Introduction / Setting

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Nonlinear parabolic eqn with rough r.h.s. f

$$\partial_t u - a(u)\partial_x^2 u = \sigma(u) f$$

1 space dim for notat. simplicity

Treat x,t similarly:

$(x,t) \rightsquigarrow (x_1,x_2)$, 1-periodic in x_1 and x_2 $\rightsquigarrow$

Hence

$$\partial_2 u - P a(u)\partial_1^2 u = P\sigma(u) f \quad , \quad Pu = u$$

where

$$Pu := u - \int_{[0,1)^2} u \, dx_1 dx_2$$

Standing assumptions on nonlinearities a, σ:
uniformly elliptic and bdd up to 3rd derivative, i.e.
for some fixed $\lambda > 0$

$$a \in [\lambda, \tfrac{1}{\lambda}], \ \underset{\text{sup norm}}{\|a'\|}, \underbrace{\|a''\|, \|a'''\|, \|\sigma\|, \|\sigma'\|, \|\sigma''\|, \|\sigma'''\|}_{\text{this amount of regularity will become clear later}} \le \tfrac{1}{\lambda}$$

state later before Step 1

What means "rough"?
Use Hölder scale w.r.t. adapted (Carnot-Carathéodory) metric

$$d(x,y) = |x_1-y_1| + \sqrt{|x_2-y_2|}$$

For $\alpha \in (0,1)$

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(y)-u(x)|}{d^\alpha(y,x)}$$

For $\alpha \in (1,2)$

$$[u]_\alpha := \sup_{x\neq y} \frac{|u(y)-u(x)-\partial_1 u(x)(y-x)_1|}{d^\alpha(y,x)}$$

$$\sim [\partial_1 u]_{\alpha-1} + \sup_{\substack{x\neq y \\ x_1 = y_1}} \frac{|u(x)-u(y)|}{d^\alpha(x,y)}$$

$\rightsquigarrow C^\alpha$

Argument later

For $\alpha \in (0,2)$

$$C^{\alpha-2} := \partial_1^2 C^\alpha + \partial_2 C^\alpha$$

alternative definition below

One distinguishes

- regular case: $f \in C^{\alpha-2}$ with $\alpha > 1$
  fairly classical: solution theory in $u \in C^\alpha$,
  see next section
- singular case: $f \in C^{\alpha-2}$ with $\alpha < 1$
  Under further assumptions we will build
  solution theory in $u \in C^\alpha$ for $\alpha > \frac{2}{3}$.

What separates regular from singular?

Product of fct u and distribution f:

$$\left.\begin{array}{l} u \in C^\alpha, \alpha > 0 \\ f \in C^\beta, \beta < 0 \end{array}\right\} \overset{\alpha+\beta>0}{\Longrightarrow} uf \in C^\beta$$

Obvious in integer spaces

$u \in C^2$, i.e. $\partial_1^2 u, \partial_2 u \in C^0$, $\partial_1 u \in C^0$

$f \in C^{-2}$, i.e. $f = \partial_1^2 g + \partial_2 h$ with $g, h \in C^0$.

$$u\,\partial_1^2 g = \partial_1^2(ug) - 2\partial_1 u\,\partial_1 g - \partial_1^2 u\, g$$
$$= \partial_1(\partial_1 u\, g) - \partial_1^2 u\, g$$

$$\begin{aligned} u(\partial_1^2 g + \partial_2 h) = \;& \partial_1^2(ug + uh) && \in C^{-2} \\ & - 2\partial_1((\partial_1 u)g) && \in C^{-1} \\ & + (\partial_1^2 u)\, g && \in C^0 \end{aligned}$$

Exact formulation and proof in fractional case below

Self-consistency in regular case:

$$\left.\begin{array}{l} f \in C^{\alpha-2} \\ u \in C^{\alpha} \Rightarrow \sigma(u) \in C^{\alpha} \end{array}\right\} \underset{\substack{\alpha+(\alpha-2)>0 \\ \Leftrightarrow\ \alpha>1}}{\Rightarrow} \sigma(u) f \in C^{\alpha-2}$$

also

$$u \in C^{\alpha} \Rightarrow \left\{\begin{array}{l} a(u) \in C^{\alpha} \\ \partial_1^2 u \in C^{\alpha-2} \end{array}\right\} a(u)\,\partial_1^2 u \in C^{\alpha-2}$$

Proof later.

Why not, hopeless in singular case?

Simple situation: no x_1-dependence (no periodicity in x_2)

(but in vertical)

$$\frac{du}{dx_2} = \sigma(u) f$$

Special case:

f = white noise in x_2

Consider

$$\frac{dv}{dx_2} = f \quad \text{, i.e. } v = \text{Brownian motion in } x_2$$

Well-known:

v is Hölder in x_2 for all exponents $< \frac{1}{2}$ (but not $= \frac{1}{2}$)

$v \in C^{\alpha}$ for all $\alpha < 1$, but not $\alpha = 1$

$f \in C^{\alpha-2}$

However, stochastic arguments allow to give a sense to the product

$$v f \in C^{\alpha-2}.$$

T. Lyons '98 (Gubinelli '04) build a deterministic solution theory for

$$\frac{du}{dx_2} = \sigma(u) f$$

for $\alpha < 1$ provided $v f \in C^{\alpha-2}$ (needs $\alpha > \frac{2}{3}$)

For experts: For f white noise, σf can be given different reasonable senses (differ by a constant)

most symmetric definition $\rightsquigarrow$ Stratonovich interpretation of $\frac{du}{dx_2} = \sigma(u) \circ f$

other definition $\rightsquigarrow$ Itô's interpretation $du = \sigma(u) df$

Provides alternative, more deterministic/pathwise, approach to SDE.

Hairer's regularity structures '14 extend this approach to SPDE.

