

An introduction to the finite and infinite dimensional reduction method

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We give an introductory description of the two gluing methods: finite dimensional and infinite dimensional. In each case we use a model problem to illustrate the ideas.

1. *Part I: Finite-dimensional reduction method*

1.1. *Introduction: What is finite dimensional Liapunov-Schmidt reduction method?*

We briefly introduce the abstract set-up of the finite dimensional Lyapunov-Schmidt reduction (although it is always used in a framework that occurs often in bifurcation theory).

Let X, Y be Banach spaces and $S(u)$ be a C^1 nonlinear map from X to Y . To find a solution to the nonlinear equation

$$S(u) = 0, \tag{1.1}$$

a natural way is to find approximations first and then to look for genuine solutions as (small) perturbations of approximations. Assume that U_λ are the approximations, where $\lambda \in \Lambda$ is the parameter (we think of Λ as the configuration space). Writing $u = U_\lambda + \phi$, then solving $S(u) = 0$ amounts to solving

$$L[\phi] + E + N(\phi) = 0, \tag{1.2}$$

where

$$L[\phi] = S'(U_\lambda)[\phi], \quad E = S(U_\lambda), \quad \text{and} \quad N(\phi) = S(U_\lambda + \phi) - S(U_\lambda) - S'(U_\lambda)[\phi].$$

Here $S'(U_\lambda)$ stands for the Fréchet derivative of S at U_λ , E denotes the error of approximation, and $N(\phi)$ denotes the nonlinear term. In order to solve (1.2), we try to invert the linear operator L so that we can rephrase the problem as a fixed point problem. That is, when L has a uniformly bounded inverse in a suitable space, one can rewrite the equation (1.2) as

$$\phi = -L^{-1}[E + N(\phi)] = \mathcal{A}(\phi).$$

What is left is to use fixed point theorems such as contraction mapping theorem.

The finite dimensional Lyapunov-Schmidt reduction deals with the situation when the linear operator L is Fredholm and its eigenfunction space associated to small eigenvalues has finite dimensional. Assuming that $\{\mathcal{Z}_1, \dots, \mathcal{Z}_n\}$ is a basis of the eigenfunction space associated to small eigenvalues of L , we can divide the procedure of solving (1.2) into two steps:

[(i)] solving the projected problem for any $\lambda \in \Lambda$,

$$\begin{cases} L[\phi] + E + N(\phi) = \sum_{j=1}^n c_j \mathcal{Z}_j, \\ \langle \phi, \mathcal{Z}_j \rangle = 0, \quad \forall j = 1, \dots, n, \end{cases}$$

where c_j may be constant or function depending on the form of $\langle \phi, \mathcal{Z}_j \rangle$.

[(ii)] solving the reduced problem

$$c_j(\lambda) = 0, \quad \forall j = 1, \dots, n,$$

by adjusting λ in the configuration space.

The original finite dimensional Liapunov-Schmidt reduction method was first introduced in a seminal paper by Floer and Weinstein [29] in their construction of single bump solutions to one dimensional nonlinear Schrodinger equations (Oh [57] generalized to high dimensional case)

$$\epsilon^2 \Delta u - V(x)u + u^p = 0, u > 0, u \in H^1(\mathbb{R}^N). \tag{1.3}$$

On the other hand, Bahri [4] and Bahri-Coron [6] developed the reduction method for critical exponent problems. In the last fifteen years, there are renewed efforts in refining the finite dimensional reduction method by many authors. When combined with variational methods, this reduction becomes "localized energy method". For subcritical exponent problems, we refer to Ambrosetti-Malchiodi [1], Gui-Wei [30], Malchiodi [50], Li-Nirenberg [43], Lin-Ni-Wei [44], Ao-Wei-Zeng [2], Wei-Yan [67] and the references therein. The localized energy method in degenerate setting is done by Byeon-Tanaka

[8, ?]. For critical exponents, we refer to Bahri-Li-Rey [7], Del Pino-Felmer-Musso [19], Del Pino-Kowalczyk-Musso [20], Li-Wei-Xu [42], Rey-Wei [59, ?] and Wei-Yan [68] and the references therein. Many new features of the finite dimensional reduction are found in the references mentioned.

In the following we shall use the following model problem (1.3) to give an introductory description of this method

$$\Delta u - u + u^3 = 0, \quad u \in H^1(\mathbf{R}^2) \quad (1.4)$$

1.2. Model Problem and Theorems

In this lecture series, we develop tools to construct infinitely many entire solutions of

$$\Delta u - u + u^3 = 0, \quad u \in H^1(\mathbf{R}^2) \quad (1.5)$$

which are defined in \mathbf{R}^2 and in particular, solutions whose maximal group of symmetry (i.e. the largest subgroup of isometries of \mathbf{R}^2 leaving the solution u fixed) is discrete. The solutions we are interested in can be either positive, negative or may change sign but they have finite energy in the sense that their energy

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbf{R}^2} (|\nabla u|^2 + u^2) dx - \frac{1}{4} \int_{\mathbf{R}^2} u^4 dx,$$

is finite.

Equations like (1.5), in dimension 2 or in higher dimensions, have been thoroughly studied over the last decades since they are ubiquitous in various models in physics, mathematical physics or biology. For example, the study of standing waves (or solitary waves) for the nonlinear Klein-Gordon or Schrödinger equations reduces to (1.5).

Let us remind the reader of a few classical and well known results concerning the solvability of (1.5) and the properties of the solutions of this equation. We will restrict our attention to the results which are relevant to the 2-dimensional case and we have deliberately chosen not to mention results which hold in higher dimensions since the list of results and contributors is by now fairly long.

It is known that there exists a unique positive, radial solution of (1.5). This solution, which will be denoted by w , has the property that it decays exponentially to 0 at infinity. More precisely, w behaves at infinity like one of the modified Bessel's functions of the second kind and hence, that there

exists a constant $C > 0$ such that

$$w(r) = C e^{-r} r^{-1/2} \left(1 + \mathcal{O}\left(\frac{1}{r}\right) \right), \quad (1.6)$$

as r tends to infinity. The classical result of Gidas, Ni and Nirenberg asserts that any finite energy, positive solution of (1.5) is (up to a translation) radially symmetric and hence finite energy, positive solutions of (1.5) are all congruent to w .

As far as sign changing solutions are concerned, Berestycki and Lions [5] have proved that (1.5) has infinitely many radial solutions which change sign. Again, these solutions do have finite energy. An open question we want to answer is whether or not there are *nonradial* sign-changing solutions. In the following I will present two results, the first one having some polygonal symmetry, the second one having no symmetry at all.

Theorem 1.1: ([51]) *Given an integer $k \geq 7$, there exist infinitely many solutions of (1.5) whose group of symmetry is the dihedral group of symmetry leaving a regular k -polygon fixed.*

Theorem 1.2: ([3]) *There exist infinitely many solutions of (1.5) which have finite energy but whose maximal group of symmetry reduces to the identity.*

Let us observe that solutions of (1.5) which have *infinite energy* do exist in abundance and it is even known that positive solutions without any symmetry do exist in this context, i.e. if the finite energy assumption is relaxed. Concerning infinite energy solutions there are two different classes of interest depending on the behavior of

$$\mathcal{E}_R(u) := \frac{1}{2} \int_{D(0,R)} (|\nabla u|^2 + u^2) dx - \frac{1}{4} \int_{D(0,R)} u^4 dx,$$

as R tends to infinity, where the integrals are understood over the disc of radius R , centered at the origin. For example, non constant, doubly periodic solutions are easy to construct using variational methods. These solutions have the property that $\mathcal{E}_R(u) \sim R^2$ as R tends to infinity. Non constant singly periodic solutions are also known to exist and they correspond to solutions for which $\mathcal{E}_R(u) \sim R$ as R tends to infinity. Solutions sharing this later property have been constructed by Malchiodi in [50] and geometrically different solutions were also obtained in [23] and in [63].

In the following I will also present a proof of Malchiodi's result, which has applications in other geometric problems, such as singular Yamabe metrics.

Theorem 1.3: ([50]) *Fixing any three rays with angle greater than $\frac{\pi}{3}$, there exist positive solutions of (1.5) with infinitely many bumps on the three rays.*

1.3. Description of the constructions

The proofs of Theorem 1.1-Theorem 1.3 will be done using the finite dimensional reduction method. Note that equation (1.5) has **NO** parameters nor there exists boundary conditions. The existence of these solution is purely due to the internal mechanism of the equations.

To help the reader, we now spend some time to briefly describe the main ideas behind the construction, without paying much attention on technical details such as estimates and functions spaces which will be used. Since we are working in \mathbf{R}^2 , it will be convenient to identify \mathbf{R}^2 with the complex plane \mathbf{C} . The scalar product in \mathbf{C} will be denoted by $\langle \cdot, \cdot \rangle_{\mathbf{C}}$ so that

$$\langle z, z' \rangle_{\mathbf{C}} := \Re(\bar{z} z').$$

In a nutshell, the idea of the construction is to start with two *finite* sets of points

$$Z^+ := \{z_j^+ \in \mathbf{C} : j = 1, \dots, n^+\} \quad \text{and} \quad Z^- := \{z_j^- \in \mathbf{C} : j = 1, \dots, n^-\},$$

and define an approximate solution to (1.5) by simply adding copies of w centered at the points z_j^+ and copies of $-w$ centered at the points z_j^- . More precisely, with these notations, we define an approximate solution \tilde{u} by the formula

$$\tilde{u} := \sum_{z \in Z^+} w(\cdot - z) - \sum_{z' \in Z^-} w(\cdot - z'). \tag{1.7}$$

We set

$$Z := Z^+ \cup Z^-,$$

and we agree that

$$\ell := \min_{z \neq z' \in Z} |z - z'|,$$

denotes the minimum of the distances between the points of Z (we assume that the points of Z are all distinct so that $\ell > 0$). Since the solution w is exponentially decreasing to 0 at infinity, the fact that \tilde{u} is a fairly good approximate solution of (1.5) as ℓ tends to infinity should not come as a surprise. Indeed, if

$$\tilde{E} := \Delta \tilde{u} - \tilde{u} + \tilde{u}^3,$$

it is not hard to check that

$$\|\tilde{E}\|_{L^\infty(\mathbf{C})} \leq C e^{-\ell} \ell^{-1/2}.$$

for some constant $C > 0$ which does not depend on $\ell \gg 1$.

The natural idea is then to let ℓ tend to infinity and to look for a solution u of (1.5) as a (small) perturbation of \tilde{u} . Writing $u = \tilde{u} + v$, this amounts to solve a nonlinear problem of the form

$$\tilde{L}v + \tilde{E} + \tilde{Q}(v) = 0, \tag{1.8}$$

where

$$\tilde{L} := \Delta - 1 + 3\tilde{u}^2,$$

is the linearized operator about \tilde{u} and where

$$\tilde{Q}(v) := v^3 + 3\tilde{u}v^2,$$

collects all the nonlinear terms. In order to solve (1.8), we try to invert \tilde{L} so that we can rephrase the problem as a fixed point problem which we solve using a fixed point theorem for a contraction mapping. It turns out that this part of the argument is rather delicate due to the presence of small eigenvalues associated to the operator \tilde{L} . Indeed, the bounded kernel of the operator

$$L_0 := \Delta - 1 + 3w^2,$$

clearly contains the functions $\partial_x w$ and $\partial_y w$ and, transplanting these functions at any of the points of Z , one can prove that there exist $2(n^+ + n^-)$ eigenfunctions of \tilde{L} which are associated to small eigenvalues which in addition tend to 0 as ℓ tends to infinity (in fact, in absolute value, these small eigenvalues can be seen to tend to 0 exponentially fast as ℓ tends to infinity). As usual when this phenomenon happens, one is lead to work orthogonally to the space of eigenfunctions associated to small eigenvalues of \tilde{L} since, on such a space, the operator \tilde{L} is invertible and has inverse whose norm can be controlled uniformly as ℓ tends to infinity. This amounts to replace the equation $\tilde{L}v = f$ by

$$\tilde{L}v + \sum_{z \in Z} \langle c_z, \nabla w(\cdot - z) \rangle_{\mathbf{C}} = f,$$

where the solution is now the function v and the complex numbers $c_z \in \mathbf{C}$. Once this is understood, one can make use of a fixed point theorem for

contraction mappings to perturb \tilde{u} into $u := \tilde{u} + v$ (where v is a small function) solution of

$$\Delta u - u + u^3 = \sum_{z \in Z} \langle F_z, \nabla w(\cdot - z) \rangle_{\mathbf{C}}, \tag{1.9}$$

where, for each $z \in Z$, the complex number $F_z \in \mathbf{C}$ depends on all the coordinates of the points of Z .

At this stage, the solvability of (1.5) reduces to the search of a set of points Z (which become *parameters* of the construction) in such a way that

$$F_z = 0, \quad \text{for all } z \in Z. \tag{1.10}$$

Observe that, *a priori* the number of equations and the number of unknowns are both equal to $2(n^+ + n^-)$ which gives some hope for the solvability of the system (1.10), even if we will see later on that the story is not that simple. This procedure is what is usually called a Liapunov-Schmidt type argument : the solvability of a nonlinear partial differential equation is reduced to the solvability of a system of equations in finite dimension.

As one can suspect, it is not possible to derive the exact expression of the complex numbers F_z in terms of the coordinates of the points of Z , but it is nevertheless possible to get a nice expansion of F_z as ℓ , the minimum of the distances between the points of Z , tends to infinity and we find, in essence, that

$$F_z \sim \sum_{z' \in Z - \{z\}} \eta_z \eta_{z'} \Upsilon(|z' - z|) \frac{z' - z}{|z' - z|}, \tag{1.11}$$

where the *interaction function* Υ is explicitly known and satisfies

$$\Upsilon(s) \sim e^{-s} s^{-1/2}, \tag{1.12}$$

as s tends to infinity and where $\eta_z = +1$ if, in the definition of \tilde{u} , there is a positive copy of w centered at the point z and $\eta_z = -1$ if, in the definition of \tilde{u} , there is negative copy of w centered at the point z .

At this stage, even if we assume that ℓ is large, finding the sets of points of Z in such a way that $F_z = 0$ for all $z \in Z$ seems to be a rather difficult and even hopeless task. However, in view of the asymptotic behavior of Υ , one quickly realizes that, in the expression of F_z given by (1.11), only the *closest neighbors of z in Z* are of interest since the influence of the other points will be of higher order and hence, will be negligible. This suggests that we should restrict our attention to the sets of points Z satisfying the following condition :

There exists $C > 0$ and $\delta > 0$ such that, if $z \neq z' \in Z$, then

$$\text{either } \ell \leq |z' - z| \leq \ell + C, \quad \text{or } |z' - z| \geq (1 + \delta)\ell. \tag{1.13}$$

Here, ℓ is considered as a parameter which will be taken very large, while $C > 0$ and $\delta > 0$ are constants which are fixed (large enough) independently of ℓ (in particular, we assume that $C \ll \delta \ell$). Under this condition, we define, for all $z \in Z$

$$N_z := \{z' \in Z - \{z\} : |z' - z| \leq \ell + C\},$$

to be the set of *closest neighbors of z in Z* and, for each $z' \in N_z$, we define $\lambda_{zz'} \in \mathbf{R}$ by

$$|z' - z| = \ell - \lambda_{zz'}.$$

Under condition (1.13) and using these notations, we find that, at main order

$$e^\ell \ell^{1/2} F_z \sim \sum_{z' \in N_z} \eta_z \eta_{z'} e^{\lambda_{zz'}} \frac{z' - z}{|z' - z|}.$$

Therefore, in order to find a set of points satisfying (1.10), it is reasonable to perturb a set Z for which

$$\sum_{z' \in N_z} a_{zz'} \frac{z' - z}{|z' - z|} = 0, \tag{1.14}$$

for all $z \in Z$, where we have defined

$$a_{zz'} := \eta_z \eta_{z'} e^{\lambda_{zz'}} \in \mathbf{R} - \{0\}.$$

In other words, the question reduces now to be able to find a set of points Z , as well as parameters $a_{zz'} \in \mathbf{R} - \{0\}$ for each $z, z' \in Z$ such that $z' \in N_z$, in such a way that (1.14) holds. But, we also need to require that

$$|z' - z| = \ell - \ln |a_{zz'}|, \tag{1.15}$$

for all $z \neq z' \in Z$ such that $z' \in N_z$. As we will see, finding a configuration of points Z satisfying (1.14) and (1.15) is not an easy task but there is an explicit algorithm that leads to configurations of such points. This is what we will explain later.

The above procedure works well if the cardinality of Z is finite (even though large). This is the case of Theorem ?? and Theorem ?. However in the case of the cardinality of Z is infinite, as in Theorem ??, new technical

difficulty arises and new ideas are needed. In particular we need to introduce the so-called Dancer’s solution.

To summarize, the method of finite dimensional reduction method applied to the equation (1.5) consists of two main steps:

Step 1: Fixing Z such that $l = \min_{z, z' \in Z, z \neq z'} |z - z'| \gg 1$, we solve the following nonlinear equation

$$\Delta u - u + u^3 = \sum_{z \in Z} \langle F_z, \nabla w(\cdot - z) \rangle_{\mathbf{C}}, \tag{1.16}$$

where F_z are Lagrange multipliers and

$$u = \tilde{u} + \phi, \quad \|\phi\| \ll 1$$

Step 2: Adjusting the set Z appropriately to find a true solution to the reduced problem

$$F_z = 0, \quad z \in Z \tag{1.17}$$

1.4. Building block and its nondegeneracy

The basic building block that we consider is

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\ 0 < w \text{ in } \mathbb{R}^N, \quad w(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty \end{cases} \tag{1.18}$$

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

$$\begin{cases} w'' + \frac{N-1}{r} w' - w + w^p = 0 & r \in (0, \infty) \\ w'(0) = 0, \quad 0 < w \text{ in } (0, \infty) \quad w(|x|) \rightarrow 0, \text{ as } |x| \rightarrow \infty \end{cases} \tag{1.19}$$

We collect the following basic properties of w , whose proof can be found in the appendix of the book [66].

- Proposition 1.1:** (a) There exist a solution $w(r)$ to (1.19);
 (b) $w(r)$ satisfies the decay estimate $w(r) = A_0 r^{-\frac{N-1}{2}} e^r (1 + O(\frac{1}{r}))$;
 (c) $w(r)$ is non-degenerate, i.e., the only bounded solution to

$$L_0(\phi) = \Delta \phi + p w(x)^{p-1} \phi - \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^N) \tag{1.20}$$

is a linear combination of the functions $\frac{\partial w}{\partial x_j}(x)$, $j = 1, \dots, N$.

In our applications below, we choose $N = 2$ and $p = 3$.

Next we consider the mapping properties of L_0 .

Given $\delta \in \mathbf{R}$, we define the weighted space

$$L_\delta^\infty(\mathbf{C}) := e^{\delta \sqrt{1+|z|^2}} L^\infty(\mathbf{C}),$$

and the norm

$$\|v\|_{L_\delta^\infty(\mathbf{C})} := \left\| e^{-\delta \sqrt{1+|z|^2}} v \right\|_{L^\infty(\mathbf{C})}.$$

We have the :

Proposition 1.4: *Assume that $\delta \in (-1, 0)$. Then, for all $f \in L_\delta^\infty(\mathbf{C})$ there exists a unique $v \in L_\delta^\infty(\mathbf{C})$ and $c \in \mathbf{C}$ such that*

$$L_0 v + \langle c, \nabla w \rangle_{\mathbf{C}} = f,$$

in \mathbf{C} and

$$\iint_{\mathbf{C}} v \partial_x w \, dx \, dy = \iint_{\mathbf{C}} v \partial_y w \, dx \, dy = 0.$$

Moreover,

$$\|v\|_{L_\delta^\infty(\mathbf{C})} + |c| \leq C \|f\|_{L_\delta^\infty(\mathbf{C})},$$

for some constant $C > 0$ which does not depend on f .

Proof: We consider the Hilbert space

$$H := \left\{ v \in H^1(\mathbf{C}) : \iint_{\mathbf{C}} \partial_x w v \, dx \, dy = \iint_{\mathbf{C}} \partial_y w v \, dx \, dy = 0 \right\}.$$

Assume that we are given $h \in L^2(\mathbf{C})$. Standard arguments (i.e. Lax-Milgram's Theorem) imply that

$$v \in H \mapsto \frac{1}{2} \iint_{\mathbf{C}} (|\nabla v|^2 + v^2 - v h) \, dx \, dy,$$

has a unique minimizer $v \in H$ (here we implicitly use the fact that $\delta < 0$ so that the last term is a continuous linear functional defined in H). Then, v is the unique weak solution of

$$\Delta v - v - h \in \text{Span} \{ \partial_x w, \partial_y w \},$$

which belongs to H . In other words, if we define the operator

$$L_0^b(v, c) := \Delta v - v + \langle c, \nabla w \rangle_{\mathbf{C}},$$

we have obtained the existence and uniqueness of a solution of

$$L_0^b(v, c) = h,$$

with $v \in H$ and $c \in \mathbf{C}$. The solvability of

$$\Delta v - v + 3w^2 v + \langle c, \nabla w \rangle_{\mathbf{C}} = h,$$

in $H \times \mathbf{C}$ can then be rephrased in the invertibility of the operator $I + K$, where by definition

$$K(v, c) := (L_0^b)^{-1}(3w^2 v). \tag{1.21}$$

Using the fact that w decays exponentially at infinity, it is easy to check that the operator K is compact, hence the invertibility of (1.21) follows from the application of Fredholm theory. Since injectivity follows from the results of Proposition 1.1. Fredholm alternative implies that $I + K$ is therefore an isomorphism.

So far, we have obtained a function v solution of $L_0 v + \langle c, \nabla w \rangle_{\mathbf{C}} = h$ which belongs to $H^1(\mathbf{C})$ but elliptic regularity implies that $v \in L^\infty(\mathbf{C})$ and that

$$\|v\|_{L^\infty(\mathbf{C})} \leq C \|f\|_{L_\delta^\infty(\mathbf{C})},$$

for some constant $C > 0$. We need to check that the solution v has the correct behavior at infinity. To this aim, just remark that if we define

$$v_\epsilon(r) := e^{\delta r} + \epsilon e^{-\delta r},$$

then

$$L_0 v_\epsilon \geq -\frac{(1 - \delta^2)}{2} v_0$$

on the complement of the ball of radius r_0 , provided is fixed large enough. Hence, the function $\left(\|v\|_{L^\infty} + \frac{2}{1-\delta^2} \|f\|_{L_\delta^\infty}\right) v_\epsilon$ is certainly a super-solution for our problem on the complement of the disc of radius $r_0 > 0$ and, passing to the limit as ϵ tends to 0, this proves that there exists a constant $C > 0$ such that

$$\|v\|_{L_\delta^\infty(\mathbf{C})} \leq C \|f\|_{L_\delta^\infty(\mathbf{C})},$$

This completes the proof of the existence of the solution. The uniqueness and the corresponding estimate follow at once from the result of Lemma ? ? ?

1.5. Linear analysis for the perturbed operator

In this section, we study the perturbed operator

$$\tilde{L} := \Delta - 1 + 3\tilde{u}^2,$$

where \tilde{u} is defined in (1.7).

Building on the previous result, we prove a similar result for the operator \tilde{L} . First we need to define weighted spaces adapted to \tilde{L} . Given $\delta < 0$, we define the weighted space

$$\mathbb{L}_\delta^\infty(\mathbf{C}) := \left(\sum_{z \in Z} e^{\delta \sqrt{1+|\cdot-z|^2}} \right) L^\infty(\mathbf{C}),$$

with the natural associated norm which is defined to be

$$\|v\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} := \left\| \left(\sum_{z \in Z} e^{\delta \sqrt{1+|\cdot-z|^2}} \right)^{-1} v \right\|_{L^\infty(\mathbf{C})}.$$

Observe that we could have used the points of \tilde{Z} instead of the points of Z to define these spaces and this would not have changed anything since the respective norms would have been uniformly equivalent independently of $\ell \gg 1$.

We define a cutoff function

$$\chi(s) := \begin{cases} 1 & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 1, \end{cases}$$

and, for all $\bar{s} > 0$ we define

$$\chi_{\bar{s}}(s) := \chi(s - \bar{s}).$$

We also define for all $z \in \tilde{Z}$, the vector field

$$\Xi_z := \chi_{\ell/4}(|\cdot - z|) \nabla w(\cdot - z), \tag{1.22}$$

being understood that we identify vectors in \mathbf{R}^2 with complex numbers.

The main result of this section reads as :

Proposition 1.5: *Assume that $\delta \in (-1, 0)$. Then, there exists $\ell_* > 0$ and, for all $\ell \geq \ell_*$, there exists a linear operator*

$$\tilde{G} : \mathbb{L}_\delta^\infty(\mathbf{C}) \rightarrow \mathbb{L}_\delta^\infty(\mathbf{C}) \times \mathbf{C}^n,$$

where n is the cardinal of Z , such that, for all $f \in \mathbb{L}_\delta^\infty(\mathbf{C})$, $\tilde{G}f =: (v, (c_z)_{z \in Z})$ satisfies

$$\tilde{L}v + \sum_{z \in \tilde{Z}} \langle c_z, \Xi_z \rangle_{\mathbf{C}} = f,$$

in \mathbf{C} . Moreover,

$$\|v\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} + \sup_{z \in \tilde{Z}} |c_z| \leq C \|f\|_{\mathbb{L}_\delta^\infty(\mathbf{C})},$$

for some constant $C > 0$ which does not depend on f and, if, for $i = 1, 2$, $\tilde{G}^{(i)}$ is the right inverse corresponding to $\tilde{z}_j^{[p,q],(i)}$, we have

$$\|\tilde{G}^{(2)} - \tilde{G}^{(1)}\| \leq C \sup_{[p,q] \in \mathcal{E}} \sup_{j=1, \dots, 2m_{[p,q]}-1} \left| \tilde{z}_j^{[p,q],(2)} - \tilde{z}_j^{[p,q],(1)} \right|, \quad (1.23)$$

for some constant $C > 0$.

Proof: We decompose f as

$$f = \left(1 - \sum_{z \in Z} \chi_{\ell/4}(|\cdot - z|) \right) f + \sum_{z \in Z} \chi_{\ell/4}(|\cdot - z|) f.$$

For each $z \in Z$, we use the result of Proposition 1.4 to solve

$$L_0 v_z + \langle c_z, \nabla w \rangle_{\mathbf{C}} = \chi_{\ell/4} f(\cdot + z).$$

We know that we have

$$\|v_z\|_{L_\delta^\infty(\mathbf{C})} + |c_z| \leq C \|f\|_{L_\delta^\infty(\mathbf{C}; \mathbf{Z})},$$

with similar estimates for the first partial derivatives of v_z .

Next, we solve

$$(\Delta - 1)v_\infty = \left(1 - \sum_{z \in Z} \chi_{\ell/4}(|\cdot - z|) \right) f - \sum_{z \in Z} [L_0, \chi_{\ell/2}] v_z(\cdot - z),$$

where $[A, B]$ denotes the commutator of A and B . Since $(\Delta - 1)1 = -1$, the maximum principle implies that

$$\|v_\infty\|_{L^\infty(\mathbf{C})} \leq C e^{\delta \ell/4} \|f\|_{\mathbb{L}_\delta^\infty(\mathbf{C})}.$$

Now, observe that, provided ℓ is chosen large enough, the function

$$v_1(z) := \sum_{z' \in Z} e^{\delta \sqrt{1+|z-z'|^2}},$$

satisfies

$$(\Delta - 1)v_1 \leq -\frac{(1 - \delta^2)}{2}v_1,$$

away from the discs of radius $\ell/4$ centered at the points of Z and the maximum principle implies that

$$e^{-\delta\ell/4} \|v_\infty\|_{L^\infty(\mathbf{C})} + \|v_\infty\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq C \|f\|_{\mathbb{L}_\delta^\infty(\mathbf{C})},$$

with similar estimates for the first partial derivatives of v_∞ .

