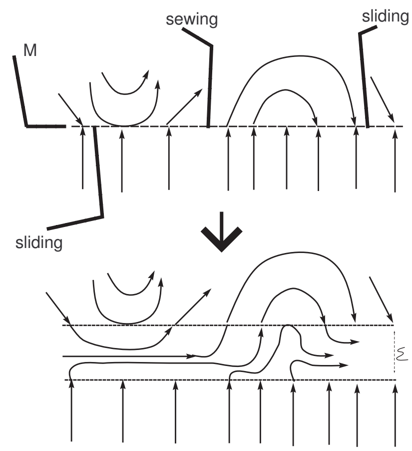

FIGURE 1. A discontinuous system and its regularization.

When the vectors $X(p)$ and $Y(p)$, with $p \in M_2 \bigcup M_3$ are linearly dependent then $Z^M(p) = 0$. In this case we say that p is a simple singularity of Z. The others singularities of Z are concentrated outside the set O.

We finish this subsection with a $3-$*dimensional* example:
Let $Z = (X, Y) \in \Omega(3)$ with $h(x, y, z) = z$, $X = (1, 0, x)$ and $Y = (0, 1, y)$. The system determines around 0 four quadrants, bounded by $\tau_X = \{x = 0\}$ and $\tau_Y = \{y = 0\}$. They are: $Q_1^+ = \{x > 0, y > 0\}$, $Q_1^- = \{x < 0, y < 0\}$, $Q_2 = \{x < 0, y > 0\}$ (sliding region) and $Q_3 = \{x > 0, y < 0\}$ (escaping region). Observe that $M_1 = Q_1^+ \bigcup Q_1^-$.

The sliding vector field defined in Q_2 is expressed by:

$$Z^s(x, y, z) = (y - x)^{-1}(x + y, \frac{y + x}{8}, 0))$$

Such system is (in Q_2) equivalent to $G(x, y, z) = (x+y, \frac{y+x}{8}, 0))$. In our terminology we consider G a smooth extension of Z^s, that is defined in a whole neighborhood of 0. It is worth to say that G is in fact a system which is equivalent to the original system in Q_2.

In [50] a generic classification of one-parameter families of sliding vector fields is presented.

IV.2. **Singular perturbation problem.** A singular perturbation problem is expressed by a differential equation $z' = \alpha(z, \varepsilon)$ (refer to [18], [15] and [30]) where $z \in R^{n+m}$, ε is a small non-negative real number and α is a C^∞ mapping.

Let $z = (x, y) \in R^{n+m}$ and $f : R^{m+n} \to R^m, g : R^{m+n} \to R^n$ be smooth mappings. We deal with equations that may be written in the form

$$\left\{ \begin{array}{l} x' = f(x, y, \varepsilon) \\ y' = \varepsilon g(x, y, \varepsilon) \end{array} \right. \quad x = x(\tau), y = y(\tau). \tag{2}$$

An interesting model of such systems can be obtained from the singular van der Pol's equation

$$\varepsilon x'' + (x^2 + x)x' + x - a = 0. \tag{3}$$

The main trick in the geometric singular perturbation (GSP) is to consider the family (2) in addition to the family

$$\left\{\begin{array}{l}\varepsilon\dot{x}=f\left(x,y,\varepsilon\right)\\ \dot{y}=g\left(x,y,\varepsilon\right)\end{array}\right.\quad x=x\left(t\right),y=y\left(t\right)\tag{4}$$

obtained after the time rescaling $t=\varepsilon\tau$.

Equation (2) is called the *fast system* and (4) the *slow system*. Observe that for $\varepsilon>0$ the phase portrait of fast and slow systems coincide.

For $\varepsilon=0$, let $\mathcal{S}$ be the set of all singular points of (2). We call $\mathcal{S}$ the slow manifold of the singular perturbation problem and it is important to notice that equation (4) defines a dynamical system on S called the *reduced problem.*

Combining results on the dynamics of these two limiting problems (2) and (4), with $\varepsilon=0$, one obtains information on the dynamics for small values of ε. In fact, such techniques can be exploited to formally construct approximated solutions on pieces of curves that satisfy some limiting version of the original equation as ε goes to zero.

Definition 5. *Let $A,B\subset R^{n+m}$ be compact sets. The Hausdorff distance between A and B is $D\left(A,B\right)=\max_{z_1\in A,z_2\in B}\{d\left(z_1,B\right),d\left(z_2,A\right)\}$.*

The main question in GSP-theory is to exhibit conditions under which a singular orbit can be approximated by regular orbits for $\varepsilon\downarrow 0$, with respect to the Hausdorff distance.

IV.3. **Regularization Process.** An approximation of the discontinuous vector field $Z=(X,Y)$ by a one-parameter family of continuous vector fields will be called a regularization of Z. In [44], Sotomayor and Teixeira introduced the regularization procedure of a discontinuous vector field. A transition function is used to average X and Y in order to get a family of continuous vector fields that approximates the discontinuous one. Figure 1 gives a clear illustration of the regularization process.

Let $Z=(X,Y)\in\Omega(n+1)$.