Hairer-Pardoux '15 develop a solution theory

$$\partial_2 u - a_0 \partial_1^2 u = \sigma(u) f$$

even for $\alpha = -\frac{1}{2}-$ (so that it includes f = space-time white noise)

New elements in this course

- quasilinear
$$\partial_2 u - a(u) \partial_1^2 u = \sigma(u) f$$
- simpler tools

Tools / regular case

We seek a family $\{\psi_T\}_{T>0}$ of convolution kernels
the two properties

* compatibility with parabolic scaling

$$\frac{1}{\ell^3}\psi_T\Big(\frac{x_1}{\ell},\frac{x_2}{\ell^2}\Big) = \psi_{\tilde T}(x_1,x_2) \quad \text{for some } \tilde T$$

* semi group property

$$(f_T)_t = f_{T+t} \quad \text{where} \quad f_T := \psi_T * f$$

Achieved by

$$\mathcal{F}\psi_T(k_1,k_2) = \exp\big(-T(k_1^4 + k_2^2)\big),$$

i.e.

$$(\cdot)_T = \text{semigroup generated by } \partial_1^4 - \partial_2^2 .$$

Compatibility with parabolic scaling specifies to

$$\frac{1}{\ell^3}\psi_T\Big(\frac{x_1}{\ell},\frac{x_2}{\ell^2}\Big) = \psi_{T/\ell^4}(x_1,x_2),$$

in particular

$$\frac{1}{T^{3/4}}\psi_T\Big(\frac{x_1}{T^{1/4}},\frac{x_2}{T^{1/2}}\Big) = \psi_1(x_1,x_2)$$

i.e. $T^{1/4}$ plays role of x_1-scale
($T^{1/2}$ —— x_2 ——)

Note that ψ_1 is Schwartz fct: $\mathcal{F}\psi_1(k_1,k_2) = \exp(-(k_1^4+k_2^2))$ is Schwartz

Definition

For $\beta < 0$

$$[f]_\beta := \sup_{T\le 1} (T^{1/4})^{-\beta}\|f_T\|$$

Compatible with earlier understanding

Lemma 1

For $\alpha \in (0,1)\cup(1,2)$:

$$[\{\partial_1^2, \partial_2\}u]_{\alpha-2} \lesssim [u]_\alpha$$

Proof of Lemma 1:

"moment bounds"

(1) $\int d^{\beta}(x,0)|\partial_1^k \psi_T(x)|\,dx = (T^{1/4})^{\beta-k} \int d(\hat{x})|\hat{\partial}_1^k \psi_1(\hat{x})|\,d\hat{x} \lesssim (T^{1/4})^{\beta-k}$

$\int d^{\beta}(x,0)|\partial_2^k \psi_T(x)|\,dx \lesssim (T^{1/4})^{\beta-2k}$

(2) $\|(\partial_1^2, \partial_2)u_T\| \lesssim (T^{1/4})^{\alpha-2}[u]_\alpha$

Argument for (2):

Wlog only ∂_1^2:

$(\partial_1^2 u)_T = \partial_1^2 \psi_T * u$

$(\partial_1^2 u_T)(x) = \int \partial_1^2 \psi_T(x-y)\,u(y)\,dy$

$$= \begin{cases} \int \partial_1^2 \psi_T(x-y)\,(u(y)-u(x))\,dy & \alpha \in (0,1) \\ \int \partial_1^2 \psi_T(x-y)\,(u(y)-u(x)-\partial_1 u(x)(y-x)_1)\,dy & \alpha \in (1,2) \end{cases}$$

$$|\partial_1^2 u_T(x)| \le \underbrace{\int \underbrace{d^{\alpha}(x,y)}_{\le d^{\alpha}(x-y,0)}|\partial_1^2 \psi_T(x-y)|\,dy}_{\lesssim (T^{1/4})^{\alpha-2} \text{ by (1).}}\,[u]_\alpha$$

Lemma 2

Let $\alpha \in (0,1) \cup (1,2)$ and $\beta < 0$ with $\alpha + \beta > 0$.

Let u be a fct and f be a distrib s.t.

$[u]_\alpha < \infty$ and $[f]_\beta < \infty$.

Then there exists a unique distribution $u \diamond f$ s.t.

$$\sup_{T \le 1} (T^{1/4})^{-(\alpha+\beta)} \| \underbrace{[u,(\cdot)_T]\diamond f}_{:= u f_T - (u\diamond f)_T} - \overbrace{\partial_1 u\,\underbrace{[x_1,(\cdot)_T]}_{\substack{\text{commutator, i.e.} \\ := x_1 f_T - (x_1 f)_T}} f}^{\text{only for } \alpha\in(1,2)} \| \lesssim [u]_\alpha [f]_\beta$$

$(x_1 f)_T$: C^∞ makes sense

Comment on uniqueness: If $\widetilde{u \diamond f}$ another such distribution, then by $\triangle$-ineq

$$\sup_{T\le 1} (T^{1/4})^{-\underbrace{(\alpha+\beta)}_{>0}} \|(u\diamond f - \widetilde{u\diamond f})_T\| \lesssim [u]_\alpha [f]_\beta$$

$$\Rightarrow \lim_{T\downarrow 0} \|(u\diamond f - \widetilde{u\diamond f})_T\| = 0$$

$$\Rightarrow u\diamond f = \widetilde{u\diamond f} \text{ as distr.}$$

Furthermore

$$[u\diamond f]_\beta \lesssim ([u]_\alpha + \|u\|)[F]_\beta .$$

<u>Proof of Lemma 2</u>

Only case $\alpha \in (0,1)$ besides

(1) $$\sup_{T\le 1} (T^{1/4})^{-(\beta-1)} \|[x_1,(\cdot)_T]f\| \lesssim [f]_\beta$$ (proof after (2))

Argument for (1): Enough to show

$$\|[x_1,(\cdot)_{2T}]f\| \lesssim (T^{1/4})^1 \|f_T\|$$

Note

$$([x_1,(\cdot)_T]f)(x) = x_1\int \psi_T(x-y)f(y)dy - \int \psi_T(x-y)y_1 f(y)dy$$
$$= \int (x_1-y_1)\psi_T(x-y)f(y)dy$$
$$= T^{1/4}\, \tilde\psi_T * f$$

where

$$\tilde\psi_T(x) = \frac{x_1}{T^{1/4}}\psi_T(x) = \frac{1}{T^{3/4}}\tilde\psi_1\left(\frac{x_1}{T^{1/4}}, \frac{x_2}{T^{1/2}}\right) \text{ with } \tilde\psi_1(x) = x_1\psi_1(x)$$