We then define

$$v := \left(1 - \sum_{z \in Z} \chi_{\ell/8}(|\cdot - z|)\right) v_\infty + \sum_{z \in Z} \chi_{\ell/2}(|\cdot - z|) v_z(\cdot + z).$$

Using the equations satisfied by v_z and v_∞ , one gets

$$\tilde{L}v - \sum_{z \in \tilde{Z}} \langle c_z, \Xi_z \rangle_{\mathbf{C}} - f = [\tilde{L}, \chi_{\ell/8}]v_\infty + 3\tilde{u}^2(1 - \chi_{\ell/8})v_\infty + \sum_{z \in \tilde{Z}} 3(\tilde{u}^2 - u_z^2)\chi_{\ell/2}v_z,$$

and, using the estimates satisfied by v_z and v_∞ , one checks that

$$\left\| \tilde{L}v - \sum_{z \in \tilde{Z}} \langle c_z, \Xi_z \rangle_{\mathbf{C}} - f \right\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq C e^{-\kappa\ell} \|f\|_{\mathbb{L}_\delta^\infty(\mathbf{C})},$$

for some $\kappa > 0$ and also that

$$\|v\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} + \sup_{z \in \tilde{Z}} |c_z| \leq C \|f\|_{\mathbb{L}_\delta^\infty(\mathbf{C})},$$

for some constant $C > 0$ which does not depend on f . The result then follows from a simple perturbation argument, provided ℓ is taken large enough. \square

1.6. Perturbation of the approximate solution

In this section, we keep the assumptions and notations introduced in §?? and we assume that $\ell \geq \ell_*$ and $\kappa \geq \kappa_* \ell^3$ so that the results of the previous sections do hold. The solution to (1.5) we are looking for has the form $u = \tilde{u} + v$, where v is a small function, in a sense to be made precise later on and where \tilde{u} is defined in (1.7). We have already defined

$$\tilde{L} := \Delta - 1 + 3\tilde{u}^2,$$

and we now define the error

$$\tilde{E} := \Delta\tilde{u} - \tilde{u} + \tilde{u}^3,$$

as well as the nonlinear functional

$$\tilde{Q}(v) := (\tilde{u} + v)^3 - \tilde{u}^3 - 3\tilde{u}^2 v,$$

which, given our nonlinearity simplifies into

$$\tilde{Q}(v) := 3\tilde{u} v^2 + v^3.$$

With these notations, the solvability of (1.5) reduces to find a function v and complex numbers c_z , for $z \in \tilde{Z}$, solutions of the nonlinear problem

$$\tilde{L} v + \tilde{E} + \tilde{Q}(v) = \sum_{z \in \tilde{Z}} \langle c_z, \Xi_z \rangle_{\mathbf{C}}.$$

where Ξ_z has been defined in (1.22). Then, we will explain how to find the points $\tilde{z}_j^{[p,q]}$ as defined in (??) and the forces $\mathbf{f}_r^p \in \mathbf{C}$ so that $F_z = 0$ for all $z \in \tilde{Z}$.

For the time being, the main purpose of this section is to prove the :

Proposition 1.6: *There exists $\ell_* > 0$ (larger than or equal to the one defined in Proposition 1.5) such that for all $\ell \geq \ell_*$, there exists $v \in \mathbb{L}_\delta^\infty(\mathbf{C}; Z)$ and, for each $z \in \tilde{Z}$ there exists $F_z \in \mathbf{C}$ such that the function $u := \tilde{u} + v$ solves*

$$\Delta u - u + u^3 = \sum_{z \in \tilde{Z}} \langle F_z, \Xi_z \rangle_{\mathbf{C}},$$

and

$$\|v\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} + \sup_{z \in \tilde{Z}} |F_z| \leq C \Upsilon(\ell),$$

for some constant $C > 0$. Moreover the function v and the vectors F_z depend continuously on the forces \mathbf{f}_r^p given in the statement of Proposition ?? and depend smoothly on the points $\tilde{z}_j^{[p,q]}$ satisfying (??). In particular, if the function $v^{(i)}$ is the solution corresponding to the points $\tilde{z}_j^{[p,q],(i)}$, we have

$$\|v^{(2)} - v^{(1)}\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq C \Upsilon(\ell) \sup_{[p,q] \in \mathcal{E}} \sup_{j=1, \dots, 2m_{[p,q]}-1} \left| \tilde{z}_j^{[p,q],(2)} - \tilde{z}_j^{[p,q],(1)} \right|, \tag{1.24}$$

for some constant $C > 0$.

We begin with the :

Lemma 1.7: *Assume that $\delta \in (-1, 0)$ is fixed. Then, there exists a constant $C_0 > 0$, independent of $\ell \geq \ell_*$ and all parameters of the construction, such that*

$$\|\tilde{E}\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq C_0 \Upsilon(\ell).$$

Proof: We start from the fact that

$$\tilde{E} = \left(\sum_{z' \in \tilde{Z}} \eta_{z'} w(\cdot - z') \right)^3 - \sum_{z' \in \tilde{Z}} (\eta_{z'} w(\cdot - z'))^3.$$

We then estimate \tilde{E} near a given point $z \in \tilde{Z}$. In a ball of radius $\ell/2$ centered at z , we can write

$$\tilde{E} = \left(\eta_z w(\cdot - z) + \sum_{z' \neq z} \eta_{z'} w(\cdot - z') \right)^3 - (\eta_z w(\cdot - z))^3 - \sum_{z' \neq z} (\eta_{z'} w(\cdot - z'))^3,$$

and hence, we get

$$|\tilde{E}| \leq C \Upsilon(\ell) e^{\delta|\cdot - z|} \leq C \Upsilon(\ell) \sum_{z' \in \tilde{Z}} e^{\delta|\cdot - z'|},$$

for some constant $C > 0$. While, away from the balls of radius $\ell/2$ centered at the points of \tilde{Z} , we take the advantage that w decays exponentially fast to 0 at infinity, to prove that

$$|\tilde{E}| \leq C \sum_{z \in \tilde{Z}} \ell^{-3/2} e^{-3|\cdot - z|} \leq C \Upsilon(\ell) \sum_{z \in \tilde{Z}} e^{\delta|\cdot - z|},$$

for some constant $C > 0$. The estimate for \tilde{E} then follows at once. Observe that the estimate is achieved near the points of \tilde{Z} . \square

We will also need the

Lemma 1.8: *Assume that $\delta \in (-1, 0)$ is fixed. Then, there exists a constant $C_1 > 0$, independent of $\ell \geq \ell_*$ and all parameters of the construction, such that*

$$\|\tilde{Q}(v') - \tilde{Q}(v)\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq C_1 \Upsilon(\ell) \|v' - v\|_{\mathbb{L}_\delta^\infty(\mathbf{C})},$$

provided $\|v'\|_{\mathbb{L}_\delta^\infty(\mathbf{C})} \leq 2C_0 \Upsilon(\ell)$,

Proof: The estimate follows from the expression

$$\tilde{Q}(v) = v^3 + 3\tilde{u}v^2,$$

we leave the details to the reader. \square

The statement of Proposition 1.6 then follows from these two results, the result of Proposition 1.5 and a simple application of a fixed point theorem for contraction mappings in the closed ball of radius $2C_0 \Upsilon(\ell)$ in $\mathbb{L}_\delta^\infty(\mathbf{C})$,

provided ℓ is chosen large enough. Proofs with all details are given in [51]. The estimate (1.24) follows from taking the difference between the equations satisfied by the two solutions and using (1.23).

1.7. Projection of the error

Again, we keep the assumptions and notations introduced in §?? and we assume that $\ell \geq \ell_*$ and $\kappa \geq \kappa_* \ell^3$ so that the results of the previous sections do hold. As explained in the introduction, we now give the expansion of the vectors F_z as ℓ tends to infinity. In the above statements, quantities of the form $\mathcal{O}(e^{-\gamma\ell})$ depend continuously on the forces \mathbf{f}_r^p and depend smoothly on the points $\tilde{z}_j^{[p,q]}$.

We start with the general :

Lemma 1.9: *There exists $\gamma_1 > 0$ such that, for all $z \in \tilde{Z}$, we have*

$$F_z = -C_* \sum_{z' \in \tilde{N}_z} \eta_{z'} \Upsilon(|z' - z|) \frac{z' - z}{|z' - z|} + \Upsilon(\ell) \mathcal{O}(e^{-\gamma_1\ell}),$$

where \tilde{N}_z denotes the set of closest neighbors of z in \tilde{Z} and $C_* > 0$ is explicitly given by

$$\frac{1}{C_*} := \iint_{\mathbf{C}} |\partial_x w|^2 dx dy.$$

Proof: We start from the fact that, by construction, the solution u given by the result of Proposition 1.6 can be decomposed as $u = \tilde{u} + v$ where \tilde{u} is defined in (??) and where v is a solution of

$$\tilde{L}v + \tilde{E} + \tilde{Q}(v) = \sum_{z' \in \tilde{Z}} \langle F_{z'}, \Xi_{z'} \rangle_{\mathbf{C}}.$$

To obtain the expansion of F_z , it is enough to integrate the above equation against Ξ_z , for some given $z \in \tilde{Z}$. One immediately gets from Proposition 1.6, that there exists $\gamma > 0$ such that

$$\iint_{\mathbf{C}} \tilde{Q}(v) \langle c, \Xi_z \rangle_{\mathbf{C}} dx dy = \Upsilon(\ell) \mathcal{O}(e^{-\gamma\ell}).$$

for any unit vector $c \in \mathbf{C}$. Next, an integration by parts leads to

$$\iint_{\mathbf{C}} \tilde{L}v \langle c, \Xi_z \rangle_{\mathbf{C}} dx dy = \iint_{\mathbf{C}} v \tilde{L} \langle c, \Xi_z \rangle_{\mathbf{C}} dx dy.$$

Since $L_0 \langle c, \nabla w \rangle_{\mathbf{C}} = 0$, we can write

$$\begin{aligned} \iint_{\mathbf{C}} \tilde{L} v \langle c, \Xi_z \rangle_{\mathbf{C}} dx dy &= 3 \iint_{\mathbf{C}} v (\tilde{u}^2 - w^2(\cdot - z)) \langle c, \chi_{\ell/4} \nabla w(\cdot - z) \rangle_{\mathbf{C}} dx dy \\ &\quad + \iint_{\mathbf{C}} v \tilde{L} ((1 - \chi_{\ell/4}) \langle c, \nabla w(\cdot - z) \rangle_{\mathbf{C}}) dx dy, \end{aligned}$$

and it is then easy to conclude that there exists $\gamma > 0$ such that

$$\iint_{\mathbf{C}} \tilde{L} v \langle c, \Xi_z \rangle_{\mathbf{C}} dx dy = \Upsilon(\ell) \mathcal{O}(e^{-\gamma\ell}),$$

for any unit vector $c \in \mathbf{C}$.

Finally, to estimate the last term, we write

$$\tilde{E} = \left(\sum_{z' \in \tilde{Z}} \eta_{z'} w(\cdot - z') \right)^3 - \sum_{z' \in \tilde{Z}} (\eta_{z'} w(\cdot - z'))^3.$$

Since Ξ_z is supported in the disc of radius $\ell/4 + 1$, centered at z , we distinguish the closest neighbors of z and the other points of Z . Hence, we can write

$$\tilde{E} = 3 w^2(\cdot - z) \sum_{z' \in \tilde{N}_z} \eta_{z'} w(\cdot - z') + \Upsilon(\ell) \mathcal{O}(e^{-\gamma\ell}),$$

in $D(z, \ell/2)$, for some $\gamma > 0$. The result then follows from the definition of Υ . Then γ_1 in the statement of the result is the least of the γ which appear in the above estimates. \square

1.8. Reduction to the nodes

There are two different consequences according to whether $z \in \tilde{Z}$ is one of the vertices of z_r^p for some $r \in \mathcal{V}^p$ (these are called **nodes**) or one of the $\tilde{z}_j^{[p,q]}$ for some $[p, q] \in \mathcal{V}$ and some $j = 1, \dots, 2m_{[p,q]} - 1$.

We first show the we can always solve the interior balancing condition.

In the former case, we have :

Corollary 1.10: *There exists $\gamma_1 > 0$ such that, if $z \in \tilde{Z}$ is one of the z_r^p for some $r \in \mathcal{V}^p$ and some $p \in \mathcal{V}$, then*

$$F_z = -C_* \eta_z \Upsilon(\ell) \left(\mathbf{f}_r^p + \frac{\mathbf{e} + it p}{n_p} \right) + \Upsilon(\ell) (\mathcal{O}(e^{-\gamma_0\ell}) + \mathcal{O}(e^{-\gamma_1\ell})).$$

Observe that, in this expansion, according to the result of Proposition ??, the \mathbf{f}_r^p are vectors which can be prescribed arbitrarily while $\mathbf{e} \in \mathbf{C}$ and $t \in \mathbf{R}$ cannot be prescribed. Also, n_p is the number of vertices of \mathcal{V}^p .

Now, when $z \in \tilde{Z}$ is one of the $z = z_j^{[p,q]}$ for some $[p, q] \in \mathcal{V}$, because of (??) and (??), the estimate in Lemma 1.9 reduces to

$$F_z = \Upsilon(\ell) \left(\mathcal{O}(e^{-\gamma_0 \ell}) + \mathcal{O}(e^{-\gamma_1 \ell}) \right),$$

where γ_0 is the constant used in (??). Hence, in this case we need to be more precise and expand the first term in the estimate of Lemma 1.9.

Recall that we have defined in section 5.2

$$\mathbf{e}_{pq} := \frac{r_q^p - r_p^q}{|r_q^p - r_p^q|}.$$

We decompose

$$\tilde{z}_j^{[p,q]} - z_j^{[p,q]} = \dot{z}_j \mathbf{e}_{pq},$$

where $\dot{z}_j \in \mathbf{C}$. We set $\dot{z}_0 = \dot{z}_{2m_{[p,q]}} = 0$ in agreement with the fact that we do not want to modify the end points r_q^p and r_p^q . Finally, we set

$$\ell_{[p,q]} := \ell (1 - \alpha_\ell(a_{[p,q]})).$$

Then we have the :

Corollary 1.11: *There exists $\gamma_1 > 0$ and $C > 0$, such that, if we assume that $z \in \tilde{Z}$ is one of the $\tilde{z}_j^{[p,q]}$ for some $[p, q] \in \mathcal{V}$ and some $j = 1, \dots, 2m_{[p,q]} - 1$, then*

$$F_z = \pm C_* \left(\Upsilon'(\ell_{[p,q]}) \Re(\dot{z}_{j+1} - 2\dot{z}_j + \dot{z}_{j-1}) + i \frac{\Upsilon(\ell_{[p,q]})}{\ell_{[p,q]}} \Im(\dot{z}_{j+1} - 2\dot{z}_j + \dot{z}_{j-1}) \right) \mathbf{e}_{pq} + \Upsilon(\ell) \left(\mathcal{O}(e^{-\gamma_1 \ell}) + \mathcal{O}(e^{-2\gamma_0 \ell}) \right),$$

where the \pm depends on the sign of $\eta_{z'}$ where z' is one of the closest neighbors of z in \tilde{Z} .

Proof: Observe that z has only two closest neighbors which we denote by $z' = \tilde{z}_{j-1}^{[p,q]}$ and $z'' := \tilde{z}_{j+1}^{[p,q]}$. According to Lemma 1.9, we have

$$F_z = -C_* \left(\eta_{z'} \Upsilon(|z' - z|) \frac{z' - z}{|z' - z|} + \eta_{z''} \Upsilon(|z'' - z|) \frac{z'' - z}{|z'' - z|} \right) + \Upsilon(\ell) \mathcal{O}(e^{-\gamma_1 \ell}).$$

The result follows at once from the expansions of Υ given in (??) and (??), the \pm which appears in the statement of the Lemma depends on the sign of $\eta_{z'}$. For a more detailed proof of this expansion, we refer to [51], Section 5. \square

As a consequence, the set of equations $F_z = 0$, for $z = \tilde{z}_1^{[p,q]}, \dots, \tilde{z}_{2m_{[p,q]}-1}^{[p,q]}$, reduces to solving a system of the form

$$\dot{z}_{j+1} - 2\dot{z}_j + \dot{z}_{j-1} = \mathcal{O}(\ell e^{-\gamma_1 \ell}) + \mathcal{O}(\ell e^{-2\gamma_0 \ell}),$$

where we recall that, by assumption, $\dot{z}_0 = \dot{z}_{2m_{[p,q]}} = 0$.

For all $m \geq 2$, we define the $m \times m$ matrix

$$\mathbb{T} := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \in M_{m \times m}. \tag{1.25}$$

It is easy to check that the inverse of \mathbb{T} is the matrix \mathbb{T}^{-1} whose entries are given by

$$\mathbb{T}^{ij} := \min(i, j) - \frac{ij}{m+1}.$$

Hence, the above system of equations can also be written as

$$\dot{z}_j = \mathcal{O}(m \ell e^{-\gamma_1 \ell}) + \mathcal{O}(m \ell e^{-2\gamma_0 \ell}).$$

where

$$m := \max_{[p,q] \in \mathcal{E}} m_{[p,q]}.$$

We choose

$$\gamma_0 := \gamma_1/4.$$

As a consequence, it is easy to apply a fixed point theorem for contraction mappings to prove the :

Proposition 1.12: *There exists $\ell_* > 0$ (larger than or equal to the ℓ_* which appears in Proposition 1.6) such that if $\ell \geq \ell_*$ and if $m \leq e^{\gamma_1 \ell/4}$, there exist $\tilde{z}_j^{[p,q]}$, for $[p, q] \in \mathcal{E}$ and $j = 1, \dots, 2m_{[p,q]} - 1$, such that*

$$F_z = 0,$$

for all $z \in \tilde{Z}$ of the form $\tilde{z}_j^{[p,q]}$ for some $[p, q] \in \mathcal{E}$ and some $j = 1, \dots, 2m_{[p,q]} - 1$. Moreover,

$$\left| \tilde{z}_j^{[p,q]} - z_j^{[p,q]} \right| \leq e^{-\gamma_0 \ell}.$$

and the $\tilde{z}_j^{[p,q]}$ depend continuously on the \mathbf{f}_r^p .

Proof: It is enough to choose $\gamma_0 > 0$ close enough to 0. This result is then a consequence of a fixed point theorem for contraction mappings. \square

1.9. Proof of Theorem 1.1

By the reduction of the nodes method, we only discuss the balancing conditions at the nodes.

Let us consider the case of regular polygon: we may restrict to the following class of functions

$$u(z e^{\frac{2\pi}{k}}) = u(z), u(\bar{z}) = u(z)$$

where $k \geq 7$.

1.10. Proof of Theorem 1.2: The existence of infinitely many solutions of (1.5)

As usual, we keep the assumptions and notations introduced in §?? and we assume that $\ell \geq \ell_*$ and $\kappa \geq \kappa_* \ell^3$ so that the results of the previous sections do hold. Building on the previous analysis, we prove the :

Proposition 1.13: *There exist \mathbf{f}_r^p for all $p \in \mathcal{V}$ and all $r \in \mathcal{V}^p$ and there exists $\mathbf{e} \in \mathbf{C}$ and $t \in \mathbf{R}$, such that*

$$F_z = C_* \eta_z \Upsilon(\ell) \left(\frac{\mathbf{e} + it p}{n_p} \right).$$

for all $z \in \tilde{Z}$ of the form $z_r^p \in \mathcal{V}^p$ for some $p \in \mathcal{V}$. Moreover,

$$|\mathbf{f}_r^p| \leq e^{-\gamma_3 \ell},$$

for some constant $\gamma_3 > 0$.

Proof: This result is just a consequence of Brower’s fixed point theorem \square

To complete the proof of the existence of a solution of (1.5) close to \tilde{u} given by (1.7), we use a Pohozaev type argument. To explain this, let us assume that the function u solves

$$\Delta u - u + u^3 = f, \tag{1.26}$$

in \mathbf{C} and further assume that both u and f are tending to 0 exponentially fast at infinity. Then, we have the following result which is a consequence of Pohozaev identity.

Lemma 1.14: *Given any Killing vector field Ξ (i.e. a vector field which generates a group of isometries of \mathbf{C}), the following identity holds*

$$\iint_{\mathbf{C}} \langle \Xi, \nabla u \rangle_{\mathbf{C}} f \, dx \, dy = 0. \tag{1.27}$$

Proof: Multiplying (1.26) by $\Xi \cdot \nabla u$ and using simple manipulations, we get

$$\operatorname{div} \left((\Xi \cdot \nabla u) \nabla u - \frac{1}{2} (|\nabla u|^2 + u^2) \Xi + \frac{1}{4} u^4 \Xi \right) = \langle \Xi, \nabla u \rangle_{\mathbf{C}} f,$$

Then, the divergence theorem implies that

$$\iint_{\mathbf{C}} \langle \Xi, \nabla u \rangle_{\mathbf{C}} f \, dx \, dy = 0,$$

provided u and f decay fast enough at infinity. \square

In our case,

$$f := C_* \sum_{p \in \mathcal{V}} \sum_{r \in \mathcal{V}^p} \eta_r \Upsilon(\ell) \chi_{\ell/4} \left\langle \frac{\mathbf{e} + i t p}{n_p}, \nabla w(\cdot - r) \right\rangle_{\mathbf{C}}.$$

Plugging this expression into (1.27), one concludes that $\mathbf{e} = 0$ and $t = 0$ provided ℓ is chosen large enough.

Let us describe the general existence result we have obtained.

Theorem 1.15: *Assume that $(\mathcal{N}, \mathring{a})$ is a closable, flexible network and further assume that, for each $p \in \mathcal{V}$, there exists a flexible unitary network $(\mathcal{N}^p, \mathring{a}^p)$ such that properties (i)-(vii) in § ?? are fulfilled. Then, there exists $\ell_* > 0$ and $\kappa_* > 0$ such that, for all $\ell \geq \ell_*$ and $\kappa \geq \kappa_* \ell^3$, there exist a network (\mathcal{N}, a) and subnetworks (\mathcal{N}^p, a^p) and a solution of (1.5) which is close to the approximate solution \tilde{u} defined in (??).*

Remark 1.16: Observe that, in our construction, we need to assume that the integers $m_{[p,q]}$ which appear in (??) do satisfy

$$\ell^3 \ll \max_{[p,q] \in \mathcal{E}} m_{[p,q]} \ll e^{\gamma \ell},$$

for some $\gamma > 0$. The inequality on the left comes from Proposition ?? while the inequality on the right comes from Proposition 1.12. The constraint $m_{[p,q]} \ll e^{\gamma \ell}$ is purely technical and can be removed in the case where one is dealing with (1.5), however, for other applications it is not clear that this constraint can be removed.

1.11. Proof of Theorem ??: Malchiodi’s Solution

In this section, I will describe the positive solutions constructed by Malchiodi [50].

1.11.1. Invertibility of ∞ -Toda matrix

We first define a matrix $T : (x_0, (x_i)_i) \in \mathbb{R} \times \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ as

$$(Tx_0)_j = \begin{cases} -x_0, & \text{if } j = 1; \\ 0 & \text{otherwise} \end{cases}$$

$$(Tx_i)_j = \begin{cases} -2x_i, & \text{if } j = i; \\ -x_i & \text{if } j = i \pm 1; \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{T} := \begin{pmatrix} \dots & \dots & 0 & -1 & 0 & \dots & \dots & \dots & \vdots \\ \vdots & 0 & -1 & 2 & -1 & 0 & \dots & \dots & \vdots \\ \vdots & \dots & 0 & -1 & 2 & -1 & 0 & \dots & \vdots \\ \vdots & \dots & \dots & 0 & -1 & 2 & -1 & 0 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.28}$$

For $\tau > 0$, we define the weighted norm and space

$$|(x_0, (x_i)_i)|_\tau = |x_0| + \sup_i e^{i\tau} |x_i|; \quad l_\tau = \{(x_0, (x_i)_i) : |(x_0, (x_i)_i)|_\tau < \infty\}. \tag{1.29}$$

We have

Lemma 1.17: *The operator T has an inverse, whose norm is bounded by $Ce^{-\tau}$, where C is independent of τ .*

Proof: Given $(f_j)_j$ with $|(f_j)_j|_\tau < +\infty$, our goal is to solve $T(x_0, (x_i)_i) = (f_j)_j$. Define

$$x_0 = -2 \sum_{l=1}^\infty l f_l; \quad x_i = -2 \sum_{l=i}^\infty (l-i) f_l \tag{1.30}$$

one can easily check that the couple $(x_0, (x_i)_i)$ satisfies the required conditions, and that this operator is an inverse of \mathbb{T} both from the left and the right. Moreover one has

$$|x_0| \leq 2|(f_j)_j|_\tau \sum_{l=1}^\infty l e^{-l\tau} \leq C e^{-\tau} |(f_j)_j|_\tau \tag{1.31}$$

$$|x_i| \leq 2|(f_j)|_\tau \sum_{l=1}^{\infty} (l-i)e^{-l\tau} \leq Ce^{-(i+1)\tau} |(f_j)|_\tau \tag{1.32}$$

Remark: Since the operator T represents a discretization of the Laplacian in one dimension, it is natural that the inverse has the form in (1.30): indeed the Greens function of $-\Delta$ in R is piecewise affine. The specific form above guarantees that solutions tend to zero at infinity, as required. Indeed, for the following ODE,

$$-u'' = f(x)$$

the unique decaying solution is given by

$$u(x) = \int_x^{+\infty} (y-x)f(y)dy \tag{1.33}$$

1.12. Solving the reduced problem: Variational Approach

In the above sections, the reduced problem

$$F_z = 0, z \in Z \tag{1.34}$$

is solved *directly*. Namely we find the good configurations first and then perturbed around it. A key property needed for this approach is the *non-degeneracy*, i.e. the flexibility.

In the following we describe another approach, based on variational methods, which requires minimal conditions.

We start with the following model problem to illustrate the idea of variational reduction method:

$$\begin{cases} \varepsilon^2 \Delta u - V(x)u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \text{ in } \mathbb{R}^N, & u(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty. \end{cases} \tag{1.35}$$

The basic assumption on the exponent is that $1 < p < \infty$ if $N \leq 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. (More general nonlinearity can be dealt with similarly.) Without loss of generality we assume that the function $V(x)$ is a positive function satisfying

$$0 < \alpha \leq V(x) \leq \beta < +\infty. \tag{1.36}$$

We want to solve the problem

$$\begin{cases} \varepsilon^2 \Delta \tilde{u} - V(x)\tilde{u} + \tilde{u}^p = 0 & \text{in } \mathbb{R}^N \\ 0 < \tilde{u} \text{ in } \mathbb{R}^N & \tilde{u}(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty \end{cases} \tag{1.37}$$

We fix a point $\xi \in \mathbb{R}^N$. Observe that $U_{\varepsilon,\xi}(y) := V(\xi)^{\frac{1}{p-1}} w\left(\sqrt{V(\xi)} \frac{y-\xi}{\varepsilon}\right)$, is a solution of the rescaled equation

$$\varepsilon^2 \Delta u - V(\xi)u + u^p = 0.$$

We will look for a solution of (1.37) such $u_\varepsilon(x) \approx U_{\varepsilon,\xi}(y)$ for some $\xi \in \mathbb{R}^N$. We define $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda}x)$.

Let us observe that if \tilde{u} satisfies (1.37), then $u(x) = \tilde{u}(\varepsilon z)$ satisfies the problem

$$\begin{cases} \Delta u - V(\varepsilon z)u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u & \text{in } \mathbb{R}^N \end{cases} \quad u(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty \quad (1.38)$$

Let $\xi' = \frac{\xi}{\varepsilon}$. We want a solution of (1.38) with the form $u(z) = w_\lambda(z - \xi') + \tilde{\phi}(z)$, with $\lambda = V(\xi)$ and $\tilde{\phi}$ being small compared with $w_\lambda(z - \xi')$.

1.13. Equation in terms of ϕ .

Let $\phi(x) = \tilde{\phi}(x - \xi')$. Then ϕ satisfies the equation

$$\Delta_x [w_\lambda(x) + \phi(x)] - V(\xi + \varepsilon x)[w_\lambda(x) + \phi(x)] + [w_\lambda(x) + \phi(x)]^p = 0.$$

We can write this equation as

$$\Delta \phi - V(\xi)\phi + pw_\lambda^{p-1}(x)\phi - E + B(\phi) + N(\phi) = 0 \quad (1.39)$$

where $E = (V(\xi + \varepsilon x) - V(\xi))w_\lambda(x)$, $B(\phi) = (V(\xi) - V(\xi + \varepsilon x))\phi$ and $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$.