Definition 6. *A C^∞ function $\varphi:R\longrightarrow R$ is a transition function if $\varphi(x)=-1$ for $x\leq-1$, $\varphi(x)=1$ for $x\geq 1$ and $\varphi'(x)>0$ if $x\in(-1,1)$. The φ–regularization of $Z=(X,Y)$ is the 1–parameter family $X_\varepsilon\in C^r$ given by*

$$Z_\varepsilon(q)=\left(\frac{1}{2}+\frac{\varphi_\varepsilon(h(q))}{2}\right)X(q)+\left(\frac{1}{2}-\frac{\varphi_\varepsilon(h(q))}{2}\right)Y(q).\tag{5}$$

with h given in subsection 2.1 and $\varphi_\varepsilon(x)=\varphi(x/\varepsilon)$, for $\varepsilon>0$.

As already said before, a point in the phase space which moves on an orbit of Z crosses M when it reaches the region M_1. Solutions of Z through points of M_3 , will remain in M in forward time. Analogously, solutions of Z through points of M_2 will remain in M in backward time. In [34] and [44] such conventions are justified by the regularization method in dimensions 2 and 3 respectively.

V. Vector fields near the boundary

In this section we discuss the behavior of smooth vector fields in R^{n+1} relative to a codimension one submanifold (say, the above defined M). We base our approach on the concepts and results contained in [43] and [53]. The principal advantage of this setting is that the generic contact between a smooth vector field and M

can often be easily recognized. As an application the typical singularities of a discontinuous system can be further classified in a straightforward way.

We say that $X, Y \in \chi(n+1)$ are $M-equivalent$ if there exists an $M-preserving$ homeomorphism $h : R^{n+1}, 0 \longrightarrow R^{n+1}, 0$ that sends orbits of X into orbits of Y. In this way we get the concept of $M-$structural stability in $\chi(n+1)$.

We call $\Gamma_0(n+1)$ the set of elements X in $\chi(n+1)$ satisfying one of the following conditions:
0) $Xh(0) \neq 0$ (0 is a regular point of X). In this case X is transversal to M at 0.
1) $Xh(0) = 0$ and $X^2h(0) \neq 0$ (0 is a 2-*fold* point of X;)
2) $Xh(0) = X^2h(0) = 0$, $X^3h(0) \neq 0$ and the set $\{Dh(0), DXh(0), DX^2h(0)\}$ is linearly independent (0 is a *cusp* point of X;)
....
n) $Xh(0) = X^2h(0) = ... = X^nh(0) = 0$ and $X^{n+1}h(0) \neq 0$. Moreover the set $\{Dh(0), DXh(0), DX^2h(0), ..., DX^nh(0)\}$ is linearly independent, and 0 is a regular point of the mapping $Xh_{|M}$.

We say that 0 is an $M-$singularity of X if $h(0) = Xh(0) = 0$. It is a $codimension-zero$ $M-$singularity provided that $X \in \Gamma_0(n+1)$.

We know that $\Gamma_0(n+1)$ is an open and dense set in $\chi(n+1)$ and it coincides with the $M-$structurally stable vector fields in $\chi(n+1)$ (see [53]).

Denote by $\tau_X \subset M$ the $M-$singular set of $X \in \chi(n+1)$; this set is represented by the equations $h = Xh = 0$. It is worth to point out that, generically, all $2-folds$ constitute an open and dense subset of τ_X. Observe that if $X(0) = 0$ then $X \notin \Gamma_0(n+1)$.

The $M-$bifurcation set is represented by $\chi_1(n+1) = \chi(n+1) - \Gamma_0(n+1)$

Vishik in [53] exhibited the normal forms of a $codimension-zero$ $M-$singularity. They are:

I) Straightened vector field

$$X = (1, 0, ..., 0)$$

and

$$h(x) = x_1^{k+1} + x_2 x_1^{k-1} + x_3 x_1^{k-2} + ... + x_{k+1} \, k = 0, 1...n$$

or

II) Straightened boundary

$$h(x) = x_1$$

and

$$X(x) = (x_2, x_3, ..., x_k, 1, 0, 0...0)$$

We now discuss an important interaction between vector fields near M and singularities of mapping theory. We discuss how singularity-theoretic techniques help the understanding of the dynamics of our systems.

We outline this setting, which will be very useful in the sequel. The starting point is the following construction.

V.1. **A construction.** Let $X \in \chi(n+1)$. Consider a coordinate system $x = (x_1, x_2, ..., x_{n+1})$ in $R^{n+1}, 0$ such that

$$M = \{x_1 = 0\}$$

and

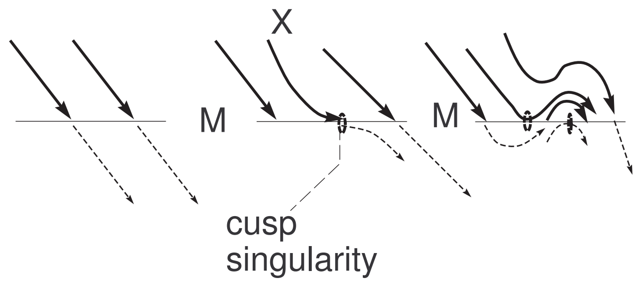

FIGURE 2. The cusp singularity and its unfolding.

$$X = (X^1, X^2, ..., X^{n+1})$$

Assume that $X(0) \neq 0$ and $X^1(0) = 0$. Let N_0 be any transversal section to X at 0.