Hence enough to show

$$2^{1/4}\tilde\psi_{2T} = 2\tilde\psi_T * \psi_T,$$

by scaling

$$2^{1/4}\tilde\psi_2 = 2\tilde\psi_1 * \psi_1$$

i.e

$$x_1\psi_2(x) = 2(\tilde\psi_1 * \psi_1)(x)$$

by semigroup reduces to

$$x_1(\psi_1 * \psi_1)(x) = 2(\tilde\psi_1 * \psi_1)(x)$$

By symmetry of convolution we have

$$
\begin{array}{ccc}
(\tilde{\varphi}_1 * \varphi_1)(x) & = & (\varphi_1 * \tilde{\varphi}_1)(x) \\
\| & & \| \\
\int \tilde{\varphi}_1(x-y)\varphi_1(y)\,dy & = & \int \varphi_1(x-y)\tilde{\varphi}_1(y)\,dy \\
\| & & \| \\
\int (x_1-y_1)\varphi_1(x-y)\varphi_1(y)\,dy & & \int \varphi_1(x-y)\,y_1\varphi_1(y)\,dy
\end{array}
$$

which yields

$$
\begin{array}{ccc}
x_1 \int \varphi_1(x-y)\varphi_1(y)\,dy & = & 2\int \varphi_1(x-y)\,y_1\varphi_1(y)\,dy \\
\| & & \| \\
x_1\,(\varphi_1 * \varphi_1)(x) & & 2\,(\varphi_1 * \tilde{\varphi}_1)(x)
\end{array}
$$

② For $\tau < T$ (dyadically related)

$$
u f_T - (u f_\tau)_{T-\tau} = \sum_{\substack{\tau \le t < T \\ \text{dyadic}}} \left([u,(\cdot)_t] f_t\right)_{T-2t}
$$

Argument for ②: Follows from

$$
\begin{array}{l}
u f_{2t} - (u f_t)_t \overset{!}{=} [u,(\cdot)_t] f_t \\
\|\ \text{semi group} \\
u (f_t)_t
\end{array}
$$

And thus

$$
\begin{array}{ccc}
(u f_{2t})_{T-2t} - (u f_t)_{T-t} & = & ([u,(\cdot)_t] f_t)_{T-2t} \\
\quad\quad\quad\quad \|\ \text{semi group} & & \\
\quad\quad\quad\quad ((u f_t)_t)_{T-2t} & &
\end{array}
$$

by telescoping sum.

For $\alpha \in (0,1)$

(3) $\| [u, (\cdot)_T] f \| \lesssim [u]_\alpha \|f\| (T^{1/4})^\alpha$, $\|f_T\| \le \|f\|$

Argument for (3):

$[u, (\cdot)_T] f := u f_T - (uf)_T$

$([u, (\cdot)_T] f)(x) = u(x) \int \psi_T(x-y) f(y) dy - \int \psi_T(x-y) u(y) f(y) dy$

$= \int \psi_T(x-y) (u(x) - u(y)) f(y) dy$

$$|([u, (\cdot)_T] f)(x)| \le \underbrace{\int \underbrace{d^\alpha(x,y)}_{\le d^\alpha(x-y,0)} |\psi_T(x-y)| dy}_{\overset{\text{Lemma 1 (1)}}{\lesssim} (T^{1/4})^\alpha} [u]_\alpha \|f\|$$

For $\tau < T \le 1$

(4) $\| u f_T - (u f_\tau)_{T-\tau} \| \lesssim [u]_\alpha [f]_\beta (T^{1/4})^{\alpha+\beta}$

Argument for (4):

$$\| u f_T - (u f_\tau)_{T-\tau} \| \le \sum_{\tau \le t < T} \| ([u, (\cdot)_t] f_t)_{T-t} \|$$

$$\overset{(3)}{\lesssim} \sum_{\tau \le t < T} [u]_\alpha \underbrace{\|f_t\|}_{\le [f]_\beta (t^{1/4})^\beta} (t^{1/4})^\alpha$$

$$= [u]_\alpha [f]_\beta \underbrace{\sum_{\tau \le t < T} (t^{1/4})^{\overbrace{\alpha+\beta}^{>0}}}_{\lesssim (T^{1/4})^{\alpha+\beta}}.$$

(5) We have for all $\tau \ll 1$

$[u f_\tau]_\beta \lesssim ([u]_\alpha + \|u\|) [f]_\beta$

Argument for (5): for $T \le 1$

$$\|(uf_\tau)_T\| \overset{(3)}{\lesssim} \|(uf_\tau)_{T-\tau}\| \le \underbrace{\|uf_T - (uf_\tau)_T\|}_{\overset{(4)}{\lesssim} [u]_\alpha [f]_\beta (T^{1/4})^{\alpha+\beta}} + \underbrace{\|uf_T\|}_{\le \|u\| \|f_T\|}$$

$= ((uf_\tau)_{T-\tau})_\tau$ semi group

$(T^{1/4})^{\alpha+\beta} \le (T^{1/4})^\beta$ since $T \le 1$, $\alpha \ge 0$

$\|f_T\| \le [f]_\beta (T^{1/4})^\beta$ def of $[\cdot]_\beta$

Hence

$$\|(uf_\tau)_T\| \lesssim ([u]_\alpha + \|u\|)[f]_\beta (T^{1/4})^\beta \quad \text{for all } T \le 1$$

and thus by def of $[\cdot]_\beta$

$$[uf_\tau]_\beta \lesssim ([u]_\alpha + \|u\|)[f]_\beta.$$

(6) Conclusion:

Argument for (6): According to (5), uf_τ is bdd as $\tau \downarrow 0$ in C^β. Hence there exists a subsequence $\tau \downarrow 0$ and an $u \diamond f \in C^\beta$ s.t.

$$uf_\tau \longrightarrow u \diamond f \quad \text{distributionally} \qquad (*)$$

and

$$[u \diamond f]_\beta \le \liminf_{\tau \downarrow 0} [uf_\tau]_\beta \overset{(5)}{\lesssim} ([u]_\alpha + \|u\|)[f]_\beta.$$

Furthermore, $(*)$ implies for $T > 0$

$$(uf_\tau)_{T-\tau} \longrightarrow (u \diamond f)_T \quad \text{uniformly}$$

so that (4) turns into

$$\|uf_T - (u \diamond f)_T\| \le [u]_\alpha [f]_\beta (T^{1/4})^{\alpha+\beta}$$

$\beta = 11$

Lemma 3

Suppose $\alpha \in (1,2)$. Let $f \in C^{\alpha-2}$, $a \in C^\alpha$, $u \in C^\alpha$
$a \in [\lambda, \frac{1}{\lambda}]$ for some $\lambda > 0$
be related by

$$\partial_2 u - P a \diamond \partial_1^2 u = Pf \quad , \quad Pu = u$$

Suppose

$$[a]_\alpha \ll 1$$

recall: constants depend only on λ.