We first consider the linear problem for $\lambda = V(\xi)$,

$$\begin{cases} L(\phi) = \Delta \phi - V(\xi + \varepsilon x)\phi + pw_\lambda(x)\phi = g - \sum_{i=1}^N c_i \frac{\partial w}{\partial x_i} \\ \int_{\mathbb{R}^N} \phi \frac{\partial w_\lambda}{\partial x_i} = 0, \quad i = 1, \dots, N \end{cases} \quad (1.40)$$

The c_i 's are defined such that

$$\int_{\mathbb{R}^N} (L(\phi) - g) \frac{\partial w_\lambda}{\partial x_i} dx = 0, \quad i = 1, \dots, N \quad (1.41)$$

which is equivalent to

$$\int_{\mathbb{R}^N} (L(\frac{\partial w_\lambda}{\partial x_i})\phi - g \frac{\partial w_\lambda}{\partial x_i}) dx = 0, \quad i = 1, \dots, N \quad (1.42)$$

Denoting

$$L_0(\phi) = \Delta \phi - V(\xi)\phi + pw_\lambda(x)\phi$$

and using the fact that

$$L_0\left(\frac{\partial w_\lambda}{\partial x_i}\right) = 0$$

we see that (1.42) can be further simplified as follows

$$\int_{\mathbb{R}^N} ((V(\xi) - V(\xi + \epsilon x)) \frac{\partial w_\lambda}{\partial x_i} \phi - g \frac{\partial w_\lambda}{\partial x_i}) dx = 0, i = 1, \dots, N \quad (1.43)$$

Since

$$\int_{\mathbb{R}^N} \frac{\partial w_\lambda}{\partial x_i} \frac{\partial w_\lambda}{\partial x_j} = \int_{\mathbb{R}^N} \left(\frac{\partial w}{\partial x_1}\right)^2 \delta_{ij}$$

we find that

$$c_i = \frac{\int_{\mathbb{R}^N} ((V(\xi) - V(\xi + \epsilon x)) \frac{\partial w_\lambda}{\partial x_i} \phi - g \frac{\partial w_\lambda}{\partial x_i}) dx}{\int_{\mathbb{R}^N} \left(\frac{\partial w_\lambda}{\partial x_1}\right)^2}, i = 1, \dots, N \quad (1.44)$$

In the following we shall solve the following:

Problem: Given $g \in L^\infty(\mathbb{R}^N)$ we want to find $\phi \in L^\infty(\mathbb{R}^N)$ solution to the problem (1.40)-(1.16).

1.14. A priori estimates of a linear problem

Let us assume that $V \in C^1(\mathbb{R}^N)$, $\|V\|_{C^1} < \infty$. We assume in addition that $|\xi| \leq M_0$ and $0 < \alpha \leq V$. Then we have

Proposition 1.2: *There exists $\epsilon_0, C_0 > 0$ such that $\forall 0 < \epsilon \leq \epsilon_0, \forall |\xi| \leq M_0, \forall g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, there exist a unique solution $\phi \in L^\infty(\mathbb{R}^N)$ to (1.40), $\phi = T[g]$ satisfies*

$$\|\phi\|_{C^1} \leq C_0 \|g\|_\infty$$

Proof:

We divide the proof into two steps.

Step 1-a priori estimates: We first obtain *a priori estimates* of the problem (1.40) on bounded domains $B_R(0)$: There exist R_0, ϵ_0, C_0 such that $\forall \epsilon < \epsilon_0, R > R_0, |\xi| \leq M_0$ such that $\forall \phi, g \in L^\infty$ solving $L(\phi) = g - \sum_i c_i \frac{\partial w_\lambda}{\partial x_i}$ in B_R , $\int_{B_R} \phi \frac{\partial w_\lambda}{\partial x_i} = 0$ and $\phi = 0$ on ∂B_R , we have

$$\|\phi\|_{C^1(B_R)} \leq C_0 \|g\|_\infty$$

We prove first $\|\phi\|_\infty \leq C_0\|g\|_\infty$. Assuming the opposite, then there exist sequences $\phi_n, g_n, \varepsilon \rightarrow 0, R_n \rightarrow \infty, |\xi_n| \leq M_0$ such that

$$L(\phi_n) = g_n - \sum_i c_i^n \frac{\partial w_\lambda}{\partial x_i}.$$

The first fact is that $c_i^n \rightarrow 0$ as $n \rightarrow \infty$. This fact follows just after multiplying the equation against $\frac{\partial w_\lambda}{\partial x_i}$ and integrating by parts, as we did in (1.16).

We observe that if $\Delta\phi = g$ in B_2 then there exist C such that

$$\|\nabla\phi\|_{L^\infty(B_1)} \leq C[\|g\|_{L^\infty(B_2)} + \|\phi\|_{L^\infty(B_2)}]$$

where B_1 and B_2 are concentric balls. This implies that $\|\nabla\phi_n\|_{L^\infty(B)} \leq C$ a given bounded set $B, \forall n \geq n_0$. Hence passing to a subsequence we obtain $\phi_n \rightarrow \phi$ uniformly on compact sets, and $\phi \in L^\infty(\mathbb{R}^N)$. Observe that $\|\phi_n\|_\infty = 1$, and this implies that $\|\phi\|_\infty \leq 1$. We can also assume that up to a subsequence $\xi_n \rightarrow \xi_0$.

Since ϕ satisfies the equation $\Delta\phi - V(\xi_0)\phi + pw_{\lambda_0}^{p-1}(x)\phi = 0$, where $\lambda_0 = V(\xi_0)$, we have that $\phi \in \text{Span}\left\{\frac{\partial w_{\lambda_0}}{\partial x_1}, \dots, \frac{\partial w_{\lambda_0}}{\partial x_N}\right\}$. Taking limits in the orthogonality condition (1.40) we obtain that $\int_{\mathbb{R}^N} \phi(w_{\lambda_0})_{\partial x_i} = 0, i = 1, \dots, N$. This implies that $\phi = 0$ and hence $\|\phi_n\|_{L^\infty(B_M(0))} \rightarrow 0, \forall M < \infty$. Maximum principle yields that $\|\phi_n\|_{L^\infty(B_{R_n} \setminus B_{M_0})} \rightarrow 0$, since $|\phi_n| = o(1)$ on $\partial B_{R_n} \setminus B_{M_0}$ and $\|g_n\|_\infty \rightarrow 0$. Therefore we arrive at $\|\phi_n\|_\infty \rightarrow 0$, which is a contradiction. This implies that $\|\phi\|_{L^\infty(B_R)} \leq C_0\|g\|_{L^\infty(B_R)}$ uniformly on large R . The C^1 estimate follows from elliptic local boundary estimates for elliptic operators.

Step 2-Existence: Recall that $g \in L^\infty$. We want to solve (1.40). We claim that solving (1.40) is equivalent to finding

$$\phi \in X = \{\psi \in H_0^1(B_R) : \int \psi \frac{\partial w_\lambda}{\partial x_i} = 0, i = 1, \dots, N\}$$

such that

$$\int \nabla\phi \nabla\psi + \int V(\xi + \varepsilon x)\phi\psi - pw^{p-1}\phi\psi + \int g\psi = 0, \quad \forall \psi \in X.$$

Take general $\Psi \in H_0^1$. We can decompose into $\Psi = \psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$, with $\alpha_i = \frac{\int \Psi \frac{\partial w_\lambda}{\partial x_i}}{\int (\frac{\partial w_\lambda}{\partial x_i})^2}$. We have

$$- \int \Delta(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) \nabla\phi + \int V(\xi)(\sum_i \alpha_i (\frac{\partial w_\lambda}{\partial x_i})\phi - pw^{p-1}(\sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i})\phi = 0$$

which implies that

$$\begin{aligned} & \int \nabla \phi \nabla \Psi + \int V(\xi) \phi \Psi - p w^{p-1} \phi \Psi \\ & - \int (V(\xi) - V(\xi + \varepsilon x)) (\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) + \int g (\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) \\ & = \int [(V(\xi + \varepsilon x) - V(\xi)) \phi + g] (\Psi - \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}) \end{aligned}$$

Let $\Pi_X(\Psi) = \sum_i \alpha_i \frac{\partial w_\lambda}{\partial x_i}$. Then the above integral equals

$$\int \Pi_X([(V(\xi + \varepsilon x) - V(\xi)) \phi + g] \Psi)$$

This implies that

$$-\Delta \phi + V(\xi) \phi - p w^{p-1} \phi + \Pi_X([(V(\xi + \varepsilon x) - V(\xi)) \phi + g] \phi) = 0.$$

The problem is formulated weakly as

$$\int \nabla \phi \nabla \psi + \int (V(\xi + \varepsilon x) - p w^{p-1}) \phi \psi + \int g \psi = 0, \phi \in X, \forall \psi \in X$$

which can be written as $\phi = A[\phi] + \tilde{g}$, where A is a compact operator. The a priori estimate implies that the only solution when $g = 0$ of this equation is $\phi = 0$. We conclude existence by Fredholm alternative. Finally we let $R \rightarrow +\infty$ and obtain the existence in the whole space, thanks to the a priori estimate in Step 1. \square

We consider the assembly of multiple spikes. We look for a solution of (1.38) which near $x_j = \xi'_j = \xi_j/\varepsilon, j = 1, \dots, k$ looks like $v(x) \approx w_{\lambda_j}(x - \xi'_j), \lambda_j = V(\xi_j)$, where $w_\lambda = \lambda^{1/(p-1)} w(\sqrt{\lambda} y)$.

Let $\xi_1, \xi_2, \dots, \xi_k \in \mathbb{R}^N$ be such that $|\xi'_j - \xi'_l| \gg 1$, if $j \neq l$. We look for a solution $v(x) \approx \sum_{j=1}^k w_{\lambda_j}(x - \xi'_j), \lambda_j = V(\xi_j)$. We assume $V \in C^2(\mathbb{R}^N)$ and $\|V\|_{C^2} < \infty, 0 < \alpha \leq V$. We use the notation $W_j = w_{\lambda_j}(x - \xi'_j), \lambda_j = V(\xi_j)$ and $W = \sum_{j=1}^k W_j$.

Setting $v = W + \phi$, then ϕ solves the problem

$$\Delta \phi - V(\varepsilon x) \phi + p W^{p-1} \phi + E + N(\phi) = 0 \tag{1.45}$$

where

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W + \phi)^p - W^p - p W^{p-1} \phi.$$

Observe that $\Delta W = \sum_j \Delta W_j = \sum_j \lambda_j W_j - W_j^p$. So we can write

$$E = \sum_j (\lambda_j - V(\varepsilon x)) W_j + \left(\sum_j W_j \right)^p - \sum_j W_j^p.$$

Our next objective is to solve the approximate linearized projected problem.

1.15. Linearized (projected) problem

We use the following notation $Z_j^i = \frac{\partial W_j}{\partial x_i}$. The linearized projected problem is the following

$$\Delta \phi - V(\varepsilon x) \phi + p W^{p-1} \phi + g = \sum_{i,j} c_j^i Z_j^i, \tag{1.46}$$

with the orthogonality condition $\int \phi Z_j^i = 0, \forall i, j$. The Z_j^i 's are “nearly orthogonal” if the centers ξ_j^i are far away one to each other. The c_j^i 's are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta \phi - V(\varepsilon x) \phi + p W^{p-1} \phi + g) Z_{j_0}^{i_0} = \sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \dots, N, j_0 = 1, \dots, k$. The c_j^i 's are indeed uniquely determined provided that $|\xi_j^i - \xi_{j_0}^{i_0}| > R_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is “nearly diagonal”, which means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N} \int |\nabla W_j|^2 & \text{if } (i,j) = (i_0, j_0), \\ o(1) & \text{if not} \end{cases}$$

Moreover by a similar argument leading to (1.43) we have

$$|c_{j_0}^{i_0}| \leq C \sum_{i,j} \int |\phi| [|\lambda_j - V| + p |W^{p-1} - W_j^{p-1}|] |Z_j^i| + \int |g| |Z_j^i| \leq C (\|\phi\|_\infty + \|g\|_\infty)$$

with C is uniform for large R_0 . Furthermore if we rescale $x = \xi^l + y$, we get

$$|(\lambda_j - V(\varepsilon x)) Z_j^i| \leq |(V(\xi_j) - V(\xi_j + \varepsilon y))| \left| \frac{\partial w_{\lambda_j}}{\partial y_i} \right| \leq C \varepsilon e^{-\frac{\sqrt{\alpha}}{2} |y|},$$

because $\left| \frac{\partial w_{\lambda_j}}{\partial y_i} \right| \leq C e^{-|y| \sqrt{\lambda_j}} |y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1}) Z_j^i| = \left| \left(1 - \sum_{l \neq j} \frac{W_l}{W_j} \right)^{p-1} - 1 \right| W_j^{p-1} Z_j^i.$$

We estimate the interactions at each spike in two regions.

Observe that if $|x - \xi'_j| < \delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|$, then

$$\frac{W_l(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}|x-\xi'_j|}} < \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|}}$$

If $\delta_0 \ll 1$ but fixed, we conclude that $e^{-\sqrt{\lambda_l}|\xi'_j - \xi'_l| + \delta_0(\sqrt{\lambda_l} - \sqrt{\lambda_j}) \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} < e^{-\rho \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} \ll 1$. Thus we conclude that if $|x - \xi'_j| < \delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|$ then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \leq e^{-\rho \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} e^{-\frac{\rho}{2}|x-\xi'_j|}.$$

On the other hand if $|x - \xi'_j| > \delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|$, then

$$|(W^{p-1} - W_j^{p-1})Z_j^i| \leq C|Z_j^i| \leq C e^{-\rho \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} e^{-\frac{\rho}{2}|x-\xi'_j|}$$

As a conclusion we obtain the following estimate

$$|c_{j_0}^{i_0}| \leq C(\varepsilon + e^{-\rho \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|}) \|\phi\|_\infty + \|g\|_\infty \quad (1.47)$$

Lemma 1.1: *Given $k \geq 1$, there exist R_0, C_0, ε_0 such that for all points ξ'_j with $|\xi'_{j_1} - \xi'_{j_2}| > R_0, j = 1, \dots, k$ and all $\varepsilon < \varepsilon_0$ then exist a unique solution ϕ to the linearized projected problem with*

$$\|\phi\|_\infty \leq C_0 \|g\|_\infty.$$

Proof: As before we first prove the a priori estimate $\|\phi\|_\infty \leq C_0 \|g\|_\infty$. If not there exist $\varepsilon_n \rightarrow 0, \|\phi_n\|_\infty = 1, \|g_n\| \rightarrow 0, \xi_j^{i_n}$ with $\min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}| \rightarrow \infty$. We denote $W_n = \sum_j W_{j_n}$, and we have

$$\Delta \phi_n - V(\varepsilon_n x) \phi_n + p W_n^{p-1} \phi_n + g_n = \sum_{i,j} (c_j^i)_n (z_j^i)_n$$

Our first observation is that $(c_j^i)_n \rightarrow 0$ (which follows from the same estimate for $c_{j_0}^{i_0}$). Next we claim that $\forall R > 0 \|\phi_n\|_{L^\infty(B(\xi_j^{i_n}, R))} \rightarrow 0, j = 1, \dots, k$. If not, there exist $j_0 \|\phi_n\|_{L^\infty(B(\xi_{j_0}^{i_n}, R))} \geq \gamma > 0$. We denote $\tilde{\phi}_n(y) := \phi_n(\xi_{j_0}^{i_n} + y)$. We have $\|\tilde{\phi}_n\|_{L^\infty(B(0, R))} \geq \gamma > 0$. Since $|\Delta \tilde{\phi}_n| \leq C, \|\tilde{\phi}_n\|_\infty \leq 1$. This implies that $\|\nabla \tilde{\phi}_n\| \leq C$. Passing to a subsequence we may assume $\tilde{\phi}_n \rightarrow \tilde{\phi}$ uniformly on compact sets. Observe that also $V(\varepsilon_n(\xi_{j_0}^{i_n} + y)) = V(\varepsilon_n \xi_{j_0}^{i_n}) + O(\varepsilon_n |y|) \rightarrow \lambda_{j_0}$ over compact sets and $W_n(\xi_{j_0}^{i_n} + y) \rightarrow W_{\lambda_{j_0}}(y)$ uniformly on compact sets. This implies that $\tilde{\phi}$ is a solution of the problem

$$\Delta \tilde{\phi} - \lambda_{j_0} \tilde{\phi} + p w_{\lambda_0}^{p-1} \tilde{\phi} - 1 = 0, \quad \int \tilde{\phi} \frac{\partial W_{\lambda_{j_0}}}{\partial y_i} dy = 0, i = 1, \dots, N$$

Nondegeneracy of $w_{\lambda_{j_0}}$ implies that $\tilde{\phi} = \sum_i \alpha_i \frac{\partial w_{\lambda_{j_0}}}{\partial y_i}$. The orthogonality condition implies that $\alpha_i = 0, \forall i = 1, \dots, N$. This implies that $\tilde{\phi} = 0$ but $\|\tilde{\phi}\|_{L^\infty(B(0,R))} \geq \gamma > 0$, a contradiction.

Now we prove: $\|\phi_n\|_{L^\infty(\mathbb{R}^N \setminus \cup_n B(\xi_j^n, R))} \rightarrow 0$, provided that $R \gg 1$ and fixed so that $\phi_n \rightarrow 0$ in the sense of $\|\phi_n\|_\infty$ (again a contradiction). We will denote $\Omega_n = \mathbb{R}^N \setminus \cup_n B(\xi_j^n, R)$. For $R \gg 1$ the equation for ϕ_n has the form

$$\Delta \phi_n - Q_n \phi_n + g_n = 0$$

where $Q_n = V(\varepsilon x) - pW_n^{p-1} \geq \frac{\alpha}{2} > 0$ for some R sufficiently large (but fixed).

Let us take for $\sigma^2 < \alpha/2$

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi_j^n|} + \mu_n.$$

We denote $\varphi(y) = e^{\sigma|y|}, r = |y|$. Observe that $\Delta \varphi - \alpha/2 \varphi = e^{\sigma|y|}(\sigma^2 + \frac{N-1}{|y|} - \alpha/2) < 0$ if $|y| > R \gg 1$. Then

$$-\Delta \bar{\phi} + Q_n \bar{\phi} - g_n > -\Delta \bar{\phi} + \frac{\alpha}{2} \bar{\phi} - \|g_n\|_\infty > \frac{\alpha}{2} \mu_n - \|g_n\|_\infty > 0 \quad (1.48)$$

if we choose $\mu_n \geq \|g_n\|_\infty \frac{2}{\alpha}$. In addition we take $\mu_n = \sum_j \|\phi_n\|_{L^\infty(B(\xi_j^n, R))} + \|g_n\|_\infty \frac{2}{\alpha}$. Maximum principle implies that $\phi_n(x) \leq \bar{\phi}$ for all $x \in \Omega_n$. Taking $\delta \rightarrow 0$ this implies that $\phi_n(x) \leq \mu_n$, for all $x \in \Omega_n$. It is also true that $|\phi_n(x)| \leq \mu_n$ for all $x \in \Omega_n^c$, and this implies that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$. \square

Remark: If in addition we have the following decay for the error

$$\theta_n = \|g_n \left(\sum_j e^{-\rho|x-\xi_j^n|} \right)^{-1}\|_\infty \rightarrow 0$$

with $\rho < \alpha/2$, then we can use as a barrier function

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi_j^n|} + \mu_n \sum_j e^{-\rho|x-\xi_j^n|}$$

with $\mu_n = e^{\rho R} \sum_j \|\phi_n\|_{L^\infty(B(\xi_j^n, R))} + \theta_n$. It is easy to see that $\bar{\phi}$ is a super solution of the equation in $(\cup_j B(\xi_j, R))^c$ and we have $|\phi_n| \leq \bar{\phi}$. Letting

$\delta \rightarrow 0$ we get $|\phi_n(x)| \leq \mu_n \sum_j e^{-\rho|x-\xi_j^n|}$. As a conclusion we also get the a priori estimate

$$\|\phi\| \left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|} \right)^{-1} \|\infty \leq C \|g\| \left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|} \right)^{-1} \|\infty$$

provided that $0 \leq \rho < \alpha/2$, $|\xi_{j_1}' - \xi_{j_2}'| > R_0 \gg 1$, $\varepsilon < \varepsilon_0$.

We now give the proof of existence.

Proof: Let g be compactly supported smooth functions. The weak formulation for

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \quad \int \phi Z_j^i = 0, \forall i, j \quad (1.49)$$

is to find $\phi \in X = \{\phi \in H^1(\mathbb{R}^N) : \int \phi Z_j^i = 0, \forall i, j\}$ such that

$$\int_{\mathbb{R}^N} \nabla\phi \nabla\psi + V\phi\psi - pW^{p-1}\phi\psi - g\psi = 0, \quad \forall \psi \in X. \quad (1.50)$$

Assume ϕ solves (1.49). For $g \in L^2$, we decompose $g = \tilde{g} + \Pi[g]$ where $\int \tilde{g} Z_j^i = 0$ for all i, j , and Π is the orthogonal projection of g onto the space spanned by the Z_j^i 's.

Let $\psi \in H^1(\mathbb{R}^N)$. We now use $\psi - \Pi[\psi]$ as a test function in (1.50). Then if $\varphi \in C_c^\infty(\mathbb{R}^N)$, then we have

$$\int_{\mathbb{R}^N} \nabla\varphi \nabla(\Pi[\psi]) = - \int_{\mathbb{R}^N} \Delta\varphi \Pi[\psi] = - \int_{\mathbb{R}^N} \Pi[\Delta\varphi] \psi. \quad (1.51)$$

On the other hand, we have $\Pi[\Delta\varphi] = \sum_{i,j} \alpha_{i,j} Z_j^i$, where

$$\sum \alpha_{i,j} \int Z_{i,j} Z_{i_0,j_0} = \int \Delta\varphi Z_{i_0}^{j_0} = \int \varphi \Delta Z_{i_0}^{j_0} \quad (1.52)$$

Then $\|\Pi[\Delta\varphi]\|_{L^2} \leq C \|\varphi\|_{H^1}$. By density argument it is also true for $\varphi \in H^1$ where $\Delta\varphi \in H^{-1}$. Therefore

$$\int \nabla\phi \nabla\psi + \int (V\phi - pW^{p-1}\phi - g)\psi = \int \Pi(V\phi - pW^{p-1}\phi + g)\psi \quad (1.53)$$

It follows that ϕ solves in weak sense

$$-\Delta\phi + V\phi - pW^{p-1}\phi - g = \Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] \quad (1.54)$$

and $\Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] = \sum_{i,j} c_j^i Z_j^i$. Therefore by definition ϕ solves (1.50) implies that ϕ solves (1.54). Classical regularity gives that this

weak solution is solution of (1.54) in strong sense, in particular $\phi \in L^\infty$ so that

$$\|\phi\|_\infty \leq C\|g\|_\infty. \tag{1.55}$$

Now we give the proof of existence for (1.49). We take g compactly supported. The equation (1.54) can be written in the following way (using Riesz theorem):

$$\langle \phi, \psi \rangle_{H^1} + \langle B[\phi], \psi \rangle_{H^1} = \langle \tilde{g}, \psi \rangle_{H^1} \tag{1.56}$$

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$. We claim that B is a compact operator. Indeed if $\phi_n \rightarrow 0$ in X , then $\phi_n \rightarrow 0$ in L^2 over compacts and

$$|\langle B[\phi_n], \psi \rangle| \leq \left| \int pW^{p-1}\phi_n\psi \right| \leq \left(\int pw^{p-1}\phi_n^2 \right)^{1/2} \left(\int pW^{p-1}\psi^2 \right)^{1/2} \tag{1.57}$$

which yields

$$|\langle B[\phi_n], \psi \rangle| \leq c \left(\int pw^{p-1}\phi_n^2 \right)^{1/2} \|\psi\|_{H^1} \tag{1.58}$$

Take $\psi = B[\phi_n]$, which implies

$$\|B[\phi_n]\|_{H^1} \leq c \left(\int pw^{p-1}\phi_n^2 \right)^{1/2} \rightarrow 0. \tag{1.59}$$

This gives that B is a compact operator.

Now we prove existence with the aid of Fredholm alternative. Problem (1.49) is solvable if for $\tilde{g} = 0$ the only solution to (1.50) is $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (1.49)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Considering $g \in \Xi_{B_R(0)}$ we conclude that

$$\|\phi_R\|_\infty \leq \|g\|_\infty \tag{1.60}$$

Taking $R \rightarrow \infty$ then along a subsequence $\phi_R \rightarrow \phi$ uniform over compacts we obtain a solution to (1.49). \square

Next we want to study the dependence and regularity of the solution with respect to the parameters. Let $g \in L^\infty$. We denote $\phi = T_{\xi'}[g]$, where $\xi' = (\xi'_1, \dots, \xi'_k)$. We want to analyze derivatives $\partial_{\xi'_{j_i}} T_{\xi'}[g]$. We know that $\|T_{\xi'}[g]\| \leq C_0\|g\|_\infty$. First we make a formal differentiation. We denote $\Phi = \frac{\partial \phi}{\partial \xi'_{j_0}}$.

We have $\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i$ and $\int \phi Z_j^i = 0$, for all i, j . Formal differentiation yields

$$\Delta\Phi - V\Phi + pW^{p-1}\Phi + \partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0j_0}} Z_j^i = \sum_{i,j} \tilde{c}_j^i Z_j^i \quad (1.61)$$

where formally $\tilde{c}_i^j = \partial_{\xi_{i_0j_0}} c_i^j$. The orthogonality conditions is reduced to

$$\int_{\mathbb{R}^N} \Phi Z_j^i = \begin{cases} 0 & \text{if } j \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}} Z_j^i & \text{if } j = j_0 \end{cases} \quad (1.62)$$

Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j} Z_j^i$. We want $\int \tilde{\Phi} Z_j^i = 0$, for all i, j . We need

$$\sum_{i,j} \alpha_{i,j} \int Z_j^i Z_{\bar{j}}^{\bar{i}} = \begin{cases} 0 & \text{if } \bar{j} \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}} Z_j^i & \text{if } \bar{j} = j_0 \end{cases} \quad (1.63)$$

The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \tilde{\Phi} Z_j^i = 0$, for all i, j . We add to the equation the term $\sum_{i,j} \alpha_{i,j} (\Delta - V + pW^{p-1}) Z_j^i$, so $\tilde{\Phi}$ satisfies the equation $\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + g = \sum_{i,j} c_j^i Z_j^i$

$$\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c_j^i \partial_{\xi_{i_0j_0}} Z_j^i = \sum_{i,j} \tilde{c}_j^i Z_j^i - \sum_{i,j} \alpha_{i,j} (\Delta - V + pW^{p-1}) Z_j^i \quad (1.64)$$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$ and hence $\|\Phi\| \leq C\|g\|_\infty$.

The above formal procedure can be made rigorous by performing the analysis discretely, namely we consider solutions corresponding to ξ and $\xi + h$ respectively. Then we consider the quotient and pass the limit in h . We omit the details. In conclusion the map $\xi \rightarrow \partial_\xi \phi$ is well defined and continuous (into L^∞). Besides we also have $\|\partial_\xi \phi\|_\infty \leq C\|g\|_\infty$, and this implies

$$\|\partial_\xi T_\xi[\phi]\| \leq C\|g\| \quad (1.65)$$

1.16. Nonlinear projected problem

Consider now the nonlinear projected problem

$$\Delta\phi - V\phi + pw^{p-1}\phi + E + N(\phi) = \sum_{i,j} c_i^j Z_j^i, \quad \int \phi Z_i^j = 0, \quad \forall i, j \quad (1.66)$$

We solve this by fixed point. We have $\phi = T(E + N(\phi)) =: M(\phi)$. We define $\Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \|\phi\|_\infty \leq M\|E\|_\infty\}$. Remember that

$E = \sum_i (\lambda_j - V(\varepsilon x)) W_j + (\sum_j W_j)^p - \sum_j W_j^p$. Observe that

$$|E| \leq \varepsilon \sum_i e^{-\sigma|x-\xi'_j|} + ce^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} \sum_j e^{-\sigma|x-\xi'_j|} \quad (1.67)$$

so, for existence we have $\|E\| \leq C[\varepsilon + e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|}] =: \rho$ (see that ρ is small). Contraction mapping implies there exists a unique solution $\phi = \Phi(\xi)$ and $\|\Phi(\xi)\| \leq M\rho$. The proof is standard and hence omitted.

1.17. Differentiability in ξ' of $\Phi(\xi')$

As before the solutions obtained for the nonlinear projected problem has more regularity. In fact we can write the equation for Φ as

$$\Phi - T'_\xi(E'_\xi + N'_\xi(\phi)) = A(\Phi, \xi') = 0 \quad (1.68)$$

If $(D_\Phi A)(\Phi(\xi'), \xi')$ is invertible in L^∞ , then $\Phi(\xi')$ turns out to be of class C^1 . This is a consequence of the fixed point characterization, i.e., $D_\Phi A(\Phi(\xi'), \xi') = I + o(1)$ (the order $o(1)$ is a direct consequence of fixed point characterization). Then it is invertible. Contraction mapping theorem yields the existence of C^1 derivative of $A(\Phi, \xi')$ in (ϕ, ξ') . This implies $\Phi(\xi')$ is C^1 . With a little bit of more work we can show that $\|D'_\xi \Phi(\xi')\| \leq C\rho$ (just using the derivative given by the implicit function theorem).