By the implicit function theorem, we derive that:

for each $p \in M, 0$ there exists a unique $t = t(p)$ in $R, 0$ such that the orbit-solution $t \mapsto \gamma(p,t)$ of X through p meets N_0 at a point $\tilde{p} = \gamma(p, t(p))$.

We define the smooth mapping $\rho_X : R^n, 0 \longrightarrow R^n, 0$ by $\rho_X(p) = \tilde{p}$. This mapping is a powerful tool in the study of vector fields around the boundary of a manifold (refer to [21], [42] [43], [53] and [47]). We observe that τ_X coincides with the singular set of ρ_X.

The late construction implements the following method. If we are interested in finding an equivalence between two vector fields which preserve M, then the problem can be sometimes reduced to finding an equivalence between ρ_X and ρ_Y in the sense of singularities of mappings.

We recall that when 0 is a fold $M-$singularity of X then associated to the fold mapping ρ_X there is the symmetric diffeomorphism β_X that satisfies $\rho_X \circ \beta_X = \rho_X$.

Given $Z = (X, Y) \in \Omega(n+1)$ such that ρ_X and ρ_Y are fold mappings with $X^2h(0) < 0$ and $Y^2h(0) > 0$ then the composition of the associated symmetric mappings β_X and β_Y provides a first return mapping β_Z associated to Z and M. This situation is usually called a $distinguished\ fold - fold$ singularity, and the mapping β_Z plays a fundamental role in the study of the dynamics of Z.

V.2. **Codimension-one M-singularity in dimensions 2 and 3.**

V.2.1. *Case n=1.* In this case the unique codimension-zero $M-$singularity is a fold point in $R^2, 0$. The codimension-one $M-$singularities are represented by the subset $\Gamma_1(2)$ of $\chi_1(2)$ and it is defined as follows.

Definition 7. *A codimension-one $M-$singularity of $X \in \Gamma_1(2)$ is either a cusp singularity or an $M-$hyperbolic critical point p in M of the vector field X. A cusp singularity (illustrated in Figure 2) is characterized by $Xh(p) = X^2h(p) = 0$, $X^3h(p) \neq 0$. In the second case this means that p is a hyperbolic critical point (illustrated in Figure 3) of X with distinct eigenvalues and with invariant manifolds (stable, unstable and strong stable and strong unstable) transversal to M.*

In this subsubsection we consider a coordinate system in $R^2, 0$ such that $h(x, y) = y$.

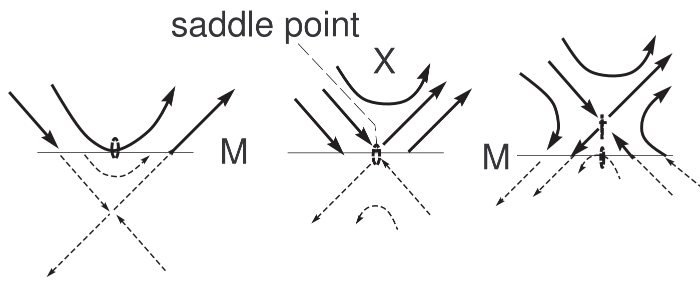

FIGURE 3. The saddle point in the boundary and its unfolding.

The next result was proved in [47]. It presents the normal forms of the codimension one singularities defined above.

Theorem 8. *Let $X \in \chi_1(2)$. The vector field X is $M-$structurally stable relative to $\chi_1(2)$ if and only if $X \in \Gamma_1(2)$. Moreover, $\Gamma_1(2)$ is an embedded codimension-one sub manifold and dense in $\chi_1(2)$. We still require that any one-parameter family X_λ, $(\lambda \in (-\varepsilon, \varepsilon))$ in $\chi(1)$ transverse to $\Gamma_1(2)$ at X_0, has one of the following normal forms:*
0.1 : $X_\lambda(x, y) = (1, 0)$ *(regular point);*
0.2 : $X_\lambda(x, y) = (1, x)$ *(fold singularity);*
1.1 $X_\lambda(x, y) = (1, \lambda + x^2)$ *(cusp singularity);*
1.2 $X_\lambda(x, y) = (ax, x + by + \lambda), a = \pm 1, b = \pm 2;$
1.3 $X_\lambda(x, y) = (x, x - y + \lambda)$;
1.4 $X_\lambda(x, y) = (x + y, -x + y + \lambda)$.

V.2.2. *Case n=2.*

Definition 9. *A vector field $X \in \chi(3)$ belongs to the set $\Gamma_1(a)$ if the following conditions hold:*

(i) $X(0) = 0$ and 0 is a hyperbolic critical point of X;

(ii) the eigenvalues of $DX(0)$ are pairwise distinct and the corresponding eigenspaces are transversal to M at 0;

(iii) each pair of non complex conjugate eigenvalues of $DX(0)$ has distinct real parts.