Then

$$[u]_\alpha \lesssim [f]_{\alpha-2}$$

No proof — we'll prove something more complicated later.

Proposition 1

Let $a = a(u)$, $\sigma = \sigma(u)$ be s.t.
$a \in [\lambda, \frac{1}{\lambda}]$, $\|\sigma\|$, $\|a'\|$, $\|\sigma'\|$, $\|a''\|$, $\|\sigma''\| \le \frac{1}{\lambda}$

Suppose $\alpha \in (1,2)$. Let $f_i \in C^{\alpha-2}$, $u_i \in C^\alpha$, $i = 0, 1$, be related by

$$\partial_2 u_i - P a(u_i) \diamond \partial_1^2 u_i = P \sigma(u_i) \diamond f_i \quad , \quad Pu_i = u_i$$

Suppose

$$[f_i]_{\alpha-2}, [u_i]_\alpha \ll 1.$$

Then

$$[u_1 - u_0]_\alpha \lesssim [f_1 - f_0]_{\alpha-2}$$

Proof of Proposition 1

By Lemma 2, $\diamond$ is bilinear; Hence

$$\partial_2 (u_1 - u_0) - P a(u_0) \diamond \partial_1^2 (u_1 - u_0)$$
$$= P \left\{ -(a(u_1) - a(u_0)) \diamond \partial_1^2 u_1 + (\sigma(u_1) - \sigma(u_0)) \diamond f_1 + \sigma(u_0) \diamond (f_1 - f_0) \right\}$$

Since $[a(u_0)]_\alpha \le \|a'\| [u_0]_\alpha \ll 1$, we may apply Lemma 3.

$$[u_1 - u_0]_\alpha \lesssim [(a(u_1) - a(u_0)) \diamond \partial_1^2 u]_{\alpha-2}$$
$$+ [(\sigma(u_1) - \sigma(u_0)) \diamond f_1]_{\alpha-2}$$
$$+ [\sigma(u_0) \diamond (f_1 - f_0)]_{\alpha-2}$$

and then Lemma 2

$$[u_1 - u_0]_\alpha \lesssim \big([a(u_1)-a(u_0)]_\alpha + \|a(u_1)-a(u_0)\|\big)\underbrace{[\partial_1^2 u_1]_{\alpha-2}}_{\lesssim [u_1]_\alpha}$$

$$+ \big([\sigma(u_1)-\sigma(u_0)]_\alpha + \|\sigma(u_1)-\sigma(u_0)\|\big)[f_1]_{\alpha-2} \quad \text{by Lemma 1}$$

$$+ \big([\sigma(u_0)]_\alpha + \|\sigma(u_0)\|\big)[f_1 - f_0]_{\alpha-2}$$

Note that because of

$$a(u_1) - a(u_0) = \int_0^1 a'(su_1 + (1-s)u_0)\,ds\,(u_1 - u_0)$$

not only

$$\|a(u_1)-a(u_0)\| \le \|a'\|\,\|u_1 - u_0\| \quad \text{but also}$$

$$[a(u_1)-a(u_0)]_\alpha \le \|a'\|\,[u_1-u_0]_\alpha + \|a''\|\max\{[u_1]_\alpha, [u_0]_\alpha\}\,\|u_1-u_0\|$$

Note that because of $\int_{(0,1)^2}(u_1-u_0)dx = 0$ we have

$$\|u_1 - u_0\| \lesssim [u_1 - u_0]_\alpha$$

Hence

$$[a(u_1)-a(u_0)]_\alpha \lesssim \big(\underbrace{[u_1]_\alpha}_{\ll 1} + \underbrace{[u_0]_\alpha}_{\ll 1} + 1\big)[u_1 - u_0]_\alpha$$

$$\lesssim [u_1 - u_0]_\alpha$$

Similarly

$$[\sigma(u_1)-\sigma(u_0)]_\alpha \lesssim [u_1 - u_0]_\alpha$$

So that

$$[u_1 - u_0]_\alpha \lesssim [u_1 - u_0]_\alpha\big(\underbrace{[u_1]_\alpha + [f_1]_{\alpha-2}}_{\ll 1}\big)$$

$$+ \big(\underbrace{[\sigma(u_0)]_\alpha}_{\substack{\le \|\sigma'\|[u_0]_\alpha \\ \ll 1}} + \underbrace{\|\sigma(u_0)\|}_{\le \|\sigma\| \lesssim 1}\big)[f_1 - f_0]_\alpha$$

$$\lesssim [f_1 - f_0]_\alpha .$$

§ 3 Singular case

Recall that we are interested in

$$\partial_2 u - P a \partial_1^2 u = P \sigma f \quad , \quad Pu = u$$

Motivates to introduce: for any constant $a_0 \in [\lambda, \frac{1}{\lambda}]$

$$(\partial_2 - a_0 \partial_1^2) V(\cdot, a_0) = Pf \quad , \quad P V(\cdot, a_0) = V(\cdot, a_0).$$

If $a \equiv a_0$, $\sigma \equiv \sigma_0$ then of course

$$u(x) = \sigma_0 V(x, a_0).$$

We will see as a particular case of Lemma 6: $[V(\cdot, a_0)]_\alpha \leq [f]_{\alpha-2}$

Naive guess for variable a, σ

$$u(x) \approx \sigma(x) V(x, a(x)) \quad \text{— wrong}$$

Better guess

$$\{\partial_1^2, \partial_2\} u \approx \sigma E \{\partial_1^2, \partial_2\} V \quad \text{distributionally,}$$

where E takes a function in (x, a_0) and evaluates it at $a_0 = a(x)$.