1.18. Solving the reduced problem: direct method

By (1.66), to solve (1.45), we need to find ξ' such that the reduced problem

$$c_j^i = 0, \forall i, j \quad (1.69)$$

to get a solution to the original problem (1.38). There are two ways to solve the reduced problem (1.69): the first one is the direct method, and the second one is the variational reduction method. We describe the first method first by proving the following

Theorem 1: (Oh [57]) *Assume that $\xi_j^0, j = 1, \dots, k$ are k distinct non-degenerate critical points of V . Then there exist a solution u_ε to the original problem with*

$$u_\varepsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon), \quad \xi_j^\varepsilon \rightarrow \xi_j^0$$

Proof: To solve the problem (1.69) we first obtain the asymptotic formula for c_j^i . To this end we multiply the equation (1.66) by $Z_{j_0}^{i_0}$ and integrate by parts. We obtain

$$\int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0} c_j^i = \int_{\mathbb{R}^N} (V(\xi_j + \epsilon x) - V(\xi_j)) w_{\xi_j} Z_{j_0}^{i_0} + O(\epsilon^2)$$

and thus

$$c_{j_0}^{i_0} \sim \partial_{i_0} V(\xi_j^0) + O(\epsilon)$$

The nondegeneracy of the critical point $\nabla V(\xi_j^0)$ and implicit function theorem yields the existence of $\xi_j = \xi_j^0 + O(\epsilon)$ such that (1.69) holds. \square

The direct method can be used to construct multiple spike solutions for problems *without variational structure*, such as Gierer-Meinhardt system. For this application we refer to [66].

1.19. Solving the reduced problem: variational reduction

If the problem concerned has a variational structure, it is more appropriate to use a variational reduction method to solve (1.69). This method gives much stronger results under very weak assumptions.

We now describe the procedure that we call Variational Reduction in which the problem of finding ξ' with $c_j^i = 0$, for all i, j , is equivalent to finding a critical point of a reduced functional of ξ' .

Define an energy functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(\epsilon x) v^2 - \frac{1}{p+1} \int_{\mathbb{R}^{N+1}} v_+^{p+1} \tag{1.70}$$

where $v \in H^1(\mathbb{R}^N)$ and $1 < p < \frac{N+2}{N-2}$. Since p is subcritical, by standard elliptic regularity arguments and Maximum Principle v is a solution of the problem

$$\Delta v - Vv + v^p = 0, v \rightarrow 0 \tag{1.71}$$

if and only if $v \in H^1(\mathbb{R}^N)$ and $J'(v) = 0$. Observe that $\langle J'(v), \varphi \rangle = \int \nabla v \nabla \varphi + Vv\varphi - v_+^p \varphi$.

We will prove the following Variational Reduction Principle

Theorem 2: $v = W_{\xi'_*} + \phi(\xi')$ is a solution of the original problem (for $\rho \ll 1$) if and only if

$$\partial_{\xi'} J(W_{\xi'} + \phi(\xi'))|_{\xi'=\xi'_*} = 0. \tag{1.72}$$

Proof: Indeed, observe that $v(\xi') := W_{\xi'} + \phi(\xi')$ solves the problem $\Delta v(\xi') - V(\varepsilon x)v(\xi') + v(\xi')^p = \sum_{i,j} c_j^i Z_j^i$ and also that

$$\partial_{\xi_{j_0^{i_0}}} J(v(\xi')) = \langle J'(v(\xi')), \partial_{\xi_{j_0^{i_0}}} v(\xi') \rangle = - \sum_{j,i} c_j^i \int Z_j^i \partial_{\xi_{j_0^{i_0}}} v = - \sum_{i,j} c_j^i \int Z_j^i (\partial_{\xi_{j_0^{i_0}}} W_{\xi'} + \partial_{\xi_{j_0^{i_0}}} \phi(\xi')). \tag{1.73}$$

Recall that $W_{\xi'} = \sum_{j=1}^k w_{\lambda_j}(x - \xi'_j)$,

$$\partial_{\xi_{j_0^{i_0}}} W_{\xi'} = \partial_{\xi_{j_0^{i_0}}} w_{\lambda_{j_0}(\xi')} (x - \xi'_j) = (\partial_{\lambda} w_{\lambda}(x - \xi'_{j_0}))|_{\lambda=\lambda_{j_0}} - \partial_{x_{i_0}} w_{\lambda_{j_0}}(x - \xi'_{j_0}) = O(e^{-\delta|x-\xi'_{j_0}|})o(\varepsilon) - Z_{j_0^{i_0}}(x) \tag{1.74}$$

This is because $\partial_{\lambda} w_{\lambda} = O(e^{-\delta|x-\xi'_{j_0}|})$. On the other hand since $\int Z_j^i \phi(\xi') = 0$ we have

$$\int Z_j^i \partial_{\xi_{j_0^{i_0}}} \phi(\xi') = - \int \phi(\xi') \partial_{\xi_{j_0^{i_0}}} Z_j^i$$

which is small thanks to the fact that $|\phi| \leq C\rho e^{-\delta|x-\xi'_{j_0}|}$. Finally, observe that

$$- \int Z_j^i (\partial_{\xi_{j_0^{i_0}}} W_{\xi'} + \partial_{\xi_{j_0^{i_0}}} \phi) = \int Z_j^i Z_{j_0^{i_0}} + O(\rho) \tag{1.75}$$

The matrix of these numbers is invertible provided $\rho \ll 1$. □

We now discuss several applications of the reduction principle.

Theorem 3: (*del Pino and Felmer [17]*) Assume that there exists an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{\partial\Lambda} V > \inf_{\Lambda} V, \tag{1.76}$$

then there exist a solution to the original problem, v_{ε} with $v_{\varepsilon}(x) = w_{V(\xi_{\varepsilon})}((x - \xi_{\varepsilon})/\varepsilon) + o(1)$ and $V(\xi_{\varepsilon}) \rightarrow \min_{\Lambda} V$, $\xi = \xi_{\varepsilon}$.

Theorem 4: (*del Pino-Felmer [18]*) Assume that $\Lambda_1, \dots, \Lambda_k$ are disjoint bounded sets with

$$\inf_{\Lambda_j} V < \inf_{\partial\Lambda_j} V, j = 1, \dots, k.$$

Then there exist a solution u_{ε} to the original problem with

$$u_{\varepsilon}(x) \approx \sum_{j=1}^k w_{V(\xi_j^{\varepsilon})}(x - \xi_j^{\varepsilon}/\varepsilon), \quad \xi_j^{\varepsilon} \in \Lambda_j$$

and $V(\xi_j^{\varepsilon}) \rightarrow \inf_{\Lambda_j} V$. The same result holds if the minimum is replaced by maximum.

Theorem 5: (Kang-Wei [41]) Let Γ be a bounded open set such that

$$\max_{\Gamma} V(x) > \max_{\partial\Gamma} V(x)$$

Then for any positive integer K there exists a solution u_ϵ such that

$$u_\epsilon(x) \approx \sum_{j=1}^k w_{V(\xi_j^\epsilon)}(x - \xi_j^\epsilon/\epsilon), \quad \xi_j^\epsilon \in \Lambda, V(\xi_j^\epsilon) \rightarrow \max_{\Lambda} V(x)$$

Proof:

Assume that $j = 1$ first so that $v(\xi') = W_{\xi'} + \phi(\xi')$. Then we can compute the reduced energy as follows:

$$J(v(\xi')) = J(W_{\xi'} + \phi(\xi')) + \langle J'(W_{\xi'} + \phi), -\phi \rangle + \frac{1}{2} J''(W_{\xi'} + (1-t)\phi)[\phi]^2 \quad (1.77)$$

(This follows from Taylor expansion of the function $\alpha(t) = J(W_{\xi'} + (1-t)\phi)$.) Observe that $\langle J'(W_{\xi'} + \phi), -\phi \rangle = \sum_{i,j} c_j^i \int Z_i^j \phi = 0$. Also observe that

$$J''(W_{\xi'} + (1-t)\phi)[\phi]^2 = \int |\nabla\phi|^2 + V(\epsilon x)\phi^2 - p(W_{\xi'} + (1-t)\phi)\phi^2 = O(\epsilon^2) \quad (1.78)$$

uniformly on ξ' because $\nabla\phi, \phi = O(\epsilon e^{-\delta|x-\xi'|})$. We call $\Phi(\xi) := J(v(\xi')) = J(W_{\xi'}) + O(\epsilon^2)$, and

$$J(W_{\xi'}) = \frac{1}{2} \int |\nabla W_{\xi'}|^2 + V(\xi)W_{\xi'}^2 - \frac{1}{p+1} \int W_{\xi'}^{p+1} + \int (V(\epsilon x) - V(\xi'))W_{\xi'}^2 \quad (1.79)$$

Taking $\lambda = V(\xi)$, we have that

$$\int |\nabla w_\lambda(x)|^2 = \lambda^{-N/2} \int |\nabla w(\lambda^{1/2}x)|^2 \lambda^{1+2/(p-1)} \lambda^{N/2} dx = \lambda^{-N/2+p+1/p-1} |\nabla w(y)|^2 dy \quad (1.80)$$

and

$$\lambda \int w_\lambda^2(x) = \lambda^{-N/2p+1/p-1} \int w(y)^{p+1} dy \quad (1.81)$$

This implies that

$$\frac{1}{2} \int |\nabla W_{\xi'}|^2 + V(\xi')W_{\xi'}^2 - \frac{1}{p+1} \int W_{\xi'}^{p+1} = V(\xi')^{p+1/p-1-N/2} c_{p,N} \quad (1.82)$$

and we also have

$$\int (V(\epsilon x) - V(\xi'))w_\lambda(x - \xi')^2 = O(\epsilon) \quad (1.83)$$

uniformly in ξ' .

In summary we have the following asymptotic expansion of the reduced energy

$$\Phi(\xi) = J(v(\xi')) = V(\xi)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon) \tag{1.84}$$

To prove Theorem 3 we observe that $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall \varepsilon \ll 1$ we have

$$\inf_{\xi \in \Lambda} \Phi(\xi) < \inf_{\xi \in \partial\Lambda} \Phi(\xi) \tag{1.85}$$

and therefore Φ has a local minimum $\xi_\varepsilon \in \Lambda$ and $V(\xi_\varepsilon) \rightarrow \min_\Lambda V$. The same procedure also works for local maximums.

For several separated local minimums, the proof is similar. In fact when $|\xi_{j_1} - \xi_{j_2}| > \delta$, for all $j_1 \neq j_2$, we have $\rho = e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} + \varepsilon \leq e^{-\delta_0 \delta / \varepsilon} + \varepsilon < 2\varepsilon$. So we obtain

$$|\nabla_x \phi(\xi')| + |\phi(\xi')| \leq C\varepsilon \sum_j e^{-\delta_0 |x - \xi'_j|} \tag{1.86}$$

Now we get

$$J(v(\xi')) = \sum_j V(\varepsilon \xi'_j)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon) \tag{1.87}$$

$\varepsilon \xi' = (\xi_1, \dots, \xi_k)$ implies for several minimal points on the Λ_j we have the result desired.

Finally we prove the existence of multiply interacting spikes. The computations are little bit involved since we have to measure precisely the interactions. The reduced energy functional takes the following form:

$$J(v(\xi')) = \sum_j V(\varepsilon \xi_j)^{p+1/p-1-N/2} (c_{p,N} + o(1)) - (1 + o(1)) \sum_{i \neq j} e^{-\min_{i \neq j} (\sqrt{V(\xi_i) V(\xi_j)}) |\xi'_i - \xi'_j|} \tag{1.88}$$

We shall take the following configuration space

$$\Sigma = \{(\xi_1, \dots, \xi_k) \mid \xi_i \in \Lambda, \min_{i \neq j} |\xi_i - \xi_j| > \rho \varepsilon \log \frac{1}{\varepsilon}\}$$

and prove that the following maximization problem attains a solution in the interior part of the set Σ :

$$\min_{(\xi_1, \dots, \xi_k) \in \Sigma} J(v(\xi')) \quad \square$$

2. Part II: Infinite-dimensional reduction method

2.1. An introduction

In Chapter one, we have dealt with the problem of constructing solutions with finitely many bumps. The idea is to first sum up these finite many bumps and solve it in the space orthogonal to the translations. Then we adjust the points to obtain a true solution. The concentrating solutions concentrate at finite number of points which accounts for zero Lebesgue measure. In this Chapter we generalize this idea to the problem of constructing solutions concentrating on higher dimensional sets, such as curves, surfaces, or minimal surfaces of codimension k . As in the finite dimensional case, we proceed in two steps. In the first step, we solve the problem along each tangent fibre. This amounts to imposing *infinitely many* orthogonal conditions. In the second step, we move the higher dimensional object in the normal direction to find a true solution. We will encounter at least three problems: the first is the uniform estimate of the error in the first step. Sometimes there may be resonances due the combined effect of tangential and instability of the profile. The second problem is the adjustment of the higher dimensional subjects, which typically involves a second order nonlocal nonlinear reduced equation. The third problem is the non-compactness of the higher dimensional object.

In the following we take the model problem of Allen-Cahn equation in \mathbb{R}^3 and the higher dimensional concentration object is minimal surfaces. For higher dimensional concentration problems with resonances we refer to papers [21], [23] and [24].

2.2. Model problem: the Allen-Cahn equation and minimal surfaces

We consider the following so-called Allen-Cahn equation in \mathbb{R}^N

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad (2.1)$$

where $f(s) = -W'(s)$ and W is a “double-well potential”, bi-stable and balanced, namely

$$W(s) > 0 \text{ if } s \neq 1, -1, \quad W(1) = 0 = W(-1), \quad W''(\pm 1) = f'(\pm 1) =: \sigma_{\pm}^2 > 0. \quad (2.2)$$

A typical example of such a nonlinearity is

$$f(u) = (1 - u^2)u \quad \text{for } W(u) = \frac{1}{4}(1 - u^2)^2, \quad (2.3)$$

while we will not make use of the special symmetries enjoyed by this example.

Equation (2.1) is a prototype for the continuous modeling of phase transition phenomena. Let us consider the energy in a subregion Ω of \mathbb{R}^N

$$J_\alpha(v) = \int_\Omega \frac{\alpha}{2} |\nabla v|^2 + \frac{1}{4\alpha} W(v),$$

whose Euler-Lagrange equation is a scaled version of (2.1),

$$\alpha^2 \Delta v + f(v) = 0 \quad \text{in } \Omega. \quad (2.4)$$

We observe that the constant functions $u = \pm 1$ minimize J_α . They are idealized as two *stable phases* of a material in Ω . It is of interest to analyze stationary configurations in which the two phases coexist. Given any subset Λ of Ω , any discontinuous function of the form

$$u_* = \chi_\Lambda - \chi_{\Omega \setminus \Lambda} \quad (2.5)$$

minimizes the second term in J_ε . The introduction of the gradient term in J_α makes an α -regularization of u_* a test function for which the energy gets bounded and proportional to the surface area of the *interface* $M = \partial\Lambda$, so that in addition to minimizing approximately the second term, stationary configurations should also select asymptotically interfaces M that are stationary for surface area, namely (generalized) minimal surfaces. This intuition on the Allen-Cahn equation gave important impulse to the calculus of variations, motivating the development of the theory of Γ -convergence in the 1970's. Modica [48] proved that a family of local minimizers u_α of J_α with uniformly bounded energy must converge in suitable sense to a function of the form (2.5) where $\partial\Lambda$ minimizes perimeter. Thus, intuitively, for each given $\lambda \in (-1, 1)$, the level sets $[v_\alpha = \lambda]$, collapse as $\alpha \rightarrow 0$ onto the interface $\partial\Lambda$. Similar result holds for critical points not necessarily minimizers, see [64]. For minimizers this convergence is known in very strong sense, see [12, 13].

If, on the other hand, we take such a critical point u_α and scale it around an interior point $0 \in \Omega$, setting $u_\alpha(x) = v_\alpha(\alpha x)$, then u_α satisfies equation (2.1) in an expanding domain,

$$\Delta u_\alpha + f(u_\alpha) = 0 \quad \text{in } \alpha^{-1}\Omega$$

so that letting formally $\alpha \rightarrow 0$ we end up with equation (2.1) in entire space. The “interface” for u_α should thus be around the (asymptotically

flat) minimal surface $M_\alpha = \alpha^{-1}M$. Modica’s result is based on the intuition that if M happens to be a smooth surface, then the transition from the equilibria -1 to 1 of u_α along the normal direction should take place in the approximate form $u_\alpha(x) \approx w(z)$ where z designates the normal coordinate to M_α . Then w should solve the ODE problem

$$w'' + f(w) = 0 \quad \text{in } \mathbb{R}, \quad w(-\infty) = -1, \quad w(+\infty) = 1. \quad (2.6)$$

This solution indeed exists thanks to assumption (2.2). It is strictly increasing and unique up to constant translations. We fix in what follows the unique w for which

$$\int_{\mathbb{R}} t w'(t)^2 dt = 0. \quad (2.7)$$

For example (2.3), we have $w(t) = \tanh(t/\sqrt{2})$. In general w approaches its limits at exponential rates,

$$w(t) - \pm 1 = O(e^{-\sigma \pm |t|}) \quad \text{as } t \rightarrow \pm\infty.$$

Observe then that

$$J_\alpha(u_\alpha) \approx \text{Area}(M) \int_{\mathbb{R}} \left[\frac{1}{2} w'^2 + W(w) \right]$$

which is what makes it plausible that M is critical for area, namely a minimal surface.

The above considerations led E. De Giorgi [26] to formulate in 1978 a celebrated conjecture on the Allen-Cahn equation (2.1), parallel to Bernstein’s theorem for minimal surfaces: The level sets $[u = \lambda]$ of a bounded entire solution u to (2.1), which is also monotone in one direction, must be hyperplanes, at least for dimension $N \leq 8$. Equivalently, up to a translation and a rotation, $u = w(x_1)$. This conjecture has been proven in dimensions $N = 2$ by Ghoussoub and Gui [31], $N = 3$ by Ambrosio and Cabré [11], and under a mild additional assumption by Savin [61]. A counterexample was built for $N \geq 9$ by M. del Pino, M.Kowalczyk and Wei in [27], see also [16, 45]. See [28] for a recent survey on the state of the art of this question.

The counter-example in [27] was built on the counter-example to the Bernstein conjecture for minimal graphs: Bernstein conjectured that all minimal graphs, i.e. graphs $\{x_N = F(x')\}$ for which F satisfies additionally the minimal graph equation

$$\nabla \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0, \quad x' \in \mathbb{R}^{N-1} \quad (2.8)$$

In 1969, Bombierie, De Giorgi and Giusti [10] built a nontrivial solution to (2.8) in dimension $N = 9$. In [27, ?] we took the opposite view of Γ -convergence: for a given nondegenerate minimal surface it is possible to build a solution to the Allen-Cahn equation which concentrates on this minimal surface. The class of minimal surfaces will include the Bombierie-De Giorgi-Giusti minimal graph, and the complete embedded minimal surfaces in \mathbb{R}^3 .

In this following we construct a new class of entire solutions to the Allen-Cahn equation in \mathbb{R}^3 whose level sets resemble a large dilation of a given complete, embedded minimal surface M , asymptotically flat in the sense that it has *finite total curvature*, namely

$$\int_M |K| dV < +\infty$$

where K denotes Gauss curvature of the manifold, which is also *non-degenerate* in a sense that we will make precise below.

2.3. Embedded minimal surfaces of finite total curvature

The theory of embedded, minimal surfaces of finite total curvature in \mathbb{R}^3 , has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [14, ?]. The *Costa surface* is a genus one minimal surface, complete and properly embedded, which outside a large ball has exactly three components (its *ends*), two of which are asymptotically catenoids with the same axis and opposite directions, the third one asymptotic to a plane perpendicular to that axis. The complete proof of embeddedness is due to Hoffman and Meeks [36]. In [37, 39] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same look as Costa's far away, but with an array of tunnels that provides arbitrary genus $k \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus k .

Many other examples of multiple-end embedded minimal surfaces have been found since, see for instance [46, ?] and references therein. In general all these surfaces look like parallel planes, slightly perturbed at their ends by asymptotically logarithmic corrections with a certain number of catenoidal

links connecting their adjacent sheets. In reality this intuitive picture is not a coincidence.

Using the Eneper-Weierstrass representation, Osserman [54] established that any embedded, complete minimal surface with finite total curvature can be described by a conformal diffeomorphism of a compact surface (actually of a Riemann surface), with a finite number of its points removed. These points correspond to the ends. Moreover, after a convenient rotation, the ends are asymptotically all either catenoids or plane, all of them with parallel axes, see Schoen [62]. The topology of the surface is thus characterized by the genus of the compact surface and the number of ends, having therefore “finite topology”.

2.4. Main results

In what follows M designates a complete, embedded minimal surface in \mathbb{R}^3 with finite total curvature (to which below we will make a further nondegeneracy assumption). As pointed out in [40], M is orientable and the set $\mathbb{R}^3 \setminus M$ has exactly two components S_+, S_- .

In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+ .

For $x = (x_1, x_2, x_3) = (x', x_3) \in \mathbb{R}^3$, we denote

$$r = r(x) = |(x_1, x_2)| = \sqrt{x_1^2 + x_2^2}.$$

After a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0 , then M decomposes into a finite number m of unbounded components M_1, \dots, M_m , its ends. From a result in [62], we know that asymptotically each end of M_k either resembles a plane or a catenoid. More precisely, M_k can be represented as the graph of a function F_k of the first two variables,

$$M_k = \{ y \in \mathbb{R}^3 / r(y) > R_0, y_3 = F_k(y') \}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \rightarrow +\infty, \quad (2.9)$$

for certain constants a_k, b_k, b_{ik} , and this relation can also be differentiated. Here

$$a_1 \leq a_2 \leq \dots \leq a_m, \quad \sum_{k=1}^m a_k = 0. \quad (2.10)$$

The direction of the normal vector $\nu(y)$ for large $r(y)$ approaches on the ends that of the x_3 axis, with alternate signs. We use the convention that for $r(y)$ large we have

$$\nu(y) = \frac{(-1)^k}{\sqrt{1 + |\nabla F_k(y')|^2}} (\nabla F_k(y'), -1) \quad \text{if } y \in M_k. \quad (2.11)$$

Let us consider the Jacobi operator of M

$$\mathcal{J}(h) := \Delta_M h + |A|^2 h \quad (2.12)$$

where $|A|^2 = -2K$ is the Euclidean norm of the second fundamental form of M . \mathcal{J} is the linearization of the mean curvature operator with respect to perturbations of M measured along its normal direction. A smooth function $z(y)$ defined on M is called a *Jacobi field* if $\mathcal{J}(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields: Associated to respectively translations along coordinates axes and rotation around the x_3 -axis, are the functions

$$\begin{aligned} z_1(y) &= \nu(y) \cdot e_i, \quad y \in M, \quad i = 1, 2, 3, \\ z_4(y) &= (-y_2, y_1, 0) \cdot \nu(y), \quad y \in M. \end{aligned} \quad (2.13)$$

We assume that M is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{ z \in L^\infty(M) / \mathcal{J}(z) = 0 \} = \text{span} \{ z_1, z_2, z_3, z_4 \}. \quad (2.14)$$

We denote in what follows by J the dimension (≤ 4) of the above vector space.

This assumption, expected to be generic for this class of surfaces, is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface which is an example of a three ended M whose genus may be of any order. See Nayatani [52, 53] and Morabito [49]. Note that for a catenoid, $z_{04} = 0$ so that $J = 3$. Non-degeneracy has been used as a tool to build new minimal surfaces for instance in Hauswirth and Pacard [35], and in Pérez and Ros [56]. It is also the basic element, in a compact-manifold version, to build solutions to the small-parameter Allen-Cahn equation in Pacard and Ritoré [55].

Let us consider a large dilation of M ,

$$M_\alpha := \alpha^{-1} M.$$

This dilated minimal surface has ends parameterized as

$$M_{k,\alpha} = \{ y \in \mathbb{R}^3 / r(\alpha y) > R_0, y_3 = \alpha^{-1} F_k(\alpha y') \} .$$

Let β be a vector of given m real numbers with

$$\beta = (\beta_1, \dots, \beta_m), \quad \sum_{i=1}^m \beta_i = 0 . \tag{2.15}$$

Our main result asserts the existence of a solution $u = u_\alpha$ defined for all sufficiently small $\alpha > 0$ such that given $\lambda \in (-1, 1)$, its level set $[u_\alpha = \lambda]$ defines an embedded surface lying at a uniformly bounded distance in α from the surface M_α , for points with $r(\alpha y) = O(1)$, while its k -th end, $k = 1, \dots, m$, lies at a uniformly bounded distance from the graph

$$r(\alpha y) > R_0, y_3 = \alpha^{-1} F_k(\alpha y') + \beta_k \log |\alpha y'| . \tag{2.16}$$

The parameters β must satisfy an additional constraint. It is clear that if two ends are parallel, say $a_{k+1} = a_k$, we need at least that $\beta_{k+1} - \beta_k \geq 0$, for otherwise the ends would eventually intersect. Our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate. We require

$$\beta_{k+1} - \beta_k > 4 \max \{ \sigma_-^{-1}, \sigma_+^{-1} \} \quad \text{if } a_{k+1} = a_k . \tag{2.17}$$

Let us consider the smooth map

$$X(y, z) = y + z\nu(\alpha y), \quad (y, t) \in M_\alpha \times \mathbb{R}. \tag{2.18}$$

$x = X(y, z)$ defines coordinates inside the image of any region where the map is one-to-one. In particular, let us consider a function $p(y)$ with

$$p(y) = (-1)^k \beta_k \log |\alpha y'| + O(1), \quad k = 1, \dots, m,$$

and β satisfying $\beta_{k+1} - \beta_k > \gamma > 0$ for all k with $a_k = a_{k+1}$. Then the map X is one-to-one for all small α in the region of points (y, z) with

$$|z - q(y)| < \frac{\delta}{\alpha} + \gamma \log(1 + |\alpha y'|)$$

provided that $\delta > 0$ is chosen sufficiently small.

Theorem 6: (*del Pino-Kowalczyk-Wei [22]*) *Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then, given β satisfying relations (2.15) and (2.17), there exists*

a bounded solution u_α of equation (2.1), defined for all sufficiently small α , such that

$$u_\alpha(x) = w(z - q(y)) + O(\alpha) \quad \text{for all } x = y + z\nu(\alpha y), \quad |z - q(y)| < \frac{\delta}{\alpha}, \tag{2.19}$$

where the function q satisfies

$$q(y) = (-1)^k \beta_k \log |\alpha y'| + O(1) \quad y \in M_{k,\alpha}, \quad k = 1, \dots, m.$$

In particular, for each given $\lambda \in (-1, 1)$, the level set $[u_\alpha = \lambda]$ is an embedded surface that decomposes for all sufficiently small α into m disjoint components (ends) outside a bounded set. The k -th end lies at $O(1)$ distance from the graph

$$y_3 = \alpha^{-1} F_k(\alpha y) + \beta_k \log |\alpha y'|.$$

We will devote the rest of this part to the proofs of Theorems 6. For the full proofs we refer to [22] in which more detailed behavior of the solutions constructed, such as finite Morse index, can be found.

3. Geometric Background

In this section we present the geometric backgrounds on the expansion of the Laplacian operator near a manifold.