Definition 10. *A vector field $X \in \chi(3)$ belongs to the set $\Gamma_1(b)$ if $X(0) \neq 0$, $Xh(0) = 0$, $X^2h(0) = 0$ and one of the following conditions hold:*
(1) $X^3h(0) \neq 0$, $rank\{Dh(0), DXh(0), DX^2h(0)\} = 2$ and 0 is a non-degenerate critical point of $Xh_{|M}$.
(2) $X^3h(0) = 0$, $X^4h(0) \neq 0$ and 0 is a regular point of $Xh_{|M}$.

The next results can be found in [43].

Theorem 11. *The following statements hold:*
(i) $\Gamma_1(3) = \Gamma_1(a) \cup \Gamma_2(b)$ is a codimension one submanifold of $\chi(3)$.

(ii) $\Gamma_1(3)$ *is open and dense set in* $\chi_1(3)$ *in the topology induced from* $\chi_1(3)$.
(iv) For a residual set of smooth curves $\gamma : R, 0 \to \chi(3)$, γ *meets* $\Gamma_1(3)$ *transversally.*

Throughout this subsubsection we fix the function $h(x, y, z) = z$.

Lemma 12. *(Classification Lemma) (1) The elements of* $\Gamma_1(3)$ *are classified as follows:*

- (a_{11}) **Nodal M-Singularity::** $X(0) = 0$, *the eigenvalues of* $DX(0)$, λ_1, λ_2, *and* λ_3, *are real, distinct,* $\lambda_1\lambda_j > 0$, $j = 2, 3$ *and the eigenspaces are transverse to* M *at* 0;
- (a_{12}) **Saddle M-Singularity::** $X(0) = 0$, *the eigenvalues of* $DX(0)$, λ_1, λ_2 *and* λ_3, *are real, distinct,* $\lambda_1\lambda_j < 0$, $j = 2$ *or* 3 *and the eigenspaces are transverse to* M *at* 0;
- (a_{13}) **Focal M-Singularity::** 0 *is a hyperbolic critical point of* X *, the eigenvalues of* $DX(0)$ *are* $\lambda_{12} = a \pm ib$, $\lambda_3 = c$, *with* a, b, c *distinct from zero and* $c \neq a$, *and the eigenspaces are transverse to* M *at* 0.
- (b_{11}) **Lips M-Singularity::** *presented in* **Definition 8, item 1**, *when* $Hess(Fh_{/S}(0)) > 0$:
- (b_{12}) **Bec to Bec M-Singularity::** *presented in* **Definition 8, item 1**, *when* $Hess(Fh_{/S}(0)) < 0$;
- (b_{13}) **Dove's Tail M-Singularity:** *presented in* **Definition 8, item 2**.

The next result is proved in [38]. It deals with the normal forms of a codimension one singularity.

Theorem 13. *i) (Generic Bifurcation and normal forms) Let* $X \in \chi(3)$. *The vector field* X *is* $M-$*structurally stable relative to* $\chi_1(3)$ *if and only if* $X \in \Gamma_1(3)$.
ii) (Versal unfolding) In the space of one-parameter families of vector fields X_α *in* $\chi(3)$, $\alpha \in (-\varepsilon, \varepsilon)$ *an everywhere dense set is formed by generic families such that their normal forms are:*

- $X_\alpha \in \Gamma_0(3)$
  $0.1 : X_\alpha(x, y, z) = (0, 0, 1)$
  $0.2 : X_\alpha(x, y, z) = (z, 0, \pm x)$
  $0.3 : X_\alpha(x, y, z) = (z, 0, x^2 + y)$
- $X_0 \in \Gamma_1(3)$
  $1.1 : X_\alpha(x, y, z) = (z, 0, \frac{-3x^2+y^2+\alpha}{2})$
  $1.2 : X_\alpha(x, y, z) = (z, 0, \frac{-3x^2-y^2+\alpha}{2})$
  $1.3 : X_\alpha(x, y, z) = (z, 0, \frac{4\delta x^3+y+\alpha x}{2})$, *with* $\delta = \pm 1$
  $1.4 : X_\alpha(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a, b, c) = \delta(3, 2, 1)$, $\delta = \pm 1$
  $1.5 : X_\alpha(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a, b, c) = \delta(1, 3, 2)$, $\delta = \pm 1$
  $1.6 : X_\alpha(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a, b, c) = \delta(1, 2, 3)$, $\delta = \pm 1$
  $1.7 : X_\alpha(x, y, z) = (xz, 2yz, \frac{x+2y-cz^2+\alpha}{2})$
  $1.8 : X_\alpha(x, y, z) = ((-x + y)z, (-x - y)z, \frac{-3x-y+z^2+\alpha}{2})$

VI. Generic Bifurcation in $\Omega^r(n+1)$

Let $Z=(X,Y)\in\Omega^r(n+1)$. Call by $\Sigma_0(n+1)$ (resp. $\Sigma_1(n+1)$) the set of all elements that are structurally stable in $\Omega^r(n+1)$ (resp. $\Omega^r_1(n+1)=\Omega^r(n+1)_{\setminus\Sigma_0(n+1)}$) in $\Omega^r(n+1)$. It is clear that a pre-classification of the generic singularities is immediately reached by:

If $Z=(X,Y)\in\Sigma_0(n+1)$ (resp. $Z=(X,Y)\in\Sigma_1(n+1)$) then X and Y are in $\Gamma_0(n+1)$ (resp. $X\in\Gamma_0(n+1)$ and $Y\in\Gamma_1(n+1)$ or vice versa). Of course, the case when both X and Y are in $\Gamma_1(n+1)$ is a codimension two phenomenon.