— $\sigma \{\partial_1^2, \partial_2\} V(\cdot, a_0)$ does not make sense for $\alpha < 1$

Good guess:

$$\{\partial_1^2, \partial_2\} u_T \approx \sigma E \{\partial_1^2, \partial_2\} V_T \quad (\text{in } \| \ \| \text{ to order } (T^{\frac{1}{4}})^{2\alpha-2})$$

Equivalent guess (equivalence see Lemma 6)

$$u(y) - u(x) \approx \sigma(x)\big(V(y, a(x)) - V(x, a(x))\big) + \nu(x)(y-x)_1$$

$$(\text{up to order } d^{2\alpha}(y, x))$$

Definition 2 (à la Gubinelli)

Let $\alpha \in (\frac{1}{2}, 1)$. We say that u is modelled after V according to a, σ provided there exists a (periodic) fct ν s.t.

$$M_u := \sup_{\substack{(y,x) \\ y \neq x}} \frac{|u(y) - u(x) - \sigma(x)\big(V(y, a(x)) - V(x, a(x))\big) - \nu(x)(y-x)_1|}{d^{2\alpha}(y, x)}$$

Remarks

Compare expression to Hölder norm for $\alpha \in (1,2)$

$$[u]_\alpha = \sup_{\substack{(y,x)\\ y \neq x}} \frac{|u(y) - u(x) - \partial_1 u(x)(y-x)_1|}{d^\alpha(y,x)}$$

This suggests in case of $\Delta_f 2$

$$v(x) = \partial_1 u(x) - \sigma(x)\partial_1 v(x, a(x)),$$

however, $\partial_1 u$, $\partial_1 v$ do not exist individually.

Lemma 4 morally speaking: $v \diamond f$ ok, u modelled after v

Lemma 4 $\Rightarrow$ $u \diamond f$ ok

Let $\alpha \in (\frac{2}{3}, 1)$. Suppose $\{v(\cdot, a_0)\}_{a_0 \in [\lambda, \frac{1}{\lambda}]}$ is a family of functions and f a distribution s.t.

$$[\{v, \tfrac{\partial v}{\partial a_0}\}(\cdot, a_0)]_\alpha \leq N_0 \quad \text{for all } a_0 \in [\lambda, \tfrac{1}{\lambda}]$$

$$[f]_{\alpha-2} \leq N_1$$

and so that $\{v, \frac{\partial v}{\partial a_0}\}(\cdot, a_0) \diamond f$ exist as a distribution s.t.

$$\|[\{v, \tfrac{\partial v}{\partial a_0}\}(\cdot, a_0), (\cdot)_T] \diamond f\| \leq N_0 N_1 (T^{1/4})^{2\alpha-2} \quad \text{for all } T \leq 1, a_0 \in [\lambda, \tfrac{1}{\lambda}]$$

for some constants $N_0, N_1 < \infty$; with the consistency $\frac{\partial}{\partial a_0}(v(\cdot, a_0) \diamond f) = \frac{\partial v}{\partial a_0}(\cdot, a_0) \diamond f$

Suppose u is modelled after v according to $a, \sigma \in C^\alpha$.

Then there exists a unique distribution $u \diamond f$ s.t.

$$\|[u, (\cdot)_T] \diamond f - \sigma E[v, (\cdot)_T] \diamond f - \nu [x_1, (\cdot)_T] f\|$$

$$\leq N_1 (M_u + N_0([\sigma]_\alpha + \|\sigma\| [a]_\alpha)) (T^{1/4})^{\overset{3\alpha-2}{>0}}$$

for all $0 < T \leq T$,

where E denotes the evaluation of a fct in (x, a_0) at $a_0 = a(x)$

In particular for $[a]_\alpha \leq 1$

$$\|[h,(\cdot)_T]\diamond f\| \lesssim N_1\left(M_h + N_0([\sigma]_\alpha + \|\sigma\|)\right)(T^{1/4})^{2\alpha-2}$$

for all $0<T\leq 1$.

<u>Proof of Lemma 4</u>

① $[v]_{2\alpha-1} \lesssim M_h + N_0(([\sigma]_\alpha + \|\sigma\|[a]_\alpha)$

Argument for ①: Introduce abbreviations

* $\ell_x(y) := u(x) - \sigma(x)V(x,a(x)) + v(x)(y-x)_1$
  $\in \operatorname{span}\{1, y_1\}$

so that

$$|u(y) - \sigma(x)V(y,a(x)) - \ell_x(y)| \leq M_h\, d^{2\alpha}(y,x) \tag{1}$$

* $f|_x^y = f(y) - f(x)$

Then we have

$$\underbrace{[\sigma(x)V(\cdot,a(x)) - \sigma(x')V(\cdot,a(x'))]_\alpha}$$

$$= (\sigma(x)-\sigma(x'))V(\cdot,a(x)) + \sigma(x')\int_0^1 \frac{\partial V}{\partial a_0}(\cdot, sa(x)+(1-s)a(x'))ds(a(x)-a(x'))$$

$$\leq \underbrace{|\sigma(x)-\sigma(x')|}_{\leq[\sigma]_\alpha d^\alpha(x,x')} \underbrace{\sup_{a_0}[V(\cdot,a_0)]_\alpha}_{\leq N_0} + \underbrace{|\sigma(x')|}_{\leq\|\sigma\|} \underbrace{|a(x)-a(x')|}_{\leq[a]_\alpha d^\alpha(x,x')} \underbrace{\sup_{a_0}\left[\frac{\partial V}{\partial a_0}(\cdot,a_0)\right]_\alpha}_{\leq N_0}$$

$$\leq N_0([\sigma]_\alpha + \|\sigma\|[a]_\alpha)\, d^\alpha(x,x'). \tag{2}$$

Writing

$$\underbrace{(\ell_x - \ell_{x'})|_{y'}^y}_{=(v(x)-v(x'))(y-y')_1} = -(u-\sigma(x)V(\cdot,a(x)) - \ell_x)|_{y'}^y$$
$$-(u-\sigma(x')V(\cdot,a(x')) - \ell_{x'})|_{y'}^y$$
$$+(\sigma(x)V(\cdot,a(x)) - \sigma(x')V(\cdot,a(x')))|_{y'}^y$$

We from (1) & (2)