3.1. Parametrization of M and its Laplace-Beltrami Operator

Let D be the set

$$D = \{y \in \mathbb{R}^2 / |y| > R_0\}.$$

We can parameterize the end M_k of M as

$$y \in D \mapsto y := Y_k(y) = y_i e_i + F_k(y) e_3. \tag{3.1}$$

and F_k is the function in (2.9). In other words, for $y = (y', y_3) \in M_k$ the coordinate y is just defined as $y = y'$. We want to represent Δ_M —the Laplace-Beltrami operator of M —with respect to these coordinates. For the coefficients of the metric g_{ij} on M_k we have

$$\partial_{y_i} Y_k = e_i + O(r^{-1}) e_3$$

so that

$$g_{ij}(y) = \langle \partial_i Y_k, \partial_j Y_k \rangle = \delta_{ij} + O(r^{-2}), \tag{3.2}$$

where $r = |y|$. The above relations “can be differentiated” in the sense that differentiation makes the terms $O(r^{-j})$ gain corresponding negative powers of r . Then we find the representation

$$\Delta_M = \frac{1}{\sqrt{\det g_{ij}}} \partial_i (\sqrt{\det g_{ij}} g^{ij} \partial_j) = \Delta_y + O(r^{-2}) \partial_{ij} + O(r^{-3}) \partial_i \quad \text{on } M_k. \quad (3.3)$$

The normal vector to M at $y \in M_k$ $k = 1, \dots, m$, corresponds to

$$\nu(y) = (-1)^k \frac{1}{\sqrt{1 + |\nabla F_k(y)|^2}} (\partial_i F_k(y) e_i - e_3), \quad y = Y_k(y) \in M_k$$

so that

$$\nu(y) = (-1)^k e_3 + \alpha_k r^{-2} y_i e_i + O(r^{-2}), \quad y = Y_k(y) \in M_k. \quad (3.4)$$

Let us observe for later reference that since $\partial_i \nu = O(r^{-2})$, then the principal curvatures of M , k_1, k_2 satisfy $k_l = O(r^{-2})$. In particular, we have that

$$|A(y)|^2 = k_1^2 + k_2^2 = O(r^{-4}). \quad (3.5)$$

To describe the entire manifold M we consider a finite number $N \geq m + 1$ of local parametrizations

$$y \in \mathcal{U}_k \subset \mathbb{R}^2 \mapsto y = Y_k(y), \quad Y_k \in C^\infty(\bar{\mathcal{U}}_k), \quad k = 1, \dots, N. \quad (3.6)$$

For $k = 1, \dots, m$ we choose them to be those in (3.1), with $\mathcal{U}_k = D$, so that $Y_k(\mathcal{U}_k) = M_k$, and $\bar{\mathcal{U}}_k$ is bounded for $k = m + 1, \dots, N$. We require then that

$$M = \bigcup_{k=1}^N Y_k(\mathcal{U}_k).$$

We remark that the Weierstrass representation of M implies that we can actually take $N = m + 1$, namely only one extra parametrization is needed to describe the bounded complement of the ends in M . We will not use this fact. In general, we represent for $y \in Y_k(\mathcal{U}_k)$,

$$\Delta_M = a_{ij}^0(y) \partial_{ij} + b_i^0(y) \partial_i, \quad y = Y_k(y), \quad y \in \mathcal{U}_k, \quad (3.7)$$

where a_{ij}^0 is a uniformly elliptic matrix and the index k is not made explicit in the coefficients. For $k = 1, \dots, m$ we have

$$a_{ij}^0(y) = \delta_{ij} + O(r^{-2}), \quad b_i^0 = O(r^{-3}), \quad \text{as } r(y) = |y| \rightarrow \infty. \quad (3.8)$$

The parametrizations set up above induce naturally a description of the expanded manifold $M_\alpha = \alpha^{-1}M$ as follows. Let us consider the functions

$$Y_{k\alpha} : \mathcal{U}_{k\alpha} := \alpha^{-1}\mathcal{U}_k \rightarrow M_\alpha, \quad y \mapsto Y_{k\alpha}(y) := \alpha^{-1}Y_k(\alpha y), \quad k = 1, \dots, N. \quad (3.9)$$

Obviously we have

$$M_\alpha = \bigcup_{k=1}^N Y_{k\alpha}(\mathcal{U}_{k\alpha}).$$

The computations above lead to the following representation for the operator Δ_{M_α} :

$$\Delta_{M_\alpha} = a_{ij}^0(\alpha y)\partial_{ij} + b_i^0(\alpha y)\partial_i, \quad y = Y_{k\alpha}(y), \quad y \in \mathcal{U}_{k\alpha}, \quad (3.10)$$

where a_{ij}^0, b_i^0 are the functions in (3.7), so that for $k = 1, \dots, m$ we have

$$a_{ij}^0 = \delta_{ij} + O(r_\alpha^{-2}), \quad b_i^0 = O(r_\alpha^{-3}), \quad \text{as } r_\alpha(y) := |\alpha y| \rightarrow \infty. \quad (3.11)$$

3.2. Coordinates near M and the Euclidean Laplacian: Fermi coordinates

Next we shall consider the parametrization of a neighborhood of M . Let us consider the smooth map

$$(y, z) \in M \times \mathbb{R} \mapsto x = \tilde{X}(y, z) = y + z\nu(y) \in \mathbb{R}^3. \quad (3.12)$$

Let us consider an open subset $\tilde{\mathcal{O}}$ of $M \times \mathbb{R}$ and assume that the map $X|_{\tilde{\mathcal{O}}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X(\tilde{\mathcal{O}})$. Certainly we can choose $\tilde{\mathcal{O}}$ such that

$$\{(y, z) \in M \times \mathbb{R} \mid |z| < \delta \log(1 + r(y))\} \subset \tilde{\mathcal{O}}.$$

Since along ends $\partial_i\nu = O(r^{-2})$ so that $z\partial_i\nu$ is uniformly small in $\tilde{\mathcal{O}}$, it follows that \tilde{X} is actually a diffeomorphism onto its image.

The Euclidean Laplacian Δ_x can be computed in such a region by the well-known formula in terms of the coordinates $(y, z) \in \tilde{\mathcal{O}}$ as

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z}\partial_z, \quad x = \tilde{X}(y, z), \quad (y, z) \in \tilde{\mathcal{O}} \quad (3.13)$$

where M_z is the manifold

$$M_z = \{y + z\nu(y) \mid y \in M\}.$$

To see the formula (3.13) we observe that

$$X_i = Y_i + z\nu_i, \quad i = 1, 2, \quad X_z = \nu$$

and hence for $i, j = 1, 2$

$$g_{ij}(x, z) = g_{0ij} + 2z\nu_i Y_j + z^2\nu_i\nu_j$$

and $g_{iz} = 0, g_{zz} = 1$. Hence the Euclidean laplacian in $\tilde{\mathcal{O}}$ becomes

$$\begin{aligned} \Delta_{M_z} h(y)|_{X=X(y,z)} &= \frac{1}{\sqrt{\det g_z}} \partial_i (\sqrt{\det(g_z)} g_z^{ij} \partial_j h)(y, z) \\ &= \partial_{zz} h + \Delta_{M_z} h + \partial_z \log(\sqrt{\det g_z}) \partial_z h \end{aligned}$$

We note that by direct computations $\det(g_z) = \prod_{j=1}^2 (1 - zk_j)^2 \det g_0$. This gives the formula (3.13).

Local coordinates $y = Y_k(\mathbf{y}), \mathbf{y} \in \mathbb{R}^2$ as in (3.1) induce natural local coordinates in M_z . The metric $g_{ij}(z)$ in M_z can then be computed as

$$g_{ij}(z) = \langle \partial_i Y, \partial_j Y \rangle + z(\langle \partial_i Y, \partial_j \nu \rangle + \langle \partial_j Y, \partial_i \nu \rangle) + z^2 \langle \partial_i \nu, \partial_j \nu \rangle \quad (3.14)$$

or

$$g_{ij}(z) = g_{ij} + zO(r^{-2}) + z^2O(r^{-4}).$$

where these relations can be differentiated. Thus we find from the expression of Δ_{M_z} in local coordinates that

$$\Delta_{M_z} = \Delta_M + za_{ij}^1(y, z)\partial_{ij} + zb_i^1(y, z)\partial_i, \quad y = Y(\mathbf{y}) \quad (3.15)$$

where a_{ij}^1, b_i^1 are smooth functions of their arguments. Let us examine this expansion closer around the ends of M_k where $y = Y_k(\mathbf{y})$ is chosen as in (3.1). In this case, from (3.14) and (3.2) we find

$$g^{ij}(z) = g^{ij} + zO(r^{-2}) + z^2O(r^4) + \dots$$

Then we find that for large r ,

$$\Delta_{M_z} = \Delta_M + zO(r^{-2})\partial_{ij} + zO(r^{-3})\partial_i. \quad (3.16)$$

Let us consider the remaining term in the expression for the Laplacian, the mean curvature H_{M_z} . We have the validity of the formula

$$H_{M_z} = \sum_{i=1}^2 \frac{k_i}{1 - k_i z} = \sum_{i=1}^2 k_i + k_i^2 z + k_i^3 z^2 + \dots$$

where $k_i, i = 1, 2$ are the principal curvatures. Since M is a minimal surface, we have that $k_1 + k_2 = 0$. Thus

$$|A|^2 = k_1^2 + k_2^2 = -2k_1 k_2 = -2K$$

where $|A|$ is the Euclidean norm of the second fundamental form, and K the Gauss curvature. As $r \rightarrow +\infty$ we have seen that $k_i = O(r^{-2})$ and hence $|A|^2 = O(r^{-4})$. More precisely, we find for large r ,

$$H_{M_z} = |A|^2 z + z^2 O(r^{-6}).$$

Thus we have found the following expansion for the Euclidean Laplacian,

$$\Delta_x = \partial_{zz} + \Delta_M - z|A|^2 \partial_z + B \tag{3.17}$$

where expressed in local coordinates in M the operator B has the form

$$B = z a_{ij}^1(y, z) \partial_{ij} + z b_i^1(y, z) \partial_i + z^2 b_3^1(y, z) \partial_z \tag{3.18}$$

with a_{ij}^1, b_i^1, b_3^1 smooth functions. Besides, we find that

$$a_{ij}^1(y, z) = O(r^{-2}), \quad b_i^1(y, z) = O(r^{-3}), \quad b_3^1(y, z) = O(r^{-6}), \tag{3.19}$$

uniformly in z for $(y, z) \in \tilde{\mathcal{O}}$. Moreover, the way these coefficients are produced from the metric yields for instance that

$$a_{ij}^1(y, z) = a_{ij}^1(y, 0) + z a_{ij}^{(2)}(y, z), \quad a_{i,j}^2(y, z) = O(r^{-3}),$$

$$b_i^1(y, z) = b_i^1(y, 0) + z b_i^{(2)}(y, z), \quad b_i^{(2)}(y, z) = O(r^{-4}).$$

We summarize the discussion above. Let us consider the parametrization in (3.12) of the region $\tilde{\mathcal{N}}$.

Lemma 3.1: *The Euclidean Laplacian can be expanded in $\tilde{\mathcal{N}}$ as*

$$\Delta_x = \partial_{zz} + \Delta_{M_z} - H_{M_z} \partial_z =$$

$$\partial_{zz} + \Delta_M - z|A|^2 \partial_z + z [a_{ij}^1(y, z) \partial_{ij} + b_i^1(y, z) \partial_i] + z^2 b_3^1(y, z) \partial_z,$$

$$\Delta_M = a_{ij}^0 \partial_{ij} + b_i^0 \partial_i, \quad x = \tilde{X}(y, z), \quad (y, z) \in \tilde{\mathcal{O}},$$

where a_{ij}^l, b_j^l are smooth, bounded functions, with the index k omitted. In addition, for $k = 1, \dots, m$,

$$a_{ij}^l = \delta_{ij} \delta_{0l} + O(r^{-2}), \quad b_i^l = O(r^{-3}), \quad b_3^l = O(r^{-6}),$$

as $r = |y| \rightarrow \infty$, uniformly in z variable.

3.3. Laplacian in expanded variables

Now we consider the expanded minimal surface $M_\alpha = \alpha^{-1}M$ for a small number α . We have that $\mathcal{N} = \alpha^{-1}\tilde{\mathcal{N}}$. We describe \mathcal{N} via the coordinates

$$x = X(y, z) := y + z\nu_\alpha(y), \quad (y, z) \in \alpha^{-1}\tilde{\mathcal{O}}. \tag{3.20}$$

Let us observe that

$$X(y, z) = \alpha^{-1}\tilde{X}(\alpha y, \alpha z)$$

where $\tilde{x} = \tilde{X}(\tilde{y}, \tilde{z}) = \tilde{y} + \tilde{z}\nu(\tilde{y})$, where the coordinates in \mathcal{N}_δ previously dealt with. We want to compute the Euclidean Laplacian in these coordinates associated to M_α . Observe that

$$\Delta_x[u(x)]|_{x=X(y,z)} = \alpha^2\Delta_{\tilde{x}}[u(\alpha^{-1}\tilde{x})]|_{\tilde{x}=\tilde{X}(\alpha y, \alpha z)}$$

and that the term in the right hand side is the one we have already computed. In fact setting $v(y, z) := u(y + z\nu_\alpha(y))$, we get

$$\Delta_x u|_{x=X(y,z)} = \alpha^2(\Delta_{\tilde{y}, M_{\tilde{z}}} + \partial_{\tilde{z}\tilde{z}} - H_{M_{\tilde{z}}}\partial_{\tilde{z}})[v(\alpha^{-1}\tilde{y}, \alpha^{-1}\tilde{z})]|_{(\tilde{y}, \tilde{z})=(\alpha y, \alpha z)}. \tag{3.21}$$

We can then use the discussion summarized in Lemma 3.1 to obtain a representation of Δ_x in \mathcal{N} via the coordinates $X(y, t)$ in (3.20). Let us consider the local coordinates $Y_{k\alpha}$ of M_α in (3.9).

Lemma 3.2: *In \mathcal{N} we have*

$$\Delta_x = \partial_{zz} + \Delta_{M_{\alpha,z}} - H_{M_{\alpha,z}}\partial_z =$$

$$\partial_{zz} + \Delta_{M_\alpha} - \alpha^2 z |A(\alpha y)|^2 \partial_z + \alpha z [a_{ij}^1(\alpha y, \alpha z)\partial_{ij} + \alpha b_i^1(\alpha y, \alpha z)\partial_i] + \alpha^3 z^2 b_3^1(\alpha y, \alpha z)\partial_z,$$

$$\Delta_{M_\alpha} = a_{ij}^0(\alpha y)\partial_{ij} + b_i^1(\alpha y)\partial_i, \quad (y, z) \in \alpha^{-1}\tilde{\mathcal{O}}, \quad y = Y_{k\alpha}(y)$$

where a_{ij}^l, b_j^l are smooth, bounded functions. In addition, for $k = 1, \dots, m$,

$$a_{ij}^l = \delta_{ij}\delta_{0l} + O(r_\alpha^{-2}), \quad b_i^l = O(r_\alpha^{-3}), \quad b_3^1 = O(r_\alpha^{-6}),$$

as $r_\alpha(y) = |\alpha y| \rightarrow \infty$, uniformly in z variable.

3.4. The Euclidean Laplacian near M_α under a perturbation

We now describe in coordinates relative to M_α the Euclidean Laplacian Δ_x , $x \in \mathbb{R}^3$, in a setting needed for the proof of our main results. The main idea is to introduce a smooth perturbation of the minimal surfaces, a priori unknown. We will need to compute the Euclidean Laplacian under this perturbation.

Let us consider a smooth function $h : M \rightarrow \mathbb{R}$, and the smooth map X_h defined as

$$X_h : M_\alpha \times \mathbb{R} \rightarrow \mathbb{R}^3, \quad (y, t) \mapsto X_h(y, t) := y + (t + h(\alpha y)) \nu(\alpha y) \quad (3.22)$$

where ν is the unit normal vector to M . Let us consider an open subset \mathcal{O} of $M_\alpha \times \mathbb{R}$ and assume that the map $X_h|_{\mathcal{O}}$ is one to one, and that it defines a diffeomorphism onto its image $\mathcal{N} = X_h(\mathcal{O})$. Then

$$x = X_h(y, t), \quad (y, t) \in \mathcal{O},$$

defines smooth coordinates to describe the open set \mathcal{N} in \mathbb{R}^3 . Moreover, the maps

$$x = X_h(Y_{k\alpha}(\mathbf{y}), t), \quad (\mathbf{y}, t) \in (\mathcal{U}_{k\alpha} \times \mathbb{R}) \cap \mathcal{O}, \quad k = 1, \dots, N,$$

define local coordinates (\mathbf{y}, t) to describe the region \mathcal{N} . We shall assume in addition that for certain small number $\delta > 0$, we have

$$\mathcal{O} \subset \{(y, t) \mid |t + h(\alpha y)| < \frac{\delta}{\alpha} \log(2 + r_\alpha(y))\}. \quad (3.23)$$

We have the validity of the following expression for the Euclidean Laplacian operator in \mathcal{N} .

Lemma 3.3: *For $x = X_h(y, t)$, $(y, t) \in \mathcal{O}$ with $y = Y_{k\alpha}(\mathbf{y})$, $\mathbf{y} \in \mathcal{U}_{k\alpha}$, we have the validity of the identity*

$$\begin{aligned} \Delta_x &= \partial_{tt} + \Delta_{M_\alpha} - \alpha^2[(t+h)|A|^2 + \Delta_M h] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\ &\alpha(t+h) [a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_i h \partial_{jt} + \alpha b_i^1 (\partial_i - \alpha \partial_i h \partial_t)] + \\ &\alpha^3(t+h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t+h) a_{ij}^1] \partial_i h \partial_j h \partial_{tt}. \end{aligned} \quad (3.24)$$

Here, in agreement with (3.10), $\Delta_{M_\alpha} = a_{ij}^0(\alpha y) \partial_{ij} + b_i^0(\alpha y) \partial_i$. The functions a_{ij}^1 , b_i^1 , b_3^1 in the above expressions appear evaluated at the pair

$(\alpha y, \alpha(t + h(\alpha y)))$, while the functions $h, \partial_i h, \Delta_M h, |A|^2, a_{ij}^0, b_i^0$ are evaluated at αy . In addition, for $k = 1, \dots, m, l = 0, 1$,

$$a_{ij}^l = \delta_{ij} \delta_{0l} + O(r_\alpha^{-2}), \quad b_i^l = O(r_\alpha^{-3}), \quad b_3^1 = O(r_\alpha^{-6}),$$

as $r_\alpha(y) = |\alpha y| \rightarrow \infty$, uniformly in their second variables. The notation $\partial_j h$ refers to $\partial_j [h \circ Y_k]$.

Proof:

Let us consider a function u defined in \mathcal{N} , expressed in coordinates $x = X(y, z)$, and consider the expression of u in the coordinates $x = X_h(y, t)$, namely the function $v(y, t)$ defined by the relation in local coordinates $y = Y_k(\mathbf{y})$,

$$v(\mathbf{y}, z - h(\alpha \mathbf{y})) = u(\mathbf{y}, z),$$

(by slight abuse of notation we are denoting just by h the function $h \circ Y_k$). Then we compute

$$\partial_i u = \partial_i v - \alpha \partial_t v \partial_i h, \quad \partial_z u = \partial_t v,$$

$$\partial_{ij} u = \partial_{ij} v - \alpha \partial_{it} v \partial_j h - \alpha \partial_{jt} v \partial_i h + \alpha^2 \partial_{tt} v \partial_i h \partial_j h - \alpha^2 \partial_t v \partial_{ij} h.$$

Observe that, in the notation for coefficients in Lemma 3.2,

$$a_{ij}^0 \partial_{ij} h + b_i^0 \partial_i h = \Delta_M h, \quad a_{ij}^0 \partial_{ij} v + \alpha b_i^0 \partial_i v = \Delta_{M_\alpha} v.$$

We find then

$$\begin{aligned} \Delta_x &= \partial_{tt} + \Delta_{M_\alpha} - \alpha^2 [(t + h)|A|^2 + \Delta_M h] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\ &\alpha(t + h) [a_{ij}^1 \partial_{ij} - 2\alpha a_{ij}^1 \partial_i h \partial_{jt} + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] + \\ &\alpha^3 (t + h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t + h)a_{ij}^1] \partial_i h \partial_j h \partial_{tt} \end{aligned} \quad (3.25)$$

where all the coefficients are understood to be evaluated at αy or $(\alpha y, \alpha(t + h(\alpha y)))$. The desired properties of the coefficients have already been established. The proof of Lemma 3.3 is concluded. \square

The proof actually yields that the coefficients a_{ij}^1 and b_i^1 can be further expanded as follows:

$$a_{ij}^1 = a_{ij}^1(\alpha y, 0) + \alpha(t + h) a_{ij}^{(2)}(\alpha y, \alpha(t + h)) =: a_{ij}^{1;0} + \alpha(t + h) a_{ij}^2,$$

with $a_{ij}^{(2)} = O(r_\alpha^{-3})$, and similarly

$$b_j^1 = b_j^1(\alpha y, 0) + \alpha(t+h) b_j^{(2)}(\alpha y, \alpha(t+h)) =: b_j^{1,0} + \alpha(t+h) b_j^2,$$

with $b_j^{(2)} = O(r_\alpha^{-4})$. As an example of the previous formula, let us compute the Laplacian of a function that separates variables t and y , that will be useful in §4.

Lemma 3.4: *Let $v(x) = k(y) \psi(t)$. Then the following holds.*

$$\begin{aligned} \Delta_x v &= k \psi'' + \psi \Delta_{M_\alpha} k - \alpha^2 [(t+h)|A|^2 + \Delta_M h] k \psi' - 2\alpha a_{ij}^0 \partial_j h \partial_i k \psi' + \\ &\alpha(t+h) [a_{ij}^{1,0} \partial_{ij} k \psi - 2\alpha a_{ij}^{1,0} \partial_j h \partial_i k \psi' + \alpha(b_i^{1,0} \partial_i k \psi - \alpha b_i^{1,0} \partial_i h k \psi')] + \\ &\alpha^2 (t+h)^2 [a_{ij}^2 \partial_{ij} k \psi - 2\alpha a_{ij}^2 \partial_j h \partial_i k \psi' + \alpha(b_i^2 \partial_i k \psi - \alpha b_i^2 \partial_i h k \psi')] + \\ &\alpha^3 (t+h)^2 b_3^1 k \psi' + \alpha^2 [a_{ij}^0 + \alpha(t+h) a_{ij}^1] \partial_i h \partial_j h k \psi'' . \end{aligned} \quad (3.26)$$

4. Approximation of the solution and preliminary discussion

4.1. Approximation of order zero and its projection

Let us consider a function h and sets \mathcal{O} and \mathcal{N} as in §3.4. Let $x = X_h(y, t)$ be the coordinates introduced in (3.22). At this point we shall make a more precise assumption about the function h . We need the following preliminary result whose proof we postpone for §7.2.

We consider a fixed m -tuple of real numbers $\beta = (\beta_1, \dots, \beta_m)$ such that

$$\sum_{i=1}^m \beta_i = 0. \quad (4.1)$$

Lemma 4.1: *Given any real numbers β_1, \dots, β_m satisfying (4.1), there exists a smooth function $h_0(y)$ defined on M such that*

$$\mathcal{J}(h_0) = \Delta_M h_0 + |A|^2 h_0 = 0 \quad \text{in } M,$$

$h_0(y) = (-1)^j \beta_j \log r + \theta$ as $r \rightarrow \infty$ in M_j for all $y \in M_j$,
 where θ satisfies

$$\|\theta\|_\infty + \|r^2 D\theta\|_\infty < +\infty. \tag{4.2}$$

We fix a function h_0 as in the above lemma and consider a function h in the form

$$h = h_0 + h_1.$$

We allow h_1 to be a parameter which we will adjust. For now we will assume that for a certain constant \mathcal{K} we have

$$\|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} \leq \mathcal{K}\alpha. \tag{4.3}$$

We want to find a solution to

$$S(u) := \Delta_x u + f(u) = 0.$$

We consider in the region \mathcal{N} the approximation

$$w(x) := w(t) = w(z - h_0(\alpha y) - h_1(\alpha y))$$

where z designates the normal coordinate to M_α . Thus, whenever $\beta_j \neq 0$, the level sets $[w = \lambda]$ for a fixed $\lambda \in (-1, 1)$ departs logarithmically from the end $\alpha^{-1}M_j$ being still asymptotically catenoidal, more precisely it is described as the graph

$$y_3 = (\alpha^{-1}a_j + \beta_j) \log r + O(1) \text{ as } r \rightarrow \infty.$$

Note that, just as in the minimal surface case, the coefficients of the ends are balanced in the sense that they add up to zero.

It is clear that if two ends are parallel, say $a_{j+1} = a_j$, we need at least that $\beta_{j+1} - \beta_j \geq 0$, for otherwise the ends of this zero level set would eventually intersect. We recall that our further condition on these numbers is that these ends in fact diverge at a sufficiently fast rate:

$$\beta_{j+1} - \beta_j > 4 \max\{\sigma_-^{-1}, \sigma_+^{-1}\} \text{ if } a_{j+1} = a_j. \tag{4.4}$$

We will explain later the role of this condition. Let us evaluate the error of approximation $S(w)$. Using Lemma 3.4 and the fact that $w'' + f(w) = 0$, we find

$$S(w) := \Delta_x w + f(w) =$$

$$\begin{aligned}
 & -\alpha^2[|A|^2 h_1 + \Delta_M h_1] w' + \\
 & -\alpha^2 |A|^2 t w' + 2 \alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 w'' + \\
 & \alpha^2 a_{ij}^0 (2 \partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_1) w'' + \\
 & 2 \alpha^3 (t + h_0 + h_1) a_{ij}^1 \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) w'' + \\
 & \alpha^3 (t + h_0 + h_1) b_i^1 \partial_i (h_0 + h_1) w' + \alpha^3 (t + h_0 + h_1)^3 b_3^1 w' \quad (4.5)
 \end{aligned}$$

where the formula above has been broken into “sizes”, keeping in mind that h_0 is fixed while $h_1 = O(\alpha)$. Since we want that w be as close as possible to be a solution of (2.1), then we would like to choose h_1 in such a way that the quantity (4.5) be as small as possible. Examining the above expression, it does not look like we can do that in absolute terms. However part of the error could be made smaller by adjusting h_1 . Let us consider the “ L^2 -projection” onto $w'(t)$ of the error for each fixed y , given by

$$\Pi(y) := \int_{-\infty}^{\infty} S(w)(y, t) w'(t) dt$$

where for now, and for simplicity we assume the coordinates are defined for all t , the difference with the integration is taken in all the actual domain for t produces only exponentially small terms in α^{-1} . Then we find

$$\begin{aligned}
 \Pi(y) = & \alpha^2 (\Delta_M h_1 + h_1 |A|^2) \int_{-\infty}^{\infty} w'^2 dt + \alpha^3 \partial_i (h_0 + h_1) \int_{-\infty}^{\infty} b_i^1 (t + h_0 + h_1) w'^2 dt + \\
 & \alpha^3 \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \int_{-\infty}^{\infty} (t + h_0 + h_1) a_{ij}^1 w'' w' dt + \alpha^3 \int_{-\infty}^{\infty} (t + h_0 + h_1)^3 b_3^1 w'^2 dt \quad (4.6)
 \end{aligned}$$

where we have used $\int_{-\infty}^{\infty} t w'^2 dt = \int_{-\infty}^{\infty} w'' w' dt = 0$ to get rid in particular of the terms of order α^2 .

Making all these “projections” equal to zero amounts to a nonlinear differential equation for h of the form

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A(y)|^2 = G_0(h_1) \quad y \in M \quad (4.7)$$

where G_0 is easily checked to be a contraction mapping of small constant in h_1 , in the ball radius $O(\alpha)$ with the C^1 norm defined by the expression in the left hand side of inequality (4.3). This is where the nondegeneracy assumption on the Jacobi operator \mathcal{J} enters, since we would like to invert it, in such a way to set up equation (4.7) as a fixed point problem for a contraction mapping of a ball of the form (4.3).

4.2. Improvement of approximation

The previous considerations are not sufficient since even after adjusting optimally h , the error in absolute value does not necessarily decrease. As we observed, the “large” term in the error,

$$-\alpha^2|A|^2tw' + \alpha^2a_{ij}^0\partial_i h_0\partial_j h_0 w''$$

did not contribute to the projection. In order to eliminate, or reduce the size of this remaining part $O(\alpha^2)$ of the error, we improve the approximation through the following argument. Let us consider the differential equation

$$\psi_0''(t) + f'(w(t))\psi_0(t) = tw'(t),$$

which has a unique bounded solution with $\psi_0(0) = 0$, given explicitly by the formula

$$\psi_0(t) = w'(t) \int_0^t w'(s)^{-2} \int_{-\infty}^s sw'(s)^2 ds.$$

Observe that this function is well defined and it is bounded since $\int_{-\infty}^{\infty} sw'(s)^2 ds = 0$ and $w'(t) \sim e^{-\sigma_{\pm}|t|}$ as $t \rightarrow \pm\infty$, with $\sigma_{\pm} > 0$. Note also that $\psi_1(t) = \frac{1}{2}tw'(t)$ solves

$$\psi_1''(t) + f'(w(t))\psi_1(t) = w''(t).$$

We consider as a second approximation

$$u_1 = w + \phi_1, \quad \phi_1(y, t) := \alpha^2|A(\alpha y)|^2\psi_0(t) - \alpha^2a_{ij}^0\partial_i h_0\partial_j h_0(\alpha y) \psi_1(t). \tag{4.8}$$

Let us observe that

$$S(w+\phi) = S(w) + \Delta_x\phi + f'(w)\phi + N_0(\phi), \quad N_0(\phi) = f(w+\phi) - f(w) - f'(w)\phi.$$

We have that

$$\partial_{tt}\phi_1 + f'(w)\phi_1 = \alpha^2|A(\alpha y)|^2tw' - \alpha^2a_{ij}^0\partial_i h_0\partial_j h_0(\alpha y) w''.$$

Hence we get that the largest remaining term in the error is canceled. Indeed, we have

$$S(u_1) = S(w) - (2\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_0 w'' - \alpha^2 |A(\alpha y)|^2 t w') + [\Delta_x - \partial_{tt}] \phi_1 + N_0(\phi_1).$$

Since ϕ_1 has size of order α^2 , a smooth dependence in αy and it is of size $O(r_\alpha^{-2} e^{-\sigma|t|})$ using Lemma 3.4, we readily check that the “error created”

$$[\Delta_x - \partial_{tt}] \phi_1 + N_0(\phi_1) := -\alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta h_1 + R_0$$

satisfies

$$|R_0(y, t)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}.$$

Hence we have eliminated the h_1 -independent term $O(\alpha^2)$ that did not contribute to the projection $\Pi(y)$, and replaced it by one smaller and with faster decay. Let us be slightly more explicit for later reference. We have

$$S(u_1) := \Delta u_1 + f(u_1) =$$

$$-\alpha^2 [|A|^2 h_1 + \Delta_M h_1] w' + \alpha^2 a_{ij}^0 (\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0 + \partial_i h_1 \partial_j h_1) w''$$

$$-\alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta_M h_1 + 2\alpha^3 (t + h) a_{ij}^1 \partial_i h \partial_j h w'' + R_1 \tag{4.9}$$

where

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_i R_1(y, t, \iota, j)| + |D_j R_1(y, t, \iota, j)| + |R_1(y, t, \iota, j)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}$$

and the constant C above possibly depends on the number \mathcal{K} of condition (4.3).