VI.1. **2-dimensional case.** The following result characterizes the structural stability in $\Omega^r(2)$.

Theorem A: (see [31] and [44]): $\Sigma_0(2)$ *is an open and dense set of* $\Omega^r(2)$. *The vector field $Z=(X,Y)$ is in $\Sigma_0(2)$ if and only one of the following conditions is satisfied:*
i) *Both elements X and Y are regular. When $0\in M$ is a simple singularity of Z then we assume that it is a hyperbolic critical point of Z^M.*
ii) *X is a fold singularity and Y is regular (and vice-versa).*

The following result still deserves a systematic proof. Following the same strategy stipulated in the generic classification of an $M-$singularity, Theorem 11 could be very useful. It is worth to mention [33] where the problem of generic bifurcation in $2D$ was also addressed.

Theorem B (Generic Bifurcation): (see [37] and [43]) $\Sigma_1(2)$ *is an open and dense set of* $\Omega^r_1(2)$. *The vector field $Z=(X,Y)$ is in $\Sigma_1(2)$ provided that one of the following conditions is satisfied:*
i) *Both elements X and Y are $M-$regular. When $0\in M$ is a simple singularity of Z then we assume that it is a codimension one critical point (saddle-node or a Bogdanov-Takens singularity) of Z^M.*
ii) *0 is a codimension-one $M-$singularity of X and Y is $M-$regular. This case includes when 0 is either a cusp $M-$singularity or a critical point. Figure 4 illustrates the case when 0 is a saddle critical point in the boundary.*
ii) *both X and Y are fold $M-$singularities at 0. In this case we have to impose that 0 is a hyperbolic critical point of the $C^r-extension$ of Z^M provided that it is in the boundary of $M_2\cup M_3$ (see example below). Moreover when 0 is a distinguished $fold-fold$ singularity of Z then 0 is a hyperbolic fixed point of the first return mapping β_Z.*

Consider in a small neighborhood of 0 in R^2, the system $Z=(X,Y)$ with $X(x,y)=(1-x^3+y^2,x)$, $Y(x,y)=(1+x+y,-x+x^2)$ and $h(x,y)=y$. The point 0 is a $fold-fold$-singularity of Z with $M_2=\{x<0\}$ and $Z^s(x,0)=(2x-x^2)^{-1}(2x-x^4+x^5)$. Observe that 0 is a hyperbolic critical point of the extended system $G(x,y)=2x-x^4+x^5$.

The classification of the codimension two singularities in $\Omega^r(2)$ is still an open problem. In this direction [51] contains information about the classification of codimension two $M-$singularities.

VI.2. **3-dimensional case.** Let $Z=(X,Y)\in\Omega^r(3)$.

The most interesting case to be analyzed is when both vector fields, X and Y are fold singularities at 0 and the tangency sets τ_X and τ_Y in M are in general position at 0. In fact they determine (in M) four quadrants, two of them are M_1-regions,

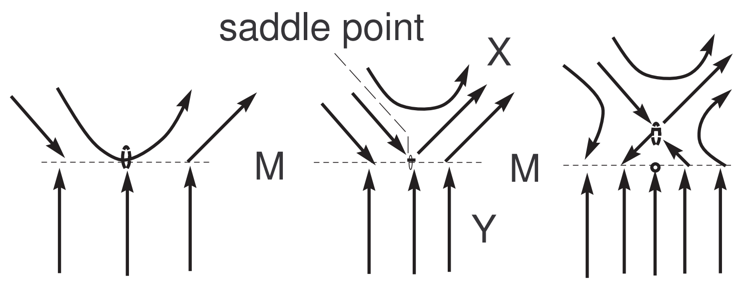

FIGURE 4. $M-$critical point for X, $M-$regular for Y and its unfolding.

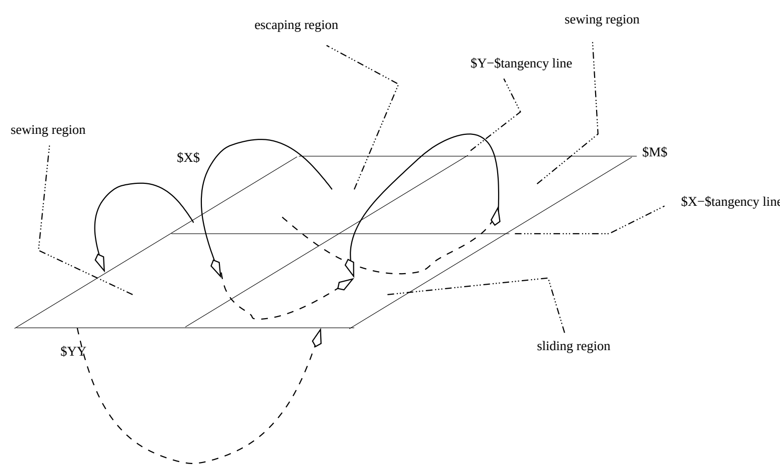

FIGURE 5. The distinguished fold-fold singularity.