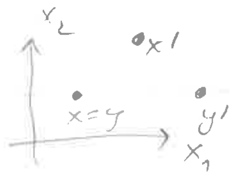
$$|(y-y')_1|\,|v(x)-v(x')| \le$$

$$\le M_h\left(d^{2\alpha}(y,x)+d^{2\alpha}(y',x)+d^{2\alpha}(y,x')+d^{2\alpha}(y',x')\right)$$

$$+\ N_0\left([\sigma]_\alpha+\|\sigma\|[a]_\alpha\right) d^\alpha(x,x')\,d^\alpha(y,y')$$

Given x, x' select

$y = x$, $y' = x \pm R(1,0)$ where $R := d(x,x')$



Then as desired

$$R\,|v(x)-v(x')| \le \left(3M_h + N_0[\sigma]_\alpha+\|\sigma\|[a]_\alpha\right)R^{2\alpha}$$

(2) Formula

$$uf_T - \sigma E[V,(\cdot)_T]\diamond f - v[x_1,(\cdot)_T]f$$
$$-(uf_\tau - \sigma E[V,(\cdot)_\tau]\diamond f - v[x_1,(\cdot)_\tau])_{T-\tau}$$
$$= \sum_{\tau\le t<T}\{ ([u,(\cdot)_t] - \sigma E[V,(\cdot)_{2t}] - v[x_1,(\cdot)_t])f_t$$
$$-[\sigma,(\cdot)_t]E[V,(\cdot)_t]\diamond f - [v,(\cdot)_t][x_1,(\cdot)_t]f$$
$$-\sigma[E,(\cdot)_t][V,(\cdot)_t]\diamond f\}$$

Argument for (2): Recall Step 2 in Lemma 2

$$uf_T - (uf_\tau)_{T-\tau} = \sum_{\tau\le t<T}([u,(\cdot)_t]f_t)_{T-2t},$$

which followed from by applying $T-2t$ to

$$u f_{2t} - (uf_t)_t = [u,(\cdot)_t]f_t \qquad (1)$$

and summing. Note l.h.s can be rewritten as

$$[u,(\cdot)_{2t}]f - ([u,(\cdot)_t]\diamond f)_t = [u,(\cdot)_t]f_t$$

Apply to $u \rightsquigarrow x_1$, regular case

$$[x_1,(\cdot)_{2t}]f - ([x_1,(\cdot)_t]f)_t = [x_1,(\cdot)_t]f_t$$

Multiply with v

$$v[x_1,(\cdot)_{2t}]f - (v[x_1,(\cdot)_t]f)_t = v[x_1,(\cdot)_t]f_t$$
$$+ [v,(\cdot)_t][x_1,(\cdot)_t]f. \qquad (2)$$

Apply to $x_1 \rightsquigarrow V(\cdot,a_0)$, $v \rightsquigarrow \sigma$ (singular case)

$$\sigma[V(\cdot,a_0),(\cdot)_{2t}]\diamond f - (\sigma[V(\cdot,a_0),(\cdot)_t]\diamond f)_t$$
$$= \sigma[V(\cdot,a_0),(\cdot)_t]f_t + [\sigma,(\cdot)_t][V(\cdot,a_0),(\cdot)_t]\diamond f$$

Apply evaluation operator E (commutes with σ, but not with $(\cdot)_t$)

$$\sigma E[V,(\cdot)_{2t}]\diamond f - (\sigma E[V,(\cdot)_t]\diamond f)_t$$
$$= \sigma E[V,(\cdot)_t]f_t + [\sigma,(\cdot)_t]E[V,(\cdot)_t]\diamond f$$
$$+ \sigma[E,(\cdot)_t][V,(\cdot)_t]\diamond f \qquad (3)$$

Take sum (1)−(3)−(2)

$$u f_{2t} - \sigma E[v,(\cdot)_{2t}]\diamond f - v[x_1,(\cdot)_{2t}]f$$
$$-(u f_t - \sigma E[v,(\cdot)_t]\diamond f - v[x_1,(\cdot)_t]f)_t$$
$$= \Big([u,(\cdot)_t] - \sigma E[v,(\cdot)_t] - v[x_1,(\cdot)_t]\Big) f_t$$
$$- [\sigma,(\cdot)_t] E[v,(\cdot)_t]\diamond f - [v,(\cdot)_t][x_1,(\cdot)_t]f$$
$$- \sigma[E,(\cdot)_t][v,(\cdot)_t]\diamond f$$

Apply $T-2t$, use semi group property on second lhs term, sum over dyadic $\tau \le t < T$.

③ $[u,(\cdot)_T]\diamond f \approx \sigma E[v,(\cdot)_T]\diamond f + v[x_1,(\cdot)_T]f$ in sense of

$$\| u f_T - \sigma E[v,(\cdot)_T]\diamond f - v[x_1,(\cdot)_T]f$$
$$-(u f_\tau - \sigma E[v,(\cdot)_\tau]\diamond f - v[x_1,(\cdot)_\tau]f)\|$$
$$\lesssim N_1\Big(M_u + N_0([\sigma]_\alpha + \|\sigma\|[a]_\alpha)\Big)(T^{1/4})^{3\alpha-2}$$

Argument for ③: Apply Δ-ineq. to ②:

$$\|\text{lhs}\| \le \sum_{\tau\le t<T}\Big\{\|([u,(\cdot)_t] - \sigma[v,(\cdot)_t] - v[x_1,(\cdot)_t])f_t\|$$
$$+ \|[\sigma,(\cdot)_t]E[v,(\cdot)_t]\diamond f\|$$
$$+ \|[v,(\cdot)_t][x_1,(\cdot)_t]f\|$$
$$+ \|\sigma\|\,\|[E,(\cdot)_t][v,(\cdot)_t]\diamond f\|\Big\}$$