The above arguments are in reality the way we will actually solve the problem: two separate, but coupled steps are involved: (1) Eliminate the parts of the error that do not contribute to the projection Π and (2) Adjust h_1 so that the projection Π becomes identically zero.

4.3. The condition of diverging ends

Let us explain the reason to introduce condition (4.4) in the parameters β_j . To fix ideas, let us assume that we have two consecutive planar ends of M , M_j and M_{j+1} , namely with $a_j = a_{j+1}$ and with $d = b_{j+1} - b_j > 0$. Assuming that the normal in M_j points upwards, the coordinate t reads approximately as

$$t = x_3 - \alpha^{-1}b_j - h \quad \text{near } M_{j\alpha}, \quad t = \alpha^{-1}b_{j+1} - x_3 - h \quad \text{near } M_{j+1\alpha}.$$

If we let $h_0 \equiv 0$ both on $M_{j\alpha}$ and $M_{j+1\alpha}$ which are separated at distance d/α , then a good approximation in the entire region between $M_{j\alpha}$ and $M_{j+1\alpha}$ that matches the parts of $w(t)$ coming both from M_j and M_{j+1} should read near M_j approximately as

$$w(t) + w(\alpha^{-1}d - t) - 1.$$

When computing the error of approximation, we observe that the following additional term arises near $M_{j\alpha}$:

$$\begin{aligned} E &:= f(w(t) + w(\alpha^{-1}d - t) - 1) - f(w(t)) - f(w(\alpha^{-1}d - t)) \sim \\ &\sim [f'(w(t)) - f'(1)](w(\alpha^{-1}d - t) - 1). \end{aligned}$$

Now in the computation of the projection of the error this would give rise to

$$\int_{-\infty}^{\infty} [f'(w(t)) - f'(1)](w(\alpha^{-1}d - t) - 1) w'(t) dt \sim c_* e^{-\sigma + \frac{d}{\alpha}}.$$

where $c_* \neq 0$ is a constant. Thus equation (4.7) for h_1 gets modified with a term which even though very tiny, it has no decay as $|y| \rightarrow +\infty$ on M_j , unlike the others involved in the operator G_0 in (4.7). That terms eventually dominates and the equation for h_1 for very large r would read in M_j as

$$\Delta_M h_1 \sim e^{-\frac{\sigma}{\alpha}} \neq 0,$$

which is inconsistent with the assumption that h is bounded. Worse yet, its solution would be quadratic thus eventually intersecting another end. This nuisance is fixed with the introduction of h_0 satisfying condition (4.4). In that case the term E created above will now read near $M_{j\alpha}$ as

$$E \sim C e^{-\sigma + \frac{d}{\alpha}} e^{-(\beta_{j+1} - \beta_j) \log r_\alpha} e^{-\sigma|t|} = O(e^{-\frac{\sigma}{\alpha}} r_\alpha^{-4} e^{-\sigma|t|})$$

which is qualitatively of the same type of the other terms involved in the computation of the error.

4.4. The global first approximation

The approximation $u_1(x)$ in (4.2) will be sufficient for our purposes, however it is so far defined only in a region of the type \mathcal{N} which we have not made precise yet. Since we are assuming that M_α is connected, the fact that M_α is properly embedded implies that $\mathbb{R}^3 \setminus M_\alpha$ consists of precisely two components S_- and S_+ . Let us use the convention that ν points in the direction of S_+ . Let us consider the function \mathbb{H} defined in $\mathbb{R}^3 \setminus M_\alpha$ as

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } x \in S_+ \\ -1 & \text{if } x \in S_- \end{cases} . \tag{4.10}$$

Then our approximation $u_1(x)$ approaches $\mathbb{H}(x)$ at an exponential rate $O(e^{-\sigma_\pm |t|})$ as $|t|$ increases. The global approximation we will use consists simply of interpolating u_1 with \mathbb{H} sufficiently well-inside $\mathbb{R}^3 \setminus M_\alpha$ through a cut-off in $|t|$. In order to avoid the problem described in §4.3 and having the coordinates (y, t) well-defined, we consider this cut-off to be supported in a region y -dependent that expands logarithmically in r_α . Thus we will actually consider a region \mathcal{N}_δ expanding at the ends, thus becoming wider as $r_\alpha \rightarrow \infty$ than the set $\mathcal{N}_\delta^\alpha$ previously considered, where the coordinates are still well-defined.

We consider the open set \mathcal{O} in $M_\alpha \times \mathbb{R}$ defined as

$$\mathcal{O} = \left\{ (y, t) \in M_\alpha \times \mathbb{R}, \quad |t + h_1(\alpha y)| < \frac{\delta}{\alpha} + 4 \max \{ \sigma_-^{-1}, \sigma_+^{-1} \} \log(1 + r_\alpha(y)) =: \rho_\alpha(y) \right\} \tag{4.11}$$

where δ is small positive number. We consider the the region $\mathcal{N} =: \mathcal{N}_\delta$ of points x of the form

$$x = X_h(y, t) = y + (t + h_0(\alpha y) + h_1(\alpha y))\nu(\alpha y), \quad (y, t) \in \mathcal{O},$$

namely $\mathcal{N}_\delta = X_h(\mathcal{O})$. The coordinates (y, t) are well-defined in \mathcal{N}_δ for any sufficiently small δ : indeed the map X_h is one to one in \mathcal{O} thanks to assumption (4.4) and the fact that $h_1 = O(\alpha)$. Moreover, Lemma 3.3 applies in \mathcal{N}_δ .

Let $\eta(s)$ be a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$. and define

$$\eta_\delta(x) := \begin{cases} \eta(|t + h_1(\alpha y)| - \rho_\alpha(y) - 3) & \text{if } x \in \mathcal{N}_\delta, \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases} \tag{4.12}$$

where ρ_α is defined in (4.11). Then we let our global approximation $\mathbf{w}(x)$ be simply defined as

$$\mathbf{w} := \eta_\delta u_1 + (1 - \eta_\delta)\mathbb{H} \tag{4.13}$$

where \mathbb{H} is given by (4.10) and $u_1(x)$ is just understood to be $\mathbb{H}(x)$ outside \mathcal{N}_δ .

Since \mathbb{H} is an exact solution in $\mathbb{R}^3 \setminus M_\delta$, the global error of approximation is simply computed as

$$S(\mathbf{w}) = \Delta \mathbf{w} + f(\mathbf{w}) = \eta_\delta S(u_1) + E \tag{4.14}$$

where

$$E = 2\nabla\eta_\delta\nabla u_1 + \Delta\eta_\delta(u_1 - \mathbb{H}) + f(\eta_\delta u_1 + (1 - \eta_\delta)\mathbb{H}) - \eta_\delta f(u_1).$$

The new error terms created are of exponentially small size $O(e^{-\frac{\alpha}{r_\alpha}})$ but have in addition decay with r_α . In fact we have

$$|E| \leq C e^{-\frac{\delta}{\alpha}} r_\alpha^{-4}.$$

Let us observe that $|t + h_1(\alpha y)| = |z - h_0(\alpha y)|$ where z is the normal coordinate to M_α , hence η_δ does not depend on h_1 , in particular the term $\Delta\eta_\delta$ does involves second derivatives of h_1 on which we have not made assumptions yet.

5. The proof of Theorem 6

The proof of Theorem 6 involves various ingredients whose detailed proofs are fairly technical. In order to keep the presentation as clear as possible, in this section we carry out the proof, skimming it from several (important) steps, which we state as lemmas or propositions, with complete proofs postponed for the subsequent sections.

We look for a solution u of the Allen Cahn equation (2.1) in the form

$$u = \mathbf{w} + \varphi \tag{5.1}$$

where \mathbf{w} is the global approximation defined in (4.13) and φ is in some suitable sense small. Thus we need to solve the following problem

$$\Delta\varphi + f'(\mathbf{w})\varphi = -S(\mathbf{w}) - N(\varphi) \tag{5.2}$$

where

$$N(\varphi) = f(\mathbf{w} + \varphi) - f(\mathbf{w}) - f'(\mathbf{w})\varphi.$$

Next we introduce various norms that we will use to set up a suitable functional analytic scheme for solving problem (5.2). For a function $g(x)$ defined in \mathbb{R}^3 , $1 < p \leq +\infty$, $\mu > 0$, and $\alpha > 0$ we write

$$\|g\|_{p,\mu,*} := \sup_{x \in \mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'|.$$

On the other hand, given numbers $\mu \geq 0$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$, $p > 3$, and functions $g(y, t)$ and $\phi(y, t)$ defined in $M_\alpha \times \mathbb{R}$ we consider the norms

$$\|g\|_{p,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} r_\alpha(y)^\mu e^{\sigma|t|} \left(\int_{B((y,t),1)} |f|^p dV_\alpha \right)^{\frac{1}{p}}. \quad (5.3)$$

Consistently we set

$$\|g\|_{\infty,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} r_\alpha(y)^\mu e^{\sigma|t|} \|f\|_{L^\infty(B((y,t),1))} \quad (5.4)$$

and let

$$\|\phi\|_{2,p,\mu,\sigma} := \|D^2\phi\|_{p,\mu,\sigma} + \|D\phi\|_{\infty,\mu,\sigma} + \|\phi\|_{\infty,\mu,\sigma}. \quad (5.5)$$

We consider also for a function $g(y)$ defined in M the L^p -weighted norm

$$\|f\|_{p,\beta} := \left(\int_M |f(y)|^p (1 + |y|^\beta)^p dV(y) \right)^{1/p} = \|(1 + |y|^\beta) f\|_{L^p(M)} \quad (5.6)$$

where $p > 1$ and $\beta > 0$.

We assume in what follows, that for a certain constant $\mathcal{K} > 0$ and $p > 3$ we have that the parameter function $h_1(y)$ satisfies

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1 + r^2) Dh_1\|_{L^\infty(M)} + \|D^2 h_1\|_{p,4-\frac{4}{p}} \leq \mathcal{K}\alpha. \quad (5.7)$$

Next we reduce problem (5.2) to solving one qualitatively similar (equation (5.20) below) for a function $\phi(y, t)$ defined in the whole space $M_\alpha \times \mathbb{R}$.

5.1. Step 1: the gluing reduction

We will follow the following procedure. Let us consider again $\eta(s)$, a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$, and define

$$\zeta_n(x) := \begin{cases} \eta(|t + h_1(\alpha y)| - \frac{\delta}{\alpha} + n) & \text{if } x \in \mathcal{N}_\delta \\ 0 & \text{if } x \notin \mathcal{N}_\delta \end{cases}. \quad (5.8)$$

We look for a solution $\varphi(x)$ of problem (5.2) of the following form

$$\varphi(x) = \zeta_2(x)\phi(y, t) + \psi(x) \quad (5.9)$$

where ϕ is defined in entire $M_\alpha \times \mathbb{R}$, $\psi(x)$ is defined in \mathbb{R}^3 and $\zeta_2(x)\phi(y, t)$ is understood as zero outside \mathcal{N}_δ .

We compute, using that $\zeta_2 \cdot \zeta_1 = \zeta_1$,

$$S(\mathbf{w} + \varphi) = \Delta\varphi + f'(\mathbf{w})\varphi + N(\varphi) + S(\mathbf{w}) =$$

$$\zeta_2 [\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) + H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1)] +$$

$$\Delta\psi - [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 \quad (5.10)$$

where $H(t)$ is any smooth, strictly negative function satisfying

$$H(t) = \begin{cases} f'(+1) & \text{if } t > 1, \\ f'(-1) & \text{if } t < -1. \end{cases}$$

Thus, we will have constructed a solution $\varphi = \zeta_2\phi + \psi$ to problem (5.2) if we require that the pair (ϕ, ψ) satisfies the following coupled system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1) = 0 \text{ for } |t| < \frac{\delta}{\alpha} + 3 \quad (5.11)$$

$$\Delta\psi + [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \text{ in } \mathbb{R}^3. \quad (5.12)$$

In order to find a solution to this system we will first extend equation (5.11) to entire $M_\alpha \times \mathbb{R}$ in the following manner. Let us set

$$\mathbf{B}(\phi) = \zeta_4[\Delta_x - \partial_{tt} - \Delta_{y, M_\alpha}] \phi \quad (5.13)$$

where Δ_x is expressed in (y, t) coordinates using expression (3.24) and $\mathbf{B}(\phi)$ is understood to be zero for $|t + h_1| > \frac{\delta}{\alpha} + 5$. The other terms in equation (5.11) are simply extended as zero beyond the support of ζ_1 . Thus we consider the extension of equation (5.11) given by

$$\partial_{tt}\phi + \Delta_{y, M_\alpha}\phi + \mathbf{B}(\phi) + f'(w(t))\phi = -\tilde{S}(u_1)$$

$$-\{[f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi)\} \text{ in } \in M_\alpha \times \mathbb{R}, \tag{5.14}$$

where we set, with reference to expression (4.9),

$$\begin{aligned} \tilde{S}(u_1) = & -\alpha^2[|A|^2 h_1 + \Delta_M h_1] w' + \alpha^2 a_{ij}^0 (2\partial_i h_0 \partial_j h_1 + \partial_i h_1 \partial_j h_0) w'' \\ & -\alpha^4 (|A|^2 t \psi'_0 - a_{ij}^0 \partial_i h_0 \partial_j h_0 t \psi'_1) \Delta h_1 + \zeta_4 [\alpha^3 (t+h) a_{ij}^1 \partial_i h \partial_j h w'' + R_1(y, t)] \end{aligned} \tag{5.15}$$

and, we recall

$$R_1 = R_1(y, t, h_1(\alpha y), \nabla_M h_1(\alpha y))$$

with

$$|D_i R_1(y, t, \iota, j)| + |D_j R_1(y, t, \iota, j)| + |R_1(y, t, \iota, j)| \leq C \alpha^3 (1 + r_\alpha(y))^{-4} e^{-\sigma|t|}. \tag{5.16}$$

In summary $\tilde{S}(u_1)$ coincides with $S(u_1)$ if $\zeta_4 = 1$ while outside the support of ζ_4 , their parts that are not defined for all t are cut-off.

To solve the resulting system (5.12)-(5.14), we find first solve equation (5.12) in ψ for a given ϕ a small function in absolute value. Noticing that the potential $[(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]$ is uniformly negative, so that the linear operator is qualitatively like $\Delta - 1$ and using contraction mapping principle, a solution $\psi = \Psi(\phi)$ is found according to the following lemma, whose detailed proof we carry out in §8.1.2.

Lemma 5.1: *For all sufficiently small α the following holds. Given ϕ with $\|\phi\|_{2,p,\mu,\sigma} \leq 1$, there exists a unique solution $\psi = \Psi(\phi)$ of problem (5.12) such that*

$$\|\psi\|_X := \|D^2\psi\|_{p,\mu,*} + \|\psi\|_{p,\mu,*} \leq C e^{-\frac{\sigma\delta}{\alpha}}. \tag{5.17}$$

Besides, Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_X \leq C e^{-\frac{\sigma\delta}{\alpha}} \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma}. \tag{5.18}$$

Thus we replace $\psi = \Psi(\phi)$ in the first equation (5.11) so that by setting $\mathbb{N}(\phi) := \mathbb{B}(\phi) + [f'(u_1) - f'(w)]\phi + \zeta_1(f'(u_1) - H(t))\Psi(\phi) + \zeta_1 N(\Psi(\phi) + \phi)$,

$$(5.19)$$

our problem is reduced to finding a solution ϕ to the following nonlinear, nonlocal problem in $M_\alpha \times \mathbb{R}$.

$$\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w)\phi = -\tilde{S}(u_1) - \mathbb{N}(\phi) \quad \text{in } M_\alpha \times \mathbb{R}. \quad (5.20)$$

Thus, we concentrate in the remaining of the proof in solving equation (5.20). As we hinted in §4.2, we will find a solution of problem (5.20) by considering two steps: (1) “Improving the approximation”, roughly solving for ϕ that eliminates the part of the error that does not contribute to the “projections” $\int [\tilde{S}(U_1) + \mathbb{N}(\phi)]w'(t)dt$, which amounts to a nonlinear problem in ϕ , and (2) Adjust h_1 in such a way that the resulting projection is actually zero. Let us set up the scheme for step (1) in a precise form.

5.2. Step 2: Eliminating terms not contributing to projections

Let us consider the problem of finding a function $\phi(y, t)$ such that for a certain function $c(y)$ defined in M_α , we have

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y,M_\alpha}\phi &= -\tilde{S}(u_1) - \mathbb{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha. \end{aligned} \quad (5.21)$$

Solving this problem for ϕ amounts to “eliminating the part of the error that does not contribute to the projection” in problem (5.20). To justify this phrase let us consider the associated linear problem in $M_\alpha \times \mathbb{R}$

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y,M_\alpha}\phi + f'(w(t))\phi &= g(y, t) + c(y)w'(t), \quad \text{for all } (y, t) \in M_\alpha \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha. \end{aligned} \quad (5.22)$$

Assuming that the corresponding operations can be carried out, let us multiply the equation by $w'(t)$ and integrate in t for fixed y . We find that

$$\Delta_{y,M_\alpha} \int_{\mathbb{R}} \phi(y, t) w' dt + \int_{\mathbb{R}} \phi(y, t) [w''' + f'(w)w'] dt = \int_{\mathbb{R}} g w' + c(y) \int_{\mathbb{R}} w'^2 .$$

The left hand side of the above identity is zero and then we find that

$$c(y) = -\frac{\int_{\mathbb{R}} g(y, t) w' dt}{\int_{\mathbb{R}} w'^2 dt}, \tag{5.23}$$

hence a ϕ solving problem (5.22). ϕ *precisely* solves or *eliminates* the part of g which does not contribute to the projections in the equation $\Delta\phi + f'(w)\phi = g$, namely the same equation with g replaced by \tilde{g} given by

$$\tilde{g}(y, t) = g(y, t) - \frac{\int_{\mathbb{R}} f(y, \cdot) w'}{\int_{\mathbb{R}} w'^2} w'(t). \tag{5.24}$$

The term $c(y)$ in problem (5.21) has a similar role, except that we cannot find it so explicitly.

In order to solve problem (5.21) we need to devise a theory to solve problem (5.22) where we consider a class of right hand sides g with a qualitative behavior similar to that of the error $S(u_1)$. As we have seen in (5.15), typical elements in this error are of the type $O((1 + r_\alpha(y))^{-\mu} e^{-\sigma|t|})$, so this is the type of functions $g(y, t)$ that we want to consider. This is actually the motivation to introduce the norms (5.3), (5.4) and (5.5). We will prove that problem (5.22) has a unique solution ϕ which respects the size of g in norm (5.3) up to its second derivatives, namely in the norm (5.5). The following fact holds.

Proposition 5.1: *Given $p > 3$, $\mu \geq 0$ and $0 < \sigma < \min\{\sigma_-, \sigma_+\}$, there exists a constant $C > 0$ such that for all sufficiently small $\alpha > 0$ the following holds. Given f with $\|g\|_{p, \mu, \sigma} < +\infty$, then Problem (5.22) with $c(y)$ given by (5.23), has a unique solution ϕ with $\|\phi\|_{\infty, \mu, \sigma} < +\infty$. This solution satisfies in addition that*

$$\|\phi\|_{2, p, \mu, \sigma} \leq C \|g\|_{p, \mu, \sigma}. \tag{5.25}$$

We will prove this result in §6. After Proposition 5.1, solving Problem (5.21) for a small ϕ is easy using the small Lipschitz character of the terms involved in the operator $N(\phi)$ in (5.19) and contraction mapping principle. The error term $\tilde{S}(u_1)$ satisfies

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p, 4, \sigma} \leq C \alpha^3. \tag{5.26}$$

Using this, and the fact that $N(\phi)$ defines a contraction mapping in a ball center zero and radius $O(\alpha^3)$ in $\|\cdot\|_{2, p, 4, \sigma}$, we conclude the existence of a unique small solution ϕ to problem (5.21) whose size is $O(\alpha^3)$ for this norm. This solution ϕ turns out to define an operator in h_1 $\phi = \Phi(h_1)$

which is Lipschitz in the norms $\|\cdot\|_*$ appearing in condition (5.7). In precise terms, we have the validity of the following result, whose detailed proof we postpone for §8.2.

Proposition 5.2: *Assume $p > 3$, $0 \leq \mu \leq 3$, $0 < \sigma < \min\{\sigma_+, \sigma_-\}$. There exists a $K > 0$ such that problem (8.8) has a unique solution $\phi = \Phi(h_1)$ such that*

$$\|\phi\|_{2,p,\mu,\sigma} \leq K\alpha^3.$$

Besides, Φ has a Lipschitz dependence on h_1 satisfying (5.7) in the sense that

$$\|\Phi(h_1) - \Phi(h_2)\|_{2,p,\mu,\sigma} \leq C\alpha^2 \|h_1 - h_2\|_*. \quad (5.27)$$

5.3. Step 3: Adjusting h_1 to make the projection zero

In order to conclude the proof of the theorem, we have to carry out the second step, namely adjusting h_1 , within a region of the form (5.7) for suitable \mathcal{K} in such a way that the “projections” are identically zero, namely making zero the function $c(y)$ found for the solution $\phi = \Phi(h_1)$ of problem (5.21). Using expression (5.23) for $c(y)$ we find that

$$c(y) \int_{\mathbb{R}} w'^2 = \int_{\mathbb{R}} \tilde{S}(u_1) w' dt + \int_{\mathbb{R}} \mathbb{N}(\Phi(h_1)) w' dt. \quad (5.28)$$

Now, setting $c_* := \int_{\mathbb{R}} w'^2 dt$ and using same computation employed to derive formula (4.6), we find from expression (5.15) that

$$\int_{\mathbb{R}} \tilde{S}(u_1)(y, t) w'(t) dt = -c_* \alpha^2 (\Delta_M h_1 + h_1 |A|^2) + c_* \alpha^2 G_1(h_1)$$

where

$$\begin{aligned} c_* G_1(h_1) = & -\alpha^2 \Delta h_1 (|A|^2 \int_{\mathbb{R}} t \psi'_0 w' dt - a_{ij}^0 \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} t \psi'_1 w' dt) + \\ & \alpha \partial_i (h_0 + h_1) \partial_j (h_0 + h_1) \int_{\mathbb{R}} \zeta_4(t+h) a_{ij}^1 w'' w' dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 R_1(y, t, h_1, \nabla_M h_1) w' dt \end{aligned} \quad (5.29)$$

and we recall that R_1 is of size $O(\alpha^3)$ in the sense (5.16). Thus, setting

$$c_* G_2(h_1) := \alpha^{-2} \int_{\mathbb{R}} \mathbb{N}(\Phi(h_1)) w' dt, \quad G(h_1) := G_1(h_1) + G_2(h_1), \quad (5.30)$$

we find that the equation $c(y) = 0$ is equivalent to the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + |A|^2 h_1 = G(h_1) \quad \text{in } M. \quad (5.31)$$

Therefore, we will have proven Theorem 6 if we find a function h_1 defined on M satisfying constraint (5.7) for a suitable \mathcal{K} that solves equation (5.31). Again, this is not so direct since the operator \mathcal{J} has a nontrivial bounded kernel. Rather than solving directly (5.31), we consider first a projected version of this problem, namely that of finding h_1 such that for certain scalars c_1, \dots, c_J we have

$$\begin{aligned} \mathcal{J}(h_1) &= G(h_1) + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M, \\ \int_M \frac{\hat{z}_i h}{1+r^4} dV &= 0, \quad i = 1, \dots, J. \end{aligned} \quad (5.32)$$

Here $\hat{z}_1, \dots, \hat{z}_J$ is a basis of the vector space of bounded Jacobi fields.

In order to solve problem (5.32) we need a corresponding linear invertibility theory. This leads us to consider the linear problem

$$\begin{aligned} \mathcal{J}(h) &= f + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M, \\ \int_M \frac{\hat{z}_i h}{1+r^4} dV &= 0, \quad i = 1, \dots, J. \end{aligned} \quad (5.33)$$

Here $\hat{z}_1, \dots, \hat{z}_J$ are bounded, linearly independent Jacobi fields, and J is the dimension of the vector space of bounded Jacobi fields.

We will prove in §7.1 the following result.

Proposition 5.3: *Given $p > 2$ and f with $\|f\|_{p,4-\frac{4}{p}} < +\infty$, there exists a unique bounded solution h of problem (5.33). Moreover, there exists a positive number $C = C(p, M)$ such that*

$$\|h\|_* := \|h\|_\infty + \|(1+|y|^2) Dh\|_\infty + \|D^2 h\|_{p,4-\frac{4}{p}} \leq C \|f\|_{p,4-\frac{4}{p}}. \quad (5.34)$$

Using the fact that G is a small operator of size $O(\alpha)$ uniformly on functions h_1 satisfying (5.7), Proposition 5.3 and contraction mapping principle yield the following result, whose detailed proof we carry out in §9.

Proposition 5.4: *Given $p > 3$, there exists a number $\mathcal{K} > 0$ such that for all sufficiently small $\alpha > 0$ there is a unique solution h_1 of problem (5.32) that satisfies constraint (5.7).*

5.4. Step 3: Conclusion

At the last step we prove that the constants c_i found in equation (5.32) are in reality all zero, without the need of adjusting any further parameters but rather as a consequence of the natural invariances of the of the full equation. The key point is to realize what equation has been solved so far.

First we observe the following. For each h_1 satysfying (5.7), the pair (ϕ, ψ) with $\phi = \Phi(h_1)$, $\psi = \Psi(\phi)$, solves the system

$$\Delta\phi + f'(u_1)\phi + \zeta_1(f'(u_1) - H(t))\psi + \zeta_1 N(\psi + \phi) + S(u_1) = c(y)w'(t) \text{ for } |t| < \frac{\delta}{\alpha} + 3$$

$$\Delta\psi + [(1 - \zeta_1)f'(u_1) + \zeta_1 H(t)]\psi +$$

$$(1 - \zeta_2)S(\mathbf{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^3.$$

Thus setting

$$\varphi(x) = \zeta_2(x)\phi(y, t) + \psi(x), \quad u = \mathbf{w} + \varphi,$$

we find from formula (5.10) that

$$\Delta u + f(u) = S(\mathbf{w} + \varphi) = \zeta_2 c(y) w'(t).$$

On the other hand choosing h_1 as that given in Proposition 5.4 which solves problem (5.32), amounts precisely to making

$$c(y) = c_* \alpha^2 \sum_{i=1}^J c_i \frac{\hat{z}_i(\alpha y)}{1 + r_\alpha(y)^4}$$

for certain scalars c_i . In summary, we have found h_1 satisfying constraint (5.7) such that

$$u = \mathbf{w} + \zeta_2(x)\Phi(h_1) + \Psi(\Phi(h_1)) \tag{5.35}$$

solves the equation

$$\Delta u + f(u) = \sum_{j=1}^J \frac{\tilde{c}_j}{1 + r_\alpha^4} \hat{z}_j(\alpha y) w'(t) \tag{5.36}$$

where $\tilde{c}_i = c_* \alpha^2 c_i$. Testing equation (5.36) against the generators of the rigid motions $\partial_i u$ $i = 1, 2, 3$, $-x_2 \partial_1 u + x_1 \partial_2 u$, and using the balancing formula for the minimal surface and the zero average of the numbers β_j in

the definition of h_0 , we find a system of equations that leads us to $c_i = 0$ for all i , thus conclude the proof. We will carry out the details in §10.