one is an M_3-region and the other is an M_2-region (see Figure 5). We emphasize that the sliding vector field Z^M can be C^r-extended to a full neighborhood of 0 in M. Moreover, $Z^M(0) = 0$. Inside this class the *distinguished* $fold-fold$ singularity (as defined in Subsection 3.1) must be taken into account. Denote by A the set of all *distinguished* $fold - fold$ *singularities* $Z \in \Omega^r(3)$. Moreover, the eigenvalues of $D\beta_Z(0)$ are $\lambda = a \pm \sqrt{(a^2-1)}$. If $\lambda \in R$ we say that Z belongs to A_s. Otherwise Z is in A_e. Recall that β_Z is the first return mapping associated to Z and M at 0 as defined in Subsection 3.1.

It is evident that the elements in the open set A_e are structurally unstable in $\Omega^r(3)$. It is worthwhile to mention that in A_e we detect elements which are asymptotically stable at the origin [48]. Concerning A_s few things are known.

We have the following result:

Theorem C: *The vector field* $Z = (X, Y)$ *belongs to* $\Sigma_0(3)$ *provided that one of the following conditions occurs:*

i) *Both elements* X *and* Y *are regular. When* $0 \in M$ *is a simple singularity of* Z *then we assume that it is a hyperbolic critical point of* Z^M.

ii) X *is a fold singularity at* 0 *and* Y *is regular.*

iii) X *is a cusp singularity at* 0 *and* Y *is regular.*

iv) *Both systems X and Y are of fold type at* 0. Moreover: a) *the tangency sets τ_X and τ_Y are in general position at* 0 *in M*; b) *The eigenspaces associated to Z^M are transverse to τ_X and τ_Y at $0 \in M$* and c) Z ***is not in A***. *Moreover the real parts of non conjugate eigenvalues are distinct.*

We recall that bifurcation diagrams of sliding vector fields are presented in [50] and [52].

VII. Singular Perturbation Problem in 2D

Geometric singular perturbation theory is an important tool in the field of continuous dynamical systems. Needless to say that in this area very good surveys are available (refer to [15], [18] and [30]). Here we highligh some results (see [10]) that bridge the space between discontinuous systems in $\Omega^r(2)$ and singularly perturbed smooth systems.

Definition 14. *Let $U \subset R^2$ be an open subset and $\varepsilon \geq 0$. A singular perturbation problem in U (SP–Problem) is a differential system which can be written as*

$$x' = dx/d\tau = f(x,y,\varepsilon), \quad y' = dy/d\tau = \varepsilon g(x,y,\varepsilon) \tag{6}$$

or equivalently, after the time re-scaling $t = \varepsilon\tau$

$$\varepsilon\dot{x} = \varepsilon dx/dt = f(x,y,\varepsilon), \quad \dot{y} = dy/dt = g(x,y,\varepsilon), \tag{7}$$

with $(x,y) \in U$ and f, g smooth in all variables.

Our first result is concerned with the transition between non-smooth systems and GSPT.

Theorem D: *Consider $Z \in \Omega^r(2), Z_\varepsilon$ its φ–regularization, and $p \in M$. Suppose that φ is a polynomial of degree k in a small interval $I \subseteq (-1,1)$ with $0 \in I$. Then the trajectories of Z_ε in $V_\varepsilon = \left\{q \in R^2, 0 : h(q)/\varepsilon \in I\right\}$ are in correspondence with the solutions of an ordinary differential equation $z' = \alpha(z,\varepsilon)$, satisfying that α is smooth in both variables and $\alpha(z,0) = 0$ for any $z \in M$. Moreover, if $((X-Y)h^k)(p) \neq 0$ then we can take a C^{r-1}–local coordinate system $\{(\partial/\partial x)(p), (\partial/\partial y)(p)\}$ such that this smooth ordinary differential equation is a SP–problem.*

The understanding of the phase portrait of the vector field associated to a SP–problem is the main goal of the *geometric singular perturbation–theory* (GSP–theory). The techniques of GSP–theory can be used to obtain information on the dynamics of (6) for small values of $\varepsilon > 0$, mainly in searching minimal sets.

System (6) is called the *fast system*, and (7) the *slow system* of SP–problem. Observe that for $\varepsilon > 0$ the phase portraits of the fast and the slow systems coincide.

Theorem D says that we can transform a discontinuous vector field in a SP–problem. In general this transition can not be done explicitly. Theorem E provides an explicit formula of the SP–problem for a suitable class of vector fields. Before the statement of such a result we need to present some preliminaries.

Consider $C = \{\xi : R^2, 0 \to R\}$ with $\xi \in C^r$ and $L(\xi) = 0$ where $L(\xi)$ denotes the linear part of ξ at $(0,0)$.

Let $\Omega_d \subset \Omega^r(2)$ be the set of vector fields $Z = (X,Y)$ in $\Omega^r(2)$ such that there exists $\xi \in C$ that is a solution of

$$(8) \qquad \nabla\xi(X-Y) = \Pi_i(X-Y),$$

where $\nabla\xi$ is the gradient of the function and Π_i denote the canonical projections, for $i=1$ or $i=2$.