First r.h.s. term

$$\{([u,(\cdot)_t] - \sigma E[v,(\cdot)_t] - v[x_1,(\cdot)_t])f_t\}(x)$$

$$= \int \varphi_t(x-y)\{(u(x)-u(y)) - \sigma(x)(v(y,a(x))-v(x,a(x))) - v(x)(y-x)_1\} f_t(y)\,dy$$

and thus

$$\|1.\ r.h.s\| \le \underbrace{\int |\varphi_t(x-y)| \underbrace{d^{2\alpha}(x,y)}_{\le d^{2\alpha}(x-y,0)} dy}_{\substack{\le (t^{1/4})^{2\alpha} \\ \text{cf ① in Lemma 1}}} M_u \underbrace{\|f_t\|}_{\le (t^{1/4})^{\alpha-2} \underbrace{[f]_{\alpha-2}}_{\le N_1}}$$

$$\le N_1 M_u (t^{1/4})^{3\alpha-2}$$

Second r.h.s. term

$$\|[\sigma,(\cdot)_t] E[v,(\cdot)_t]\diamond f\| \le [\sigma]_\alpha (t^{1/4})^\alpha \|E[v,(\cdot)_t]\diamond f\| \quad \text{cf ③ Lemma 2}$$

$$\le \sup_{a_0} \underbrace{\|[v(\cdot,a_0),(\cdot)_t]\diamond f\|}_{\le N_0 N_1 (t^{1/4})^{2\alpha-2}}$$

$$\le N_1 N_0 [\sigma]_\alpha (t^{1/4})^{3\alpha-2}$$

Third r.h.s. term

$$\|[v,(\cdot)_t][x_1,(\cdot)_t]f\| \le \underbrace{[v]_{2\alpha-1}}_{\le N_0([\sigma]_\alpha + \|\sigma\|[a]_\alpha)} (t^{1/4})^{2\alpha-1} \underbrace{\|[x_1,(\cdot)_t]f\|}_{\le (t^{1/4})^{\alpha-1} \underbrace{[f]_{\alpha-2}}_{\le N_1}} \quad \text{cf ③ Lemma 2}$$

cf ① Lemma 2

$$\le N_1 N_0([\sigma]_\alpha + \|\sigma\|[a]_\alpha)(t^{1/4})^{3\alpha-2}$$

Fourth r.h.s. term

$$\{[E,(\cdot)_t][v,(\cdot)_t]\diamond f\}(x)$$

$$= \int \varphi_t(x-y)\{\underbrace{[v(\cdot,a(x)),(\cdot)_t]\diamond f - [v(\cdot,a(y)),(\cdot)_t]\diamond f}_{= \int_0^1 [\frac{\partial v}{\partial a_0}(\cdot, s a(x)+(1-s)a(y)),(\cdot)_t]\diamond f\, ds\,(a(x)-a(y))}\}(y)\,dy$$

and thus

$$\| [E, (\cdot)_t][V, (\cdot)_t] \diamond f \|$$
$$\leq \underbrace{\int |\psi_t(x-y)| d^\alpha(x,y) dy}_{\lesssim (t^{1/4})^\alpha} [a]_\alpha \underbrace{\sup_{a_0} \| [\tfrac{\partial v}{\partial a_0}(\cdot, a_0), (\cdot)_t] \diamond f \|}_{\leq N_0 N_1 (t^{1/4})^{2\alpha-2}}$$

so that
$$\|\sigma\| \, \| [E, (\cdot)_t][V, (\cdot)_t] \diamond f \|$$
$$\lesssim N_1 \, N_0 \|\sigma\| [a]_\alpha \, (t^{1/4})^{3\alpha-2}.$$

$\square$

Recap:
Big issue in Ito-integral for SDE
$$du = \sigma(u) dV, \quad \rightsquigarrow \quad \frac{du}{dx_2} = \sigma(u) f$$
is to give a sense to
$$\int_{x_2}^{x_2+h} \sigma(u) dV \rightsquigarrow \int_{x_2}^{x_2+h} (\sigma(u) - \sigma(u(x_2))) dV$$
$$\rightsquigarrow \int_{x_2}^{x_2+h} (\sigma(u(x_2')) - \sigma(u(x_2))) f(x_2') dx_2'$$
$$= -\underbrace{\left(\sigma(u(x_2)) \int_{x_2}^{x_2+h} f(x_2') dx_2' - \int_{x_2}^{x_2+h} \sigma(u(x_2')) f(x_2') dx_2' \right)}$$
$$= [\sigma(u), \int_{x_2}^{x_2+h} \cdot \, dx_2'] f$$
$$\rightsquigarrow [\sigma(u), (\cdot)_{t}] \diamond f$$

Major idea of Lyons ("rough paths")

$V \diamond f$ ($f = dV$) can be given a sense

u looks like V on small scales

$\Rightarrow \sigma(u)$ ———— $\sigma(u) V$ ————

$\Rightarrow$ $\sigma(u) \diamond f$ can be given a sense

by showing $\int_{x_2}^{x_2+h} (\sigma(u) - \sigma(u(x_2))) dV \approx \sigma'(u(x_2)) \int_{x_2}^{x_2+h} V dV$

Contribution of Gubinelli: Give a nice sense to u looks like V on small scales "controlled rough paths"

Contribution of Hairer ("regularity structures")
from time variable to multiple variables
SDE to SPDE

Hairer: wavelets
Gubinelli, Imkeller, Perkowski: para products } harmonic analysis
here: convolutions with semigroup property
Lemma 4 deals with $\sigma \diamond f$
Lemma 4, Lemma 5 } deals with $b \diamond \partial_1^2 u$ for { b-factor, $\partial_1^2 u$-factor

Lemma 5

Let $\alpha \in (\frac{2}{3}, 1)$. Suppose b is a function and $\{V(\cdot, a_0)\}_{a_0 \in [\lambda, \frac{1}{\lambda}]}$ a family of functions such that

$$[b]_\alpha \leq N_1$$

$$[\{V, \tfrac{\partial V}{\partial a_0}\}(\cdot, a_0)]_\alpha \leq N_0 \text{ for all } a_0 \in [\lambda, \tfrac{1}{\lambda}]$$

And so that $\{b \diamond \partial_1^2 \{V, \frac{\partial V}{\partial a_0}\}(\cdot, a_0)\}_{a_0}$ exist as distributions s.t.

$$\| [b, (\cdot)_T] \diamond \partial_1^2 \{V, \tfrac{\partial V}{\partial a_0}\}(\cdot, a_0) \| \leq N_0 N_1 (T^{1/4})^{2\alpha - 2} \text{ for all } 0 < T \leq 1, \; a_0 \in [\lambda, \tfrac{1}{\lambda}]$$

for some constants $N_0, N_1 < \infty$, with consistency in the sense of $\frac{\partial}{\partial a_0}(b \diamond \partial_1^2 V(\cdot, a_0)) = b \diamond \partial_1^2 \frac{\partial V}{\partial a_0}(\cdot, a_0)$.