In sections §6-10 we will complete the proofs of the intermediate steps of the program designed in this section.

6. The linearized operator

In this section we will prove Proposition 5.1. At the core of the proof of the stated a priori estimates is the fact that the one-variable solution w of (2.1) is *nondegenerate* in $L^\infty(\mathbb{R}^3)$ in the sense that the linearized operator

$$L(\phi) = \Delta_y \phi + \partial_{tt} \phi + f'(w(t))\phi, \quad (y, t) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R},$$

is such that the following property holds.

Lemma 6.1: *Let ϕ be a bounded, smooth solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}. \tag{6.1}$$

Then $\phi(y, t) = Cw'(t)$ for some $C \in \mathbb{R}$.

Proof: We begin by reviewing some known facts about the one-dimensional operator $L_0(\psi) = \psi'' + f'(w)\psi$. Assuming that $\psi(t)$ and its derivative decay sufficiently fast as $|t| \rightarrow +\infty$ and defining $\psi(t) = w'(t)\rho(t)$, we get that

$$\int_{\mathbb{R}} [|\psi'|^2 - f'(w)\psi^2] dt = \int_{\mathbb{R}} L_0(\psi)\psi dt = \int_{\mathbb{R}} w'^2 |\rho'|^2 dt,$$

therefore this quadratic form is positive unless ψ is a constant multiple of w' . Using this and a standard compactness argument we get that there is a constant $\gamma > 0$ such that whenever $\int_{\mathbb{R}} \psi w' = 0$ with $\psi \in H^1(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} (|\psi'|^2 - f'(w)\psi^2) dt \geq \gamma \int_{\mathbb{R}} (|\psi'|^2 + |\psi|^2) dt. \tag{6.2}$$

Now, let ϕ be a bounded solution of equation (6.1). We claim that ϕ has exponential decay in t , uniform in y . Let us consider a small number $\sigma > 0$ so that for a certain $t_0 > 0$ and all $|t| > t_0$ we have that

$$f'(w) < -2\sigma^2.$$

Let us consider for $\varepsilon > 0$ the function

$$g_\varepsilon(t, y) = e^{-\sigma(|t|-t_0)} + \varepsilon \sum_{i=1}^2 \cosh(\sigma y_i)$$

Then for $|t| > t_0$ we get that

$$L(g_\delta) < 0 \quad \text{if } |t| > t_0.$$

As a conclusion, using maximum principle, we get

$$|\phi| \leq \|\phi\|_\infty g_\varepsilon \quad \text{if } |t| > t_0,$$

and letting $\varepsilon \rightarrow 0$ we then get

$$|\phi(y, t)| \leq C \|\phi\|_\infty e^{-\sigma|t|} \quad \text{if } |t| > t_0.$$

Let us observe the following fact: the function

$$\tilde{\phi}(y, t) = \phi(y, t) - \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}$$

also satisfies $L(\tilde{\phi}) = 0$ and, in addition,

$$\int_{\mathbb{R}} w'(t) \tilde{\phi}(y, t) dt = 0 \quad \text{for all } y \in \mathbb{R}^2. \tag{6.3}$$

In view of the above discussion, it turns out that the function

$$\varphi(y) := \int_{\mathbb{R}} \tilde{\phi}^2(y, t) dt$$

is well defined. In fact so are its first and second derivatives by elliptic regularity of ϕ , and differentiation under the integral sign is thus justified. Now, let us observe that

$$\Delta_y \varphi(y) = 2 \int_{\mathbb{R}} \Delta_y \tilde{\phi} \cdot \tilde{\phi} dt + 2 \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2$$

and hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} (L(\tilde{\phi}) \cdot \tilde{\phi}) \\ &= \frac{1}{2} \Delta_y \varphi - \int_{\mathbb{R}} |\nabla_y \tilde{\phi}|^2 dz - \int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w) \tilde{\phi}^2) dt. \end{aligned} \tag{6.4}$$

Let us observe that because of relations (6.3) and (6.2), we have that

$$\int_{\mathbb{R}} (|\tilde{\phi}_t|^2 - f'(w) \tilde{\phi}^2) dt \geq \gamma \varphi.$$

It follows then that

$$\frac{1}{2} \Delta_y \varphi - \gamma \varphi \geq 0.$$

Since φ is bounded, from maximum principle we find that φ must be identically equal to zero. But this means

$$\phi(y, t) = \left(\int_{\mathbb{R}} w'(\zeta) \phi(y, \zeta) d\zeta \right) \frac{w'(t)}{\int_{\mathbb{R}} w'^2}. \quad (6.5)$$

Then the bounded function

$$g(y) = \int_{\mathbb{R}} w_{\zeta}(\zeta) \phi(y, \zeta) d\zeta$$

satisfies the equation

$$\Delta_y g = 0, \quad \text{in } \mathbb{R}^2. \quad (6.6)$$

Liouville's theorem implies that $g \equiv \text{constant}$ and relation (6.5) yields $\phi(y, t) = Cw'(t)$ for some C . This concludes the proof. \square

6.1. A priori estimates

We shall consider problem (5.22) in a slightly more general form, also in a domain finite in y -direction. For a large number $R > 0$ let us set

$$M_{\alpha}^R := \{y \in M_{\alpha} / r(\alpha y) < R\}$$

and consider the variation of Problem (5.22) given by

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_{\alpha}}\phi + f'(w(t))\phi &= g(y, t) + c(y)w'(t) \quad \text{in } M_{\alpha}^R \times \mathbb{R}, \\ \phi &= 0, \quad \text{on } \partial M_{\alpha}^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in M_{\alpha}^R, \end{aligned} \quad (6.7)$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt .$$

We begin by proving a priori estimates.

Lemma 6.2:

Let us assume that $0 < \sigma < \min\{\sigma_-, \sigma_+\}$ and $\mu \geq 0$. Then there exists a constant $C > 0$ such that for all small α and all large R , and every solution ϕ to Problem (6.13) with $\|\phi\|_{\infty, \mu, \sigma} < +\infty$ and right hand side g satisfying $\|g\|_{p, \mu, \sigma} < +\infty$ we have

$$\|D^2\phi\|_{p, \mu, \sigma} + \|D\phi\|_{\infty, \mu, \sigma} + \|\phi\|_{\infty, \mu, \sigma} \leq C\|g\|_{p, \mu, \sigma}. \quad (6.8)$$

Proof: For the purpose of the a priori estimate, it clearly suffices to consider the case $c(y) \equiv 0$. By local elliptic estimates, it is enough to show that

$$\|\phi\|_{\infty,\mu,\sigma} \leq C\|g\|_{p,\mu,\sigma}. \tag{6.9}$$

Let us assume by contradiction that (6.9) does not hold. Then we have sequences $\alpha = \alpha_n \rightarrow 0$, $R = R_n \rightarrow \infty$, g_n with $\|g_n\|_{p,\mu,\sigma} \rightarrow 0$, ϕ_n with $\|\phi_n\|_{\infty,\mu,\sigma} = 1$ such that

$$\begin{aligned} \partial_{tt}\phi_n + \Delta_{y,M_\alpha}\phi_n + f'(w(t))\phi_n &= g_n \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi_n &= 0 \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi_n(y,t) w'(t) dt &= 0 \quad \text{for all } y \in M_\alpha^R. \end{aligned} \tag{6.10}$$

Then we can find points $(y_n, t_n) \in M_\alpha^R \times \mathbb{R}$ such that

$$e^{-\sigma|t_n|}(1+r(\alpha_n y_n))^\mu |\phi_n(y_n, t_n)| \geq \frac{1}{2}.$$

We will consider different possibilities. We may assume that either $r_\alpha(y_n) = O(1)$ or $r_\alpha(y_n) \rightarrow +\infty$.

6.1.1. *Case $r(\alpha_n y_n)$ bounded.*

We have $\alpha_n y_n$ lies within a bounded subregion of M , so we may assume that

$$\alpha_n y_n \rightarrow \tilde{y}_0 \in M.$$

Assume that $\tilde{y}_0 \in Y_k(\mathcal{U}_k)$ for one of the local parametrization of M . We consider $\tilde{y}_n, \tilde{y}_0 \in \mathcal{U}_k$ with $Y_k(\tilde{y}_n) = \alpha_n y_n$, $Y_k(\tilde{y}_0) = \tilde{y}_0$.

On $\alpha_n^{-1}Y_k(\mathcal{U}_k)$, M_α is parameterized by $Y_{k,\alpha_n}(\mathbf{y}) = \alpha_n^{-1}Y_k(\alpha_n \mathbf{y})$, $\mathbf{y} \in \alpha_n^{-1}\mathcal{U}_k$. Let us consider the local change of variable,

$$\mathbf{y} = \alpha^{-1}\tilde{\mathbf{y}}_n + \mathbf{y}.$$

6.1.2. *Subcase t_n bounded*

Let us assume first that $|t_n| \leq C$. Then, setting

$$\tilde{\phi}_n(\mathbf{y}, t) := \tilde{\phi}_n(\alpha^{-1}\tilde{\mathbf{y}}_n + \mathbf{y}, t),$$

the local equation becomes

$$a_{ij}^0(\tilde{y}_n + \alpha_n \mathbf{y}) \partial_{ij} \tilde{\phi}_n + \alpha_n b_j^0(\tilde{y}_n + \alpha_n \mathbf{y}) \partial_j \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + f'(w(t)) \tilde{\phi}_n = \tilde{g}_n(\mathbf{y}, t)$$

where $\tilde{g}_n(\mathbf{y}, t) := g_n(\tilde{y}_n + \alpha \mathbf{y}, t)$. We observe that this expression is valid for \mathbf{y} well-inside the domain $\alpha^{-1} \mathcal{U}_k$ which is expanding to entire \mathbb{R}^2 . Since $\tilde{\phi}_n$ is bounded, and $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^2)$, we obtain local uniform $W^{2,p}$ -bound. Hence we may assume, passing to a subsequence, that $\tilde{\phi}_n$ converges uniformly in compact subsets of \mathbb{R}^3 to a function $\tilde{\phi}(\mathbf{y}, t)$ that satisfies

$$a_{ij}^0(\tilde{y}) \partial_{ij} \tilde{\phi} + \partial_{tt} \tilde{\phi} + f'(w(t)) \tilde{\phi} = 0.$$

Thus $\tilde{\phi}$ is non-zero and bounded. After a rotation and stretching of coordinates, the constant coefficient operator $a_{ij}^0(\tilde{y}) \partial_{ij}$ becomes $\Delta_{\mathbf{y}}$. Hence Lemma 6.1 implies that, necessarily, $\tilde{\phi}(\mathbf{y}, t) = Cw'(t)$. On the other hand, we have

$$0 = \int_{\mathbb{R}} \tilde{\phi}_n(\mathbf{y}, t) w'(t) dt \longrightarrow \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t) w'(t) dt \quad \text{as } n \rightarrow \infty.$$

Hence, necessarily $\tilde{\phi} \equiv 0$. But we have $(1 + r(\alpha_n y_n))^\mu |\tilde{\phi}_n(0, t_n)| \geq \frac{1}{2}$, and since t_n and $r(\alpha_n y_n)$ were bounded, the local uniform convergence implies $\tilde{\phi} \neq 0$. We have reached a contradiction.

6.1.3. Subcase t_n unbounded

If y_n is in the same range as above, but, say, $t_n \rightarrow +\infty$, the situation is similar. The variation is that we define now

$$\tilde{\phi}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} \phi_n(\alpha_n^{-1} \mathbf{y}_n + \mathbf{y}, t_n+t), \quad \tilde{g}_n(\mathbf{y}, t) = e^{\sigma(t_n+t)} g_n(\alpha_n^{-1} \mathbf{y}_n + \mathbf{y}, t_n+t).$$

Then $\tilde{\phi}_n$ is uniformly bounded, and $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^3)$. Now $\tilde{\phi}_n$ satisfies

$$\begin{aligned} a_{ij}^0(\mathbf{y}_n + \alpha_n \mathbf{y}) \partial_{ij} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + \alpha_n b_j(\mathbf{y}_n + \alpha_n \mathbf{y}) \partial_j \tilde{\phi}_n \\ - 2\sigma \partial_t \tilde{\phi}_n + (f'(w(t+t_n)) + \sigma^2) \tilde{\phi}_n = \tilde{g}_n. \end{aligned}$$

We fall into the limiting situation

$$a_{ij}^* \partial_{ij} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (\sigma_+^2 - \sigma^2) \tilde{\phi} = 0 \quad \text{in } \mathbb{R}^3 \quad (6.11)$$

where a_{ij}^* is a positive definite, constant matrix and $\tilde{\phi} \neq 0$. But since, by hypothesis $\sigma_+^2 - \sigma^2 > 0$, maximum principle implies that $\tilde{\phi} \equiv 0$. We obtain a contradiction.

6.1.4. Case $r(\alpha_n y_n) \rightarrow +\infty$.

In this case we may assume that the sequence $\alpha_n y_n$ diverges along one of the ends, say M_k . Considering now the parametrization associated to the end, $y = \psi_k(\mathbf{y})$, given by (3.1), which inherits that for $M_{\alpha_n, k}$, $y = \alpha_n^{-1} \psi_k(\alpha_n \mathbf{y})$. Thus in this case $a_{ij}^0(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}) \rightarrow \delta_{ij}$, uniformly in compact subsets of \mathbb{R}^2 .

6.1.5. Subcase t_n bounded

Let us assume first that the sequence t_n is bounded and set

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu \phi_n(\alpha_n^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

Then

$$\begin{aligned} \partial_j(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= -\mu \alpha r^{-\mu-1} \partial_j r \tilde{\phi} + r^{-\mu} \partial_j \tilde{\phi} \\ \partial_{ij}(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= \mu(\mu+1) \alpha^2 r^{-\mu-2} \partial_i r \partial_j r \tilde{\phi} - \mu \alpha^2 r^{-\mu-1} \partial_{ij} r \tilde{\phi} - \mu \alpha r^{-\mu-1} \partial_j r \partial_i \tilde{\phi} \\ &\quad + r^{-\mu} \partial_{ij} \tilde{\phi} - \mu \alpha r^{-\mu-1} \partial_i r \partial_j \tilde{\phi}. \end{aligned}$$

Now $\partial_i r = O(1)$, $\partial_{ij} r = O(r^{-1})$, hence we have

$$\begin{aligned} \partial_j(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= r^{-\mu} \left[\partial_j \tilde{\phi} + O(\alpha r_{\alpha}^{-1}) \tilde{\phi} \right], \\ \partial_{ij}(r_{\alpha_n}^{-\mu} \tilde{\phi}_n) &= r_{\alpha}^{-\mu} \left[\partial_{ij} \tilde{\phi} + O(\alpha r_{\alpha}^{-1}) \partial_i \tilde{\phi} + O(\alpha^2 r_{\alpha}^{-2}) \tilde{\phi} \right], \end{aligned}$$

and the equation satisfied by $\tilde{\phi}_n$ has therefore the form

$$\Delta_{\mathbf{y}} \tilde{\phi}_n + \partial_{tt} \tilde{\phi}_n + o(1) \partial_{ij} \tilde{\phi}_n + o(1) \partial_j \tilde{\phi}_n + o(1) \tilde{\phi}_n + f'(w(t)) \tilde{\phi}_n = \tilde{g}_n.$$

where $\tilde{\phi}_n$ is bounded, $\tilde{g}_n \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^3)$. From elliptic estimates, we also get uniform bounds for $\|\partial_j \tilde{\phi}_n\|_{\infty}$ and $\|\partial_{ij} \tilde{\phi}_n\|_{p,0,0}$. In the limit we obtain a $\tilde{\phi} \neq 0$ bounded, solution of

$$\Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} + f'(w(t)) \tilde{\phi} = 0, \quad \int_{\mathbb{R}} \tilde{\phi}(\mathbf{y}, t) w'(t) dt = 0, \quad (6.12)$$

a situation which is discarded in the same way as before if $\tilde{\phi}$ is defined in \mathbb{R}^3 . There is however, one more possibility which is that $r(\alpha_n y_n) - R_n = O(1)$. In such a case we would see in the limit equation (6.12) satisfied in a half-space, which after a rotation in the \mathbf{y} -plane can be assumed to be

$$H = \{(\mathbf{y}, t) \in \mathbb{R}^2 \times \mathbb{R} / y_2 < 0\}, \quad \text{with } \phi(y_1, 0, t) = 0 \quad \text{for all } (y_1, t) \in \mathbb{R}^2.$$

By Schwarz's reflection, the odd extension of $\tilde{\phi}$, which achieves for $y_2 > 0$, $\tilde{\phi}(y_1, y_2, t) = -\tilde{\phi}(y_1, -y_2, t)$, satisfies the same equation, and thus we fall into one of the previous cases, again finding a contradiction.

6.1.6. Subcase t_n unbounded

Let us assume now $|t_n| \rightarrow +\infty$. If $t_n \rightarrow +\infty$ we define

$$\tilde{\phi}_n(\mathbf{y}, t) = (1 + r(\tilde{\mathbf{y}}_n + \alpha_n \mathbf{y}))^\mu e^{t_n + t} \phi_n(\alpha_n^{-1} \tilde{\mathbf{y}}_n + \mathbf{y}, t_n + t).$$

In this case we end up in the limit with a $\tilde{\phi} \neq 0$ bounded and satisfying the equation

$$\Delta_{\mathbf{y}} \tilde{\phi} + \partial_{tt} \tilde{\phi} - 2\sigma \partial_t \tilde{\phi} - (\sigma_+^2 - \sigma^2) \tilde{\phi} = 0$$

either in entire space or in a Half-space under zero boundary condition. This implies again $\tilde{\phi} = 0$, and a contradiction has been reached that finishes the proof of the a priori estimates.

6.2. Existence: conclusion of proof of Proposition 5.1

Let us prove now existence. We assume first that g has compact support in $M_\alpha \times \mathbb{R}$.

$$\begin{aligned} \partial_{tt} \phi + \Delta_{y, M_\alpha} \phi + f'(w(t)) \phi &= g(y, t) + c(y) w'(t) \quad \text{in } M_\alpha^R \times \mathbb{R}, \\ \phi &= 0, \quad \text{on } \partial M_\alpha^R \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \phi(y, t) w'(t) dt &= 0 \quad \text{for all } y \in M_\alpha^R, \end{aligned} \tag{6.13}$$

where we allow $R = +\infty$ and

$$c(y) \int_{\mathbb{R}} w'^2 dt = - \int_{\mathbb{R}} g(y, t) w' dt .$$

Problem (6.13) has a weak formulation which is the following. Let

$$H = \{ \phi \in H_0^1(M_\alpha^R \times \mathbb{R}) / \int_{\mathbb{R}} \phi(y, t) w'(t) dt = 0 \quad \text{for all } y \in M_\alpha^R \} .$$

H is a closed subspace of $H_0^1(M_\alpha^R \times \mathbb{R})$, hence a Hilbert space when endowed with its natural norm,

$$\|\phi\|_H^2 = \int_{M_\alpha^R} \int_{\mathbb{R}} (|\partial_t \phi|^2 + |\nabla_{M_\alpha} \phi|^2 - f'(w(t)) \phi^2) dV_\alpha dt .$$

ϕ is then a weak solution of Problem (6.13) if $\phi \in H$ and satisfies

$$\begin{aligned} a(\phi, \psi) &:= \int_{M_\alpha^R \times \mathbb{R}} (\nabla_{M_\alpha} \phi \cdot \nabla_{M_\alpha} \psi - f'(w(t)) \phi \psi) dV_\alpha dt = \\ &- \int_{M_\alpha^R \times \mathbb{R}} g \psi dV_\alpha dt \quad \text{for all } \psi \in H. \end{aligned}$$

It is standard to check that a weak solution of problem (6.13) is also classical provided that g is regular enough. Let us observe that because of the orthogonality condition defining H we have that

$$\gamma \int_{M_\alpha^R \times \mathbb{R}} \psi^2 dV_\alpha dt \leq a(\psi, \psi) \quad \text{for all } \psi \in H.$$

Hence the bilinear form a is coercive in H , and existence of a unique weak solution follows from Riesz's theorem. If g is regular and compactly supported, ψ is also regular. Local elliptic regularity implies in particular that ϕ is bounded. Since for some $t_0 > 0$, the equation satisfied by ϕ is

$$\Delta \phi + f'(w(t)) \phi = c(y)w'(t), \quad |t| > t_0, \quad y \in M_\alpha^R, \quad (6.14)$$

and $c(y)$ is bounded, then enlarging t_0 if necessary, we see that for $\sigma < \min\{\sigma_+, \sigma_-\}$, the function $v(y, t) := Ce^{-\sigma|t|} + \varepsilon e^{\sigma|t|}$ is a positive supersolution of equation (6.14), for a large enough choice of C and arbitrary $\varepsilon > 0$. Hence $|\phi| \leq Ce^{-\sigma|t|}$, from maximum principle. Since M_α^R is bounded, we conclude that $\|\phi\|_{p, \mu, \sigma} < +\infty$. From Lemma 6.2 we obtain that if R is large enough then

$$\|D^2 \phi\|_{p, \mu, \sigma} + \|D\phi\|_{\infty, \mu, \sigma} + \|\phi\|_{\infty, \mu, \sigma} \leq C\|g\|_{p, \mu, \sigma} \quad (6.15)$$

Now let us consider Problem (6.13) for $R = +\infty$, allowed above, and for $\|g\|_{p, \mu, \sigma} < +\infty$. Then solving the equation for finite R and suitable compactly supported g_R , we generate a sequence of approximations ϕ_R which is uniformly controlled in R by the above estimate. If g_R is chosen so that $g_R \rightarrow g$ in $L^p_{loc}(M_\alpha \times \mathbb{R})$ and $\|g_R\|_{p, \mu, \sigma} \leq C\|g\|_{p, \mu, \sigma}$. We obtain that ϕ_R is locally uniformly bounded, and by extracting a subsequence, it converges uniformly locally over compacts to a solution ϕ to the full problem which respects the estimate (5.25). This concludes the proof of existence, and hence that of the proposition.

7. Theory of the Jacobi operator

We consider this section the problem of finding a function h such that for certain constants c_1, \dots, c_J ,

$$\mathcal{J}(h) = \Delta_M h + |A|^2 h = f + \sum_{j=1}^J \frac{c_j}{1+r^4} \hat{z}_j \quad \text{in } M, \quad (7.1)$$

$$\int_M \frac{\hat{z}_i h}{1+r^4} = 0, \quad i = 1, \dots, J \quad (7.2)$$

and prove the result of Proposition 5.3. We will also deduce the existence of Jacobi fields of logarithmic growth as in Lemma 4.1. We recall the definition of the norms $\|\cdot\|_{p,\beta}$ in (5.6).

Outside of a ball of sufficiently large radius R_0 , it is natural to parameterize each end of M , $y_3 = F_k(y_1, y_2)$ using the Euclidean coordinates $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$. The requirement in f on each end amounts to $\tilde{f} \in L^p(B(0, 1/R_0))$ where

$$\tilde{f}(\mathbf{y}) := |\mathbf{y}|^{-4} f(|\mathbf{y}|^{-2} \mathbf{y}). \tag{7.3}$$

Indeed, observe that

$$\|\tilde{f}\|_{L^p(B(0,1/R_0))}^p = \int_{B(0,1/R_0)} |\mathbf{y}|^{-4p} |f(|\mathbf{y}|^{-2} \mathbf{y})|^p d\mathbf{y} = \int_{\mathbb{R}^2 \setminus B(0, R_0)} |\mathbf{y}|^{4(p-1)} |f(\mathbf{y})|^p d\mathbf{y}.$$

In order to prove the proposition we need some a priori estimates.

Lemma 7.1: *Let $p > 2$. For each $R_0 > 0$ sufficiently large there exists a constant $C > 0$ such that if*

$$\|f\|_{p, 4-\frac{4}{p}} + \|h\|_{L^\infty(M)} < +\infty$$

and h solves

$$\Delta_M h + |A|^2 h = f, \quad y \in M, \quad |y| > R_0,$$

then

$$\begin{aligned} \|h\|_{L^\infty(|y|>2R_0)} + \||y|^2 Dh\|_{L^\infty(|y|>2R_0)} + \||y|^{4-\frac{4}{p}} D^2 h\|_{L^p(|y|>2R_0)} \leq \\ C [\|f\|_{p, 4-\frac{4}{p}} + \|h\|_{L^\infty(R_0 < |y| < 3R_0)}]. \end{aligned}$$

Proof: Along each end M_k of M , Δ_M can be expanded in the coordinate \mathbf{y} as

$$\Delta_M = \Delta + O(|\mathbf{y}|^{-2})D^2 + O(|\mathbf{y}|^{-3})D.$$

A solution of h of equation (7.1) satisfies

$$\Delta_M h + |A|^2 h = f, \quad |y| > R_0$$

for a sufficiently large R_0 . Let us consider a Kelvin's transform

$$h(\mathbf{y}) = \tilde{h}(\mathbf{y}/|\mathbf{y}|^2).$$

Then we get

$$\Delta h(\mathbf{y}) = |\mathbf{y}|^{-4} (\Delta \tilde{h})(\mathbf{y}/|\mathbf{y}|^2).$$

Besides

$$O(|y|^{-2})D^2h(y)+O(|y|^{-3})Dh(y) = O(|y|^{-6})D^2\tilde{h}(y/|y|^2)+O(|y|^{-5})D\tilde{h}(y/|y|^2).$$

Hence

$$(\Delta_M h)(y/|y|^2) = |y|^4 \left[\Delta\tilde{h}(y) + O(|y|^2)D^2\tilde{h}(y) + O(|y|)D\tilde{h}(y) \right].$$

Then \tilde{h} satisfies the equation

$$\Delta\tilde{h} + O(|y|^2)D^2\tilde{h} + O(|y|)D\tilde{h} + O(1)h = \tilde{f}(y), \quad 0 < |y| < \frac{1}{R_0}$$

where \tilde{f} is given by (7.3). The operator above satisfies maximum principle in $B(0, \frac{1}{R_0})$ if R_0 is fixed large enough. This, the fact that \tilde{h} is bounded, and L^p -elliptic regularity for $p > 2$ in two dimensional space imply that

$$\|\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))} + \|D^2\tilde{h}\|_{L^p(B(0,1/2R_0))} \leq$$

$$C[\|\tilde{f}\|_{L^p(B(0,1/R_0))} + \|\tilde{h}\|_{L^\infty(1/3R_0 < |y| < 1/R_0)}] \leq$$

$$C[\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0 < |y| < 3R_0))}].$$

Let us observe that

$$\|\tilde{h}\|_{L^\infty(B(0,1/2R_0))} = \|h\|_{L^\infty(|y| > 2R_0)},$$

$$\|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))} = \||y|^2 Dh\|_{L^\infty(|y| > 2R_0)}.$$

Since

$$|D^2h(y)| \leq C(|y|^{-4}|D^2\tilde{h}(|y|^{-2}y)| + |y|^{-3}|D\tilde{h}(|y|^{-2}y)|)$$

then

$$|y|^{4-\frac{4}{p}}|D^2h(y)| \leq C(|y|^{-4/p}|D^2\tilde{h}(|y|^{-2}y)| + |y|^{-\frac{4}{p}-1}|D\tilde{h}(|y|^{-2}y)|).$$

Hence

$$\int_{|y| > 2R_0} |y|^{4p-4}|D^2h|^p dy \leq$$

$$C\left(\int_{B(0,1/2R_0)} |D^2\tilde{h}(y)|^p dy + \|D\tilde{h}\|_{L^\infty(B(0,1/2R_0))}^p \int_{|y| > 2R_0} |y|^{-4-p} dy\right).$$

It follows that

$$\|h\|_{L^\infty(|y| > 2R_0)} + \||y|^2 Dh\|_{L^\infty(|y| > 2R_0)} + \||y|^{4-\frac{4}{p}} D^2h\|_{L^p(|y| > 2R_0)} \leq$$

$$C [\|f\|_{p,4-\frac{4}{p}} + \|h\|_{L^\infty(B(R_0 < |y| < 3R_0))}] .$$

Since this estimate holds at each end, the result of the lemma follows, after possibly changing slightly the value R_0 . \square

Lemma 7.2: *Under the conditions of Lemma 7.1, assume that h is a bounded solution of Problem (7.1)-(7.2). Then the a priori estimate (5.34) holds.*

Proof:

Let us observe that this a priori estimate in Lemma 7.1 implies in particular that the Jacobi fields \hat{z}_i satisfy

$$\nabla \hat{z}_i(y) = O(|y|^{-2}) \quad \text{as } |y| \rightarrow +\infty.$$

Using \hat{z}_i as a test function in a ball $B(0, \rho)$ in M we obtain

$$\begin{aligned} \int_{\partial B(0, \rho)} (h \partial_\nu \hat{z}_i - \hat{z}_i \partial_\nu h) + \int_{|y| < \rho} (\Delta_M \hat{z}_i + |A|^2 \hat{z}_i) h = \\ \int_{|y| < \rho} f \hat{z}_i + \sum_{j=1}^J c_j \int_M \frac{\hat{z}_i \hat{z}_j}{1+r^4}. \end{aligned}$$

Since the boundary integral in the above identity is of size $O(\rho^{-1})$ we get

$$\int_M f \hat{z}_i + \sum_{j=1}^J c_j \int_M \frac{\hat{z}_i \hat{z}_j}{1+r^4} = 0 \tag{7.4}$$

so that in particular

$$|c_j| \leq C \|f\|_{p,4-\frac{4}{p}} \quad \text{for all } j = 1, \dots, J. \tag{7.5}$$

In order to prove the desired estimate, we assume by contradiction that there are sequences h_n, f_n with $\|h_n\|_\infty = 1$ and $\|f_n\|_{p,4-\frac{4}{p}} \rightarrow 0$, such that

$$\Delta_M h_n + |A|^2 h_n = f_n + \sum_{j=1}^J \frac{c_j^n \hat{z}_j}{1+r^4}$$

$$\int_M \frac{h_n \hat{z}_i}{1+r^4} = 0 \quad \text{for all } i = 1, \dots, J.$$

Thus according estimate (7.5), we have that $c_i^n \rightarrow 0$. From Lemma 7.1 we find

$$\|h_n\|_{L^\infty(|y|>2R_0)} \leq C[o(1) + \|h_n\|_{L^\infty(B(0,3R_0))}].$$

The latter inequality implies that

$$\|h_n\|_{L^\infty(B(0,3R_0))} \geq \gamma > 0.$$

Local elliptic estimates imply a C^1 bound for h_n on bounded sets. This implies the presence of a subsequence h_n which we denote the same way such that $h_n \rightarrow h$ uniformly on compact subsets of M , where h satisfies

$$\Delta_M h + |A|^2 h = 0.$$

h is bounded hence, by the nondegeneracy assumption, it is a linear combination of the functions \hat{z}_i . Besides $h \neq 0$ and satisfies

$$\int_M \frac{h \hat{z}_i}{1+r^4} = 0 \quad \text{for all } i = 1, \dots, J.$$

The latter relations imply $h = 0$, hence a contradiction that proves the validity of the a priori estimate. \square

7.1. Proof of Proposition 5.3

Thanks to Lemma 7.2 it only remains to prove existence of a bounded solution to problem (7.1)-(7.2). Let f be as in the statement of the proposition. Let us consider the Hilbert space H of functions $h \in H_{loc}^1(M)$ with

$$\|h\|_H^2 := \int_M |\nabla h|^2 + \frac{1}{1+r^4} |h|^2 < +\infty,$$

$$\int_M \frac{1}{1+r^4} h \hat{z}_i = 0 \quad \text{for all } i = 1, \dots, J.$$

Problem (7.1)-(7.2) can be formulated in weak form as that of finding $h \in H$ with

$$\int_M \nabla h \nabla \psi - |A|^2 h \psi = - \int_M f \psi \quad \text{for all } \psi \in H.$$

In fact, a weak solution $h \in H$ of this problem must be bounded thanks to elliptic regularity, with the use of Kelvin's transform in each end for the control at infinity. Using that $|A|^2 \leq Cr^{-4}$, Riesz representation theorem and the fact that H is compactly embedded in $L^2((1+r^4)^{-1}dV)$ (which

follows for instance by inversion at each end), we see that this weak problem can be written as an equation of the form

$$h - T(h) = \tilde{f}$$

where T is a compact operator in H and $\tilde{f} \in H$ depends linearly on f . When $f = 0$, the a priori estimates found yield that necessarily $h = 0$. Existence of a solution then follows from Fredholm's alternative. The proof is complete.