Theorem E: *Consider $Z \in \Omega_d$ and Z_ε its φ–regularization. Suppose that φ is a polynomial of degree k in a small interval $I \subset R$ with $0 \in I$. Then the trajectories of Z_ε on $V_\varepsilon = \{q \in R^2, 0 : h(q)/\varepsilon \in I\}$ are solutions of a SP–problem.*

We remark that the singular problems discussed in the previous theorems, when $\varepsilon \searrow 0$, defines a dynamical system on the discontinuous set of the original problem. This fact can be very useful for problems in Control Theory.

Our third theorem says how the fast and the slow systems approximate the discontinuous vector field. Moreover, we can deduce from the proof that whereas the fast system approximates the discontinuous vector field, the slow system approaches the corresponding sliding vector field.

Consider $Z \in \Omega^r(2)$ and $\rho : R^2, 0 \longrightarrow R$ with $\rho(x,y)$ being the distance between (x,y) and M. We denote by $\widehat{Z}$ the vector field given by $\widehat{Z}(x,y) = \rho(x,y)Z(x,y)$.

In what follows we identify $\widehat{Z}_\varepsilon$ and the vector field on $\{\{R^2,0\} \setminus M \times R\} \subset R^3$ given by $\widehat{Z}(x,y,\varepsilon) = (\widehat{Z}_\varepsilon(x,y), 0)$.

Theorem F: Consider $p = 0 \in M$. Then there exist an open set $U \subset R^2$, $p \in U$, a 3–dimensional manifold $\mathcal{M}$, a smooth function $\Phi : \mathcal{M} \longrightarrow R^3$ and a SP–problem W on $\mathcal{M}$ such that Φ sends orbits of $W|_{\Phi^{-1}(U\times(0,+\infty))}$ in orbits of $\widehat{Z}|_{(U\times(0,+\infty))}$.

VII.1. **Examples.** 1- Take $X(x,y) = (1,x), Y(x,y) = (-1,-3x)$, and $h(x,y) = y$. The discontinuity set is $\{(x,0) \mid x \in R\}$. We have $Xh = x, Yh = -3x$, and then the unique non–regular point is $(0,0)$. In this case we may apply Theorem E.

2- Let $Z_\varepsilon(x,y) = (y/\varepsilon, 2xy/\varepsilon - x)$. The associated partial differential equation (refer to Theorem E) with $i=2$ given above becomes $2\,(\partial\xi/\partial x) + 4x\,(\partial\xi/\partial y) = 4x$. We take the coordinate change $\overline{x} = x, \overline{y} = y - x^2$. The trajectories of X_ε in these coordinates are the solutions of the singular system

$$\varepsilon\dot{\overline{x}} = \overline{y} + \overline{x}^2, \quad \dot{\overline{y}} = -\overline{x}.$$

3- In what follows we try, by means an example, to present a rough idea on the transition from non-smooth systems to GSPT. Consider $X(x,y) = (3y^2 - y - 2, 1)$, $Y(x,y) = (-3y^2 - y + 2, -1)$ and $h(x,y) = x$. The regularized vector field is

$$Z_\varepsilon(x,y) = \left(\frac{1}{2} + \frac{1}{2}\varphi(\frac{x}{\varepsilon})\right)(3y^2 - y - 2, 1) + \left(\frac{1}{2} - \frac{1}{2}\varphi(\frac{x}{\varepsilon})\right)(-3y^2 - y + 2, -1).$$

After performing the polar blow up coordinates $\alpha : [0,+\infty) \times [0,\pi] \times R \to R^3$ given by $x = r\cos\theta$ and $\varepsilon = r\sin\theta$ the last system is expressed by:

$$r\dot{\theta} = -\sin\theta(-y + \varphi(\cot\theta)(3y^2 - 2)), \quad \dot{y} = \varphi(\cot\theta).$$

So the slow manifold is given implicitly by $\varphi(\cot\theta) = \frac{y}{3y^2-2}$ which defines two functions $y_1(\theta) = \frac{1+\sqrt{1+24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)}$ and $y_2(\theta) = \frac{1-\sqrt{1+24\varphi^2(\cot\theta)}}{6\varphi(\cot\theta)}$. The function $y_1(\theta)$

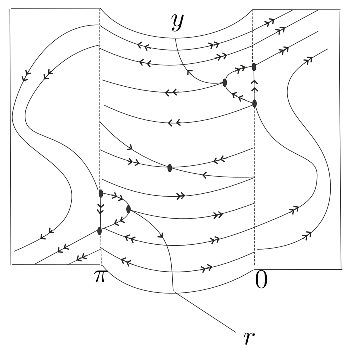

Figure 6. Example of fast and slow dynamics of the SP–Problem

is increasing, $y_1(0) = 1$, $\lim\limits_{\theta\to\frac{\pi}{2}^-} y_1(\theta) = +\infty$, $\lim\limits_{\theta\to\frac{\pi}{2}^+} y_1(\theta) = -\infty$ and $y_1(\pi) = -1$. The function $y_2(\theta)$ is increasing, $y_2(0) = -\frac{2}{3}$, $\lim\limits_{\theta\to\frac{\pi}{2}} y_2(\theta) = 0$ and $y_2(\pi) = \frac{2}{3}$. We can extend y_2 to $(0,\pi)$ as a differential function with $y_2(\frac{\pi}{2}) = 0$.