Suppose u is modelled after V according to $a, \sigma \in C^\alpha$.
Then there exists a unique distribution $b \diamond \partial_1^2 u$ s.t.

$$\| [b, (\cdot)_T] \diamond \partial_1^2 u - \sigma E [b, (\cdot)_T] \diamond \partial_1^2 V \|$$

$$\lesssim N_1 (M_u + N_0 ([\sigma]_\alpha + \|\sigma\| [a]_\alpha)) (T^{1/4})^{3\alpha - 2}$$

for all $0 < T \leq 1$.

In particular for $[a]_\alpha \leq 1$:

$$\| [b, (\cdot)_T] \diamond \partial_1^2 u \| \leq N_1 (M_u + N_0 (\|\sigma\| + [\sigma]_\alpha)) (T^{1/4})^{2\alpha - 2} \text{ for all } 0 < T \leq 1.$$

(Before Lemma 6)

We are interested in

$$\partial_2 u - P a \diamond \partial_1^2 u = P \sigma \diamond f$$

If with help of Lemma 4 & 5 we can define $a \diamond \partial_1^2 u$ and $\sigma \diamond f$ in the sense of

$$\|\{[a,(\cdot)_T]\diamond \partial_1^2 u,\ [\sigma,(\cdot)_T]\diamond f\}\| \le N_1^2 (T^{1/4})^{2\alpha-2},$$

then we have because of

$$\partial_2 u_T - P(a\,\partial_1^2 u_T + \sigma f_T) = -P([a,(\cdot)_T]\diamond \partial_1^2 u + [\sigma,(\cdot)_T]\diamond f)$$

that

$$\|\partial_2 u_T - P(a\,\partial_1^2 u_T + \sigma f_T)\| \le N_1^2 (T^{1/4})^{2\alpha-2}$$

State Lemma 6 (on next page)

The goal is a fixed point argument for

$$\bar{u} \longmapsto u \quad \text{where} \quad \partial_2 u - P\, a(\bar{u})\diamond \partial_1^2 u = P\, \sigma(\bar{u})\diamond f.$$

It turns out to be rather a fixed point argument in the expanded space

$(\bar{u}, \bar{a}, \bar{\sigma})$ (i.e. $\bar{u}$ is modelled after V according to $\bar{a}, \bar{\sigma}$)

$\longmapsto$

$(u, a(\bar{u}), \sigma(\bar{u}))$ (i.e. u is modelled after V according to $a(\bar{u}), \sigma(\bar{u})$)

State Prop. 1 (on next page)

Lemma 6

Fix $\alpha \in (\frac{1}{2}, 1)$. For $a_0 \in [\lambda, \frac{1}{\lambda}]$ let the function $V(\cdot, a_0)$ and the distribution f be related through

$$(\partial_2 - a_0 \partial_1^2) V(\cdot, a_0) = Pf, \quad PV(\cdot, a_0) = V(\cdot, a_0).$$

Suppose for some constant $N_0 < \infty$

$$[f]_{\alpha-2} \le N_0$$

Suppose that u is modelled after V according to $a, \sigma \in C^\alpha$ and that for some constant $N_1 < \infty$

$$\| \partial_2 u_T - P(a \partial_1^2 u_T + \sigma f_T) \| \le N_1^2 (T^{1/4})^{2\alpha-2}$$

[then] u solves $\partial_2 u - a \diamond \partial_1^2 u = \sigma \diamond f$ and $[a, (\cdot)_T] \diamond \partial_1^2 u$, $[\sigma, (\cdot)_T] \diamond \sigma$ are controlled

Then provided $[a]_\alpha \ll 1$

$$M_u \lesssim N_1^2 + N_0 ([\sigma] + \|\sigma\|)$$

Our goal is a fixed point argument for $\bar{u} \longmapsto u$ where $\partial_2 u - P a(\bar{u}) \diamond \partial_1^2 u = P\sigma(\bar{u}) \diamond f$

Proposition 1 (self map property)

Let $\alpha \in (\frac{2}{3}, 1)$. For $a_0 \in [\lambda, \frac{1}{\lambda}]$ let the function $V(\cdot, a_0)$ and the distribution f be related through

$$(\partial_2 - a_0 \partial_1^2) V(\cdot, a_0) = Pf, \quad PV(\cdot, a_0) = V(\cdot, a_0).$$

Suppose that for some constant $N_0 < \infty$

$$[f]_{\alpha-2} \le N_0$$

and that the six products $\{V(\cdot, a_0), \frac{\partial V}{\partial a_0}(\cdot, a_0)\} \diamond \{f, \partial_1^2 V(\cdot, a_0'), \partial_1^2 \frac{\partial V}{\partial a_0}(\cdot, a_0')\}$ exist as distributions with

$$\| [\{V(\cdot, a_0), \frac{\partial V}{\partial a_0}(\cdot, a_0)\}, (\cdot)_T] \diamond \{f, \partial_1^2 V(\cdot, a_0'), \partial_1^2 \frac{\partial V}{\partial a_0}(\cdot, a_0')\} \|$$

$$\le N_0^2 (T^{1/4})^{2\alpha-2} \quad \text{for } 0 < T \le 1, \ a_0, a_0' \in [\lambda, \tfrac{1}{\lambda}].$$

Suppose $\bar{u}$ is modelled after V according to $\bar{a}, \bar{\sigma} \in C^\alpha$.
Suppose u is modelled after V according to $a(\bar{u}), \sigma(\bar{u})$.
Then $a(\bar{u}) \diamond \partial_1^2 u$ and $\sigma(\bar{u}) \diamond f$ can be given a distributional sense; if in addition

and

$$\partial_2 u - P\, a(\bar u) \diamond \partial_1^2 u = P\, \sigma(\bar u) \diamond f \quad , \quad P u = u$$

therefore have

$$M_{\bar u} + N_0 \left([\bar\sigma]_\alpha + \|\bar\sigma\| [a]_\alpha \right) \ll 1$$

$$\Longrightarrow$$

$$M_u + N_0 \left([\sigma(\bar u)] + \|\sigma(\bar u)\| [a(\bar u)]_\alpha \right) \lesssim N_0$$