The fast vector field is $(\theta', 0)$ with $\theta' > 0$ if (θ, y) belongs to

$$\left[(0,\frac{\pi}{2})\times(y_2(\theta), y_1(\theta))\bigcup(\frac{\pi}{2},\pi)\times(y_2(\theta),+\infty)\bigcup(\frac{\pi}{2},\pi)\times(-\infty,y_1(\theta))\right]$$

and with $\theta' < 0$ if (θ, y) belongs to

$$\left[(0,\frac{\pi}{2})\times(y_1(\theta),+\infty)\bigcup(0,\frac{\pi}{2})\times(-\infty,y_2(\theta))\bigcup(\frac{\pi}{2},\pi)\times(y_1(\theta),y_2(\theta))\right].$$

The reduced flow has one singular point at $(0, 0)$ and it takes the positive direction of the y-axis if $y \in (-\frac{2}{3}, 0) \cup (1,\infty)$ and the negative direction of the y-axis if $y \in (-\infty, -1) \cup (0, \frac{2}{3})$.

One can see that the singularities $(\theta, y, r) = (0, 1, 0)$ and $(\theta, y, r) = (0, -1, 0)$ are not normally hyperbolic points. In this way, as usual, we perform additional blow ups. In Figure 6 we illustrate the fast and the slow dynamics of the SP–problem. We present a phase portrait on the blowing up locus where double arrow over a trajectory means that the trajectory belongs to the fast dynamical system, and simple arrow means that the trajectory belongs to the slow dynamical system.

VIII. Future directions

Our concluding section is devoted to an outlook. Firstly we present some open problems linked with the setting that point out future directions of research. The main task for the future seems to bring the theory of non-smooth dynamical systems to a similar maturity as that of smooth systems. Finally we briefly discuss the main results in this text.

VIII.1. **Some problems.** In connection to this present work, some theoretical problems remain open:
1- The description of the bifurcation diagram of the codimension-two singularities in $\Omega(2)$. In this last class we find some models (see [36]) where the following questions can also be addressed: a) when is a typical singularity topological equivalent to a

regular center?; b) how about the isochronicity of such a center?; c) when does a polynomial perturbation of such a system in $\Omega(2)$ produce limit cycles? The articles [13], [9] , [22], [21] and [46] can be useful auxiliary references.
2- Let $\Omega(N)$ be the set of all non-smooth vector fields on a two-dimensional compact manifold N having a codimension one compact submanifold M as its discontinuity set. The problem is to study the global generic bifurcation in $\Omega(N)$. The articles [31], [33], [47] and [40] can be useful auxiliary references.
3- Study of the bifurcation set in $\Omega^r(3)$. The articles [38], [43], [50] and [40] can be useful auxiliary references.
4- Study of the dynamics of the distinguished $fold-fold$ singularity in $\Omega^r(n+1)$. The article [48] can be an useful auxiliary reference.
5- In many applications examples of non-smooth systems where the discontinuities are located on algebraic varieties are available. For instance, consider the system $\ddot{x} + xsign(x) + sign(\dot{x}) = 0$. Motivated by such models we present the following problem. Let 0 be a non-degenerate critical point of a smooth mapping $h : R^{n+1}, 0 \to R, 0$. Let $\Phi(n+1)$ be the space of all vector fields Z on $R^{n+1}, 0$ defined in the same way as $\Omega(n+1)$. We propose: i) classify the typical singularities in that space; ii) Analyze the elements of $\Phi(2)$ by means of "regularization processes"' and the methods of GSPT, similarly to Section 5. The articles [1] and [2] can be very useful auxiliary references.
6- In [28] and [29] classes of $4D-$relay systems are considered. Conditions for the existence of one-parameter families of periodic orbits terminating at typical singularities are provided. We propose to find conditions for the existence of such families for $n-dimensional$ relay systems.

VIII.2. **Conclusion.** In this paper we have presented a compact survey of the geometric/qualitative theoretical features of non-smooth dynamical systems. We feel that our survey illustrates that this field is still in its early stages but enjoying growing interest. Given the importance and the relevance of such theme, we have pointed above some open questions and we remark that there is still a wide range of bifurcation problems to be tackled. A brief summary of the main results in the text is given below.

1- We firstly deal with $2-dimensional$ non-smooth vector fields $Z = (X, Y)$ defined around the origin in R^2, where the discontinuity set is concentrated on the line $\{y = 0\}$. The first task is to characterize those systems which are structurally stable. This characterization is a starting point to establish a bifurcation theory as indicated by the Thom-Smale program.

2- In higher dimension the problem becomes much more complicated. We have presented here sufficient conditions for the $3-dimensional$ local structural stability. Any further investigation on bifurcation in this context must pass through a deep analysis of the so called $fold-fold$ singularity.

3- We have established a bridge between discontinuous and singularly perturbed smooth systems. Many similarities between such systems were observed and a comparative study of the two categories is called for.

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