7.2. Jacobi fields of logarithmic growth. The proof of Lemma 4.1

We will use the theory developed above to construct Jacobi fields with logarithmic growth as $r \rightarrow +\infty$, whose existence we stated and use to set up the initial approximation in Lemma 4.1. One of these Jacobi fields is the generator of dilations of the surface, $z_0(y) = y \cdot \nu(y)$. We will prove next that there are another $m - 2$ linearly independent logarithmically growing Jacobi fields.

Let us consider an m -tuple of numbers β_1, \dots, β_m with $\sum_j \beta_j = 0$, and any smooth function $p(y)$ in M such that on each end M_j we have that for sufficiently large $r = r(y)$,

$$p(y) = (-1)^j \beta_j \log r(y), \quad y \in M_j$$

for certain numbers β_1, \dots, β_m that we will choose later. To prove the result of Lemma 4.1 we need to find a solution h_0 of the equation $\mathcal{J}(h_0) = 0$ of the form $h_0 = p + h$ where h is bounded. This amounts to solving

$$\mathcal{J}(h) = -\mathcal{J}(p). \tag{7.6}$$

Let us consider the cylinder $C_R = \{x \in \mathbb{R}^3 / r(x) < R\}$ for a large R . Then

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = \int_{M \cap C_R} \mathcal{J}(z_3) z_3 dV + \int_{\partial C_R \cap M} (z_3 \partial_n p - p \partial_n z_3) d\sigma(y).$$

Hence

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = \sum_{j=1}^m \int_{\partial C_R \cap M_j} (z_3 \partial_n p - p \partial_n z_3) d\sigma(y).$$

Thus using the graph coordinates on each end, we find

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV =$$

$$\sum_{j=1}^m (-1)^j \left[\frac{\beta_j}{R} \int_{|y|=R} \nu_3 d\sigma(y) - \beta_j \log R \int_{|y|=R} \partial_r \nu_3 d\sigma(y) \right] + O(R^{-1}).$$

We have that, on each end M_j ,

$$\nu_3(\mathbf{y}) = \frac{(-1)^j}{\sqrt{1 + |\nabla F_k(\mathbf{y})|^2}} = (-1)^j + O(r^{-2}), \quad \partial_r \nu_3(\mathbf{y}) = O(r^{-3}).$$

Hence we get

$$\int_{M \cap C_R} \mathcal{J}(p) z_3 dV = 2\pi \sum_{j=1}^m \beta_j + O(R^{-1}).$$

It is easy to see, using the graph coordinates that $\mathcal{J}(p) = O(r^{-4})$ and it is hence integrable. We pass to the limit $R \rightarrow +\infty$ and get

$$\int_M \mathcal{J}(p) z_3 dV = 2\pi \sum_{j=1}^m \beta_j = 0. \tag{7.7}$$

We make a similar integration for the remaining bounded Jacobi fields.

For $z_i = \nu_i(\mathbf{y})$ $i = 1, 2$ we find

$$\int_{M \cap C_R} \mathcal{J}(p) z_2 dV = \sum_{j=1}^m (-1)^j \left[\frac{\beta_j}{R} \int_{|y|=R} \nu_2 d\sigma(y) - \beta_j \log R \int_{|y|=R} \partial_r \nu_2 d\sigma(y) \right] + O(R^{-1}).$$

Now, on M_j ,

$$\nu_2(\mathbf{y}) = \frac{(-1)^j}{\sqrt{1 + |\nabla F_k(\mathbf{y})|^2}} = (-1)^j a_j \frac{x_i}{r^2} + O(r^{-3}), \quad \partial_r \nu_2(\mathbf{y}) = O(r^{-2}).$$

Hence

$$\int_M \mathcal{J}(p) z_i dV = 0 \quad i = 1, 2.$$

Finally, for $z_4(\mathbf{y}) = (-y_2, y_1, 0) \cdot \nu(\mathbf{y})$ we find on M_j ,

$$(-1)^j z_4(\mathbf{y}) = -\mathbf{y}_2 \partial_2 F_j + \mathbf{y}_1 \partial_1 F_j = b_{j1} \frac{y_2}{r^2} - b_{j2} \frac{y_1}{r^2} + O(r^{-2}), \quad \partial_r z_4 = O(r^{-2})$$

and hence again

$$\int_M \mathcal{J}(p) z_4 dV = 0.$$

From the solvability theory developed, we can then find a bounded solution to the problem

$$\mathcal{J}(h) = -\mathcal{J}(p) + \sum_{j=1}^J q c_j \hat{z}_j.$$

Since $\int_M \mathcal{J}(p) z_i dV = 0$ and hence $\int_M \mathcal{J}(p) \hat{z}_i dV = 0$, relations (7.4) imply that $c_i = 0$ for all i .

We have thus found a bounded solution to equation (7.6) and the proof is concluded.

Remark 7.3: *Observe that, in particular, the explicit Jacobi field $z_0(y) = y \cdot \nu(y)$ satisfies that*

$$z(y) = (-1)^j a_j \log r + O(1) \quad \text{for all } y \in M_j$$

and we have indeed $\sum_j a_j = 0$. Besides this one, we thus have the presence of another $m - 2$ linearly independent Jacobi fields with $|z(y)| \sim \log r$ as $r \rightarrow +\infty$, where m is the number of ends.

These are in reality all Jacobi fields with exact logarithmic growth. In fact if $\mathcal{J}(z) = 0$ and

$$|z(y)| \leq C \log r, \tag{7.8}$$

then the argument in the proof of Lemma 7.1 shows that the Kelvin's inversion $\tilde{z}(\mathbf{y})$ as in the proof of Lemma 7.2 satisfies near the origin $\Delta \tilde{z} = \tilde{f}$ where \tilde{f} belongs to any L^p near the origin, so it must equal a multiple of $\log |\mathbf{y}|$ plus a regular function. It follows that on M_j there is a number β_j with

$$z(\mathbf{y}) = (-1)^j \beta_j \log |\mathbf{y}| + h$$

where h is smooth and bounded. The computations above force $\sum_j \beta_j = 0$. It follows from Lemma 4.1 that then z must be equal to one of the elements there predicted plus a bounded Jacobi field. We conclude in particular that the dimension of the space of Jacobi fields satisfying (7.8) must be at most $m - 1 + J$, thus recovering a fact stated in Lemma 5.2 of [56].

8. Reducing the gluing system and solving the projected problem

In this section we prove Lemma 5.1, which reduces the gluing system (5.12)-(5.14) to solving the nonlocal equation (5.20) and prove Proposition 5.2 on solving the nonlinear projected problem (5.21), in which the basic element is linear theory stated in Proposition 5.1. In what follows we refer to notation and objects introduced in §5.1, §5.2.

8.1. Reducing the gluing system

Let us consider equation (5.12) in the gluing system (5.12)-(5.14),

$$\Delta\psi - W_\alpha(x)\psi + (1 - \zeta_2)S(\bar{w}) + (1 - \zeta_1)N(\psi + \zeta_2\phi) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1 = 0 \quad \text{in } \mathbb{R}^3 \tag{8.1}$$

where

$$W_\alpha(x) := [(1 - \zeta_1)f'(u_1) + \zeta_1H(t)] .$$

8.1.1. Solving the linear outer problem

We consider first the linear problem

$$\Delta\psi - W_\alpha(x)\psi + g(x) = 0 \quad \text{in } \mathbb{R}^3 \tag{8.2}$$

We observe that globally we have $0 < a < W_\alpha(x) < b$ for certain constants a and b . In fact we can take $a = \min\{\sigma_-^2, \sigma_+^2\} - \tau$ for arbitrarily small $\tau > 0$.

We consider for the purpose the norms for $1 < p \leq +\infty$,

$$\|g\|_{p,\mu} := \sup_{x \in \mathbb{R}^3} (1 + r(\alpha x))^\mu \|g\|_{L^p(B(x,1))}, \quad r(x', x_3) = |x'| .$$

Lemma 8.1: *Given $p > 3$, $\mu \geq 0$, there is a $C > 0$ such that for all sufficiently small α and any g with $\|g\|_{p,\mu} < +\infty$ there exists a unique ψ solution to Problem (8.2) with $\|\psi\|_{\infty,\mu} < +\infty$. This solution satisfies in addition,*

$$\|D^2\psi\|_{p,\mu} + \|\psi\|_{\infty,\mu} \leq C\|g\|_{p,\mu}. \tag{8.3}$$

Proof:

We claim that the a priori estimate

$$\|\psi\|_{\infty,\mu} \leq C\|g\|_{p,\mu} \tag{8.4}$$

holds for solutions ψ with $\|\psi\|_{\infty,\mu} < +\infty$ to problem (8.2) with $\|g\|_{p,\mu} < +\infty$ provided that α is small enough. This and local elliptic estimates in turn implies the validity of (8.3). To see this, let us assume the opposite, namely the existence $\alpha_n \rightarrow 0$, and solutions ψ_n to equation (8.2) with $\|\psi_n\|_{\infty,\mu} = 1$, $\|g_n\|_{p,\mu} \rightarrow 0$. Let us consider a point x_n with

$$(1 + r(\alpha_n x_n))^\mu \psi_n(x_n) \geq \frac{1}{2}$$

and define

$$\tilde{\psi}_n(x) = (1 + r(\alpha_n(x_n + x)))^\mu \psi_n(x_n + x), \quad \tilde{g}_n(x) = (1 + r(\alpha_n(x_n + x)))^\mu g_n(x_n + x),$$

$$\tilde{W}_n(x) = W_{\alpha_n}(x_n + x).$$

Then, similarly to what was done in the previous section, we check that the equation satisfied by $\tilde{\psi}_n$ has the form

$$\Delta \tilde{\psi}_n - \tilde{W}_n(x) \tilde{\psi}_n + o(1) \nabla \tilde{\psi}_n + o(1) \tilde{\psi}_n = \tilde{g}_n.$$

$\tilde{\psi}_n$ is uniformly bounded. Then elliptic estimates imply L^∞ -bounds for the gradient and the existence of a subsequence uniformly convergent over compact subsets of \mathbb{R}^3 to a bounded solution $\tilde{\psi} \neq 0$ to an equation of the form

$$\Delta \tilde{\psi} - W_*(x) \tilde{\psi} = 0 \quad \text{in } \mathbb{R}^3$$

where $0 < a \leq W_*(x) \leq b$. But maximum principle makes this situation impossible, hence estimate (8.4) holds.

Now, for existence, let us consider g with $\|g\|_{p,\mu} < +\infty$ and a collection of approximations g_n to g with $\|g_n\|_{\infty,\mu} < +\infty$, $g_n \rightarrow g$ in $L^p_{loc}(\mathbb{R}^3)$ and $\|g_n\|_{p,\mu} \leq C\|g\|_{p,\mu}$. The problem

$$\Delta \psi_n - W_n(x) \psi_n = g_n \quad \text{in } \mathbb{R}^3$$

can be solved since this equation has a positive supersolution of the form $C_n(1 + r(\alpha x))^{-\mu}$, provided that α is sufficiently small, but independently of n . Let us call ψ_n the solution thus found, which satisfies $\|\psi_n\|_{\infty,\mu} < +\infty$. The a priori estimate shows that

$$\|D^2 \psi_n\|_{p,\mu} + \|\psi_n\|_{\infty,\mu} \leq C\|g\|_{p,\mu}.$$

and passing to the local uniform limit up to a subsequence, we get a solution ψ to problem (8.2), with $\|\psi\|_{\infty,\mu} < +\infty$. The proof is complete. \square

8.1.2. The proof of Lemma 5.1

Let us call $\psi := \Upsilon(g)$ the solution of Problem (8.2) predicted by Lemma 8.1. Let us write Problem (8.1) as fixed point problem in the space X of $W^{2,p}_{loc}$ -functions ψ with $\|\psi\|_X < +\infty$,

$$\psi = \Upsilon(g_1 + K(\psi)) \tag{8.5}$$

where

$$g_1 = (1 - \zeta_2)S(\mathbf{w}) + 2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1, \quad K(\psi) = (1 - \zeta_1)N(\psi + \zeta_2\phi).$$

Let us consider a function ϕ defined in $M_\alpha \times \mathbb{R}$ such that $\|\phi\|_{2,p,\mu,\sigma} \leq 1$. Then,

$$|2\nabla\zeta_1\nabla\phi + \phi\Delta\zeta_1| \leq Ce^{-\sigma\frac{\delta}{\alpha}}(1+r(\alpha x))^{-\mu}\|\phi\|_{2,p,\mu,\sigma}.$$

We also have that $\|S(\mathbf{w})\|_{p,\mu,\sigma} \leq C\alpha^3$, hence

$$|(1-\zeta_2)S(\mathbf{w})| \leq Ce^{-\sigma\frac{\delta}{\alpha}}(1+r(\alpha x))^{-\mu}$$

and

$$\|g_1\|_{p,\mu} \leq Ce^{-\sigma\frac{\delta}{\alpha}}.$$

Let consider the set

$$\Lambda = \{\psi \in X / \|\psi\|_X \leq Ae^{-\sigma\frac{\delta}{\alpha}}\},$$

for a large number $A > 0$. Since

$$|K(\psi_1) - K(\psi_2)| \leq C(1-\zeta_1) \sup_{t \in (0,1)} |t\psi_1 + (1-t)\psi_2 + \zeta_2\phi| |\psi_1 - \psi_2|,$$

we find that

$$\|K(\psi_1) - K(\psi_2)\|_{\infty,\mu} \leq Ce^{-\sigma\frac{\delta}{\alpha}}\|\psi_1 - \psi_2\|_{\infty,\mu}$$

while $\|K(0)\|_{\infty,\mu} \leq Ce^{-\sigma\frac{\delta}{\alpha}}$. It follows that the right hand side of equation (8.5) defines a contraction mapping of Λ , and hence a unique solution $\psi = \Psi(\phi) \in \Lambda$ exists, provided that the number A in the definition of Λ is taken sufficiently large and $\|\phi\|_{2,p,\mu,\sigma} \leq 1$. In addition, it is direct to check the Lipschitz dependence of Ψ (5.18) on $\|\phi\|_{2,p,\mu,\sigma} \leq 1$.

Thus, we replace $\psi = \Psi(\phi)$ into the equation (5.14) of the gluing system (5.12)-(5.14) and get the (nonlocal) problem,

$$\partial_{tt}\phi + \Delta_{y,M_\alpha}\phi = -\tilde{S}(u_1) - N(\phi) \quad \text{in } M_\alpha \times \mathbb{R} \tag{8.6}$$

where

$$N(\phi) := \underbrace{\mathbb{B}(\phi) + [f'(u_1) - f'(w)]\phi}_{N_1(\phi)} + \underbrace{\zeta_1(f'(u_1) - H(t))\Psi(\phi)}_{N_2(\phi)} + \underbrace{\zeta_1 N(\Psi(\phi) + \phi)}_{N_3(\phi)}, \tag{8.7}$$

which is what we concentrate in solving next.

8.2. Proof of Proposition 5.2

We recall from §5.2 that Proposition 5.2 refers to solving the projected problem

$$\begin{aligned} \partial_{tt}\phi + \Delta_{y, M_\alpha}\phi &= -\tilde{S}(u_1) - \mathbf{N}(\phi) + c(y)w'(t) \quad \text{in } M_\alpha \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y, t) w'(t) dt &= 0, \quad \text{for all } y \in M_\alpha, \end{aligned} \tag{8.8}$$

and then adjust h_1 so that $c(y) \equiv 0$. Let $\phi = T(g)$ be the linear operator providing the solution in Proposition 5.1. Then Problem (8.8) can be reformulated as the fixed point problem

$$\phi = T(-\tilde{S}(u_1) - \mathbf{N}(\phi)) =: \mathcal{T}(\phi), \quad \|\phi\|_{2,p,\mu,\sigma} \leq 1 \tag{8.9}$$

which is equivalent to

$$\phi = T(-\tilde{S}(u_1) + \alpha^2 \Delta h_1 w' - \mathbf{N}(\phi)), \quad \|\phi\|_{2,p,\mu,\sigma} \leq 1, \tag{8.10}$$

since the term added has the form $\rho(y)w'$ which thus adds up to $c(y)w'$. The reason to absorb this term is that because of assumption (5.7), $\|\alpha^2 \Delta h_1 w'\|_{p,4,\sigma} = O(\alpha^{3-\frac{2}{p}})$ while the remainder has a priori size slightly smaller, $O(\alpha^3)$.

8.2.1. Lipschitz character of \mathbf{N}

We will solve Problem (8.10) using contraction mapping principle, so that we need to give account of a suitable Lipschitz property for the operator \mathcal{T} . We claim the following.

Claim. *We have that for a certain constant $C > 0$ possibly depending on \mathcal{K} in (5.7) but independent of $\alpha > 0$, such that for any ϕ_1, ϕ_2 with*

$$\|\phi_l\|_{2,p,\mu,\sigma} \leq K\alpha^3,$$

$$\|\mathbf{N}(\phi_1) - \mathbf{N}(\phi_2)\|_{p,\mu+1,\sigma} \leq C\alpha \|\phi_1 - \phi_2\|_{2,p,\mu,\sigma} \tag{8.11}$$

where the operator \mathbf{N} is defined in (8.7).

We study the Lipschitz character of the operator \mathbf{N} through analyzing each of its components. Let us start with N_1 . This is a second order linear

operator with coefficients of order α plus a decay of order at least $O(r_\alpha^{-1})$. We recall that $B = \zeta_2 B$ where in coordinates

$$\begin{aligned}
 B = & (f'(u_1) - f'(w)) - \alpha^2[(t + h_1)|A|^2 + \Delta_M h_1] \partial_t - 2\alpha a_{ij}^0 \partial_j h \partial_{it} + \\
 & \alpha(t + h) [a_{ij}^1 \partial_{ij} - \alpha a_{ij}^1 (\partial_j h \partial_{it} + \partial_i h \partial_{jt}) + \alpha(b_i^1 \partial_i - \alpha b_i^1 \partial_i h \partial_t)] + \\
 & \alpha^3(t + h)^2 b_3^1 \partial_t + \alpha^2 [a_{ij}^0 + \alpha(t + h) a_{ij}^1] \partial_i h \partial_j h \partial_{tt} \quad (8.12)
 \end{aligned}$$

where, we recall,

$$\begin{aligned}
 a_{ij}^1 = O(r_\alpha^{-2}), \quad a_{ij}^0 = O(r_\alpha^{-2}), \quad b_i^1 = O(r_\alpha^{-3}), \quad b_i^3 = O(r_\alpha^{-6}), \\
 f'(u_1) - f'(w) = O(\alpha^2 r_\alpha^{-2} e^{-\sigma|t|}) \quad \partial_j h = O(r_\alpha^{-1}), \quad |A|^2 = O(r_\alpha^{-4}).
 \end{aligned}$$

We claim that

$$\|N_1(\phi)\|_{p, \mu+1, \sigma} \leq C \alpha \|\phi\|_{2, p, \mu, \sigma}. \quad (8.13)$$

The only term of $N_1(\phi)$ that requires a bit more attention is $\alpha^2(\Delta h_1)(\alpha y) \partial_t \phi$. We have

$$\begin{aligned}
 & \int_{B((y,t),1)} |\alpha^2(\Delta h_1)(\alpha z) \partial_t \phi|^p dV_\alpha(z) d\tau \leq \\
 & C \alpha^{2p} \|\partial_t \phi\|_{L^\infty(B((y,t),1))} (1+r_\alpha(y))^{-4p+4} \int_{B((y,t),1)} |(1+r_\alpha(z))^{4-\frac{4}{p}} (\Delta h_1)(\alpha z)|^p |dV_\alpha(z)| \leq \\
 & C \alpha^{2p-2} \|\Delta h_1\|_{L^p(M)}^p e^{-p\sigma|t|} (1+r_\alpha(y))^{-p\mu-4p+4} \|\nabla \phi\|_{\infty, \mu, \sigma},
 \end{aligned}$$

and hence in particular for $p \geq 3$,

$$\|\alpha^2(\Delta h_1)(\alpha y) \partial_t \phi\|_{p, \mu+2, \sigma} \leq C \alpha^{2-\frac{2}{p}} \|h_1\|_* \|\phi\|_{2, p, \mu, \sigma} \leq C \alpha^{3-\frac{2}{p}} \|\phi\|_{2, p, \mu, \sigma}.$$

Let us consider now functions ϕ_l with

$$\|\phi_l\|_{2, p, \mu, \sigma} \leq 1, \quad l = 1, 2.$$

Now, according to Lemma 5.1, we get that

$$\|N_2(\phi_1) - N_2(\phi_2)\|_{p, \mu, \sigma} \leq C e^{-\sigma \frac{\delta}{\alpha}} \|\phi_1 - \phi_2\|_{p, \mu, \sigma}. \quad (8.14)$$

Finally, we also have that

$$|N_3(\phi_1) - N_3(\phi_2)| \leq$$

$$C\zeta_1 \sup_{t \in (0,1)} |t(\Psi(\phi_1) + \phi_1) + (1-t)(\Psi(\phi_2) + \phi_2)| [|\phi_1 - \phi_2| + |\Psi(\phi_1) - \Psi(\phi_2)|],$$

hence

$$\|\mathbb{N}_3(\phi_1) - \mathbb{N}_3(\phi_2)\|_{p,2\mu,\sigma} \leq C (\|\phi_1\|_{\infty,\mu,\sigma} + \|\phi_2\|_{\infty,\mu,\sigma} + e^{-\sigma \frac{\delta}{\alpha}}) \|\phi_1 - \phi_2\|_{\infty,\mu,\sigma}. \tag{8.15}$$

From (8.13), (8.14) and (8.15), inequality (8.11) follows. The proof of the claim is concluded.

8.2.2. Conclusion of the proof of Proposition 5.2

The first observation is that choosing $\mu \leq 3$, we get

$$\|\tilde{S}(u_1) + \alpha^2 \Delta h_1 w'\|_{p,\mu,\sigma} \leq C\alpha^3. \tag{8.16}$$

Let us assume now that $\phi_1, \phi_2 \in B_\alpha$ where

$$B_\alpha = \{\phi / \|\phi\|_{2,p,\mu,\sigma} \leq K\alpha^3\}$$

where K is a constant to be chosen. Then we observe that for small α

$$\|\mathbb{N}(\phi)\|_{p,\mu+1,\sigma} \leq C\alpha^4, \quad \text{for all } \phi \in B_\alpha,$$

where C is independent of K . Then, from relations (8.16)-(8.15) we see that if K is fixed large enough independent of α , then the right hand side of equation (8.5) defines an operator that applies B_α into itself, which is also a contraction mapping of B_α endowed with the norm $\|\cdot\|_{p,\mu\sigma}$, provided that $\mu \leq 3$. We conclude, from contraction mapping principle, the existence of ϕ as required.

The Lipschitz dependence (5.27) is a consequence of series of lengthy but straightforward considerations of the Lipschitz character in h_1 of the operator in the right hand side of equation (8.5) for the norm $\|\cdot\|_*$ defined in (5.34). Let us recall expression (8.12) for the operator B , and consider as an example, two terms that depend linearly on h_1 :

$$A(h_1, \phi) := \alpha a_{ij}^0 \partial_j h_1 \partial_{it} \phi.$$

Then

$$|A(h_1, \phi)| \leq C\alpha |\partial_j h_1| |\partial_{it} \phi|.$$

Hence

$$\|A(h_1, \phi)\|_{p,\mu+2,\sigma} \leq C\alpha \|(1+r_\alpha^2) \partial_j h_1\|_\infty \|\partial_{it} \phi\|_{p,\mu,\sigma} \leq C\alpha^4 \|h_1\|_* \|\phi\|_{2,p,\mu,\sigma}.$$

Similarly, for $A(\phi, h_1) = \alpha^2 \Delta_M h_1 \partial_t \phi$ we have

$$|A(\phi, h_1)| \leq C \alpha^2 |\Delta_M h_1(\alpha y)| (1 + r_\alpha)^{-\mu} e^{-\sigma|t|} \|\phi\|_{2,p,\mu,\sigma}.$$

Hence

$$\|\alpha^2 \Delta_M h_1 \partial_t \phi\|_{p,\mu+2,\sigma} \leq C \alpha^{5-\frac{2}{p}} \|h_1\|_* \|\phi\|_{2,p,\mu,\sigma}.$$

We should take into account that some terms involve nonlinear, however mild dependence, in h_1 . We recall for instance that $a_{ij}^1 = a_{ij}^1(\alpha y, \alpha(t + h_0 + h_1))$. Examining the rest of the terms involved we find that the whole operator \mathbb{N} produces a dependence on h_1 which is Lipschitz with small constant, and gaining decay in r_α ,

$$\|\mathbb{N}(h_1, \phi) - \mathbb{N}(h_2, \phi)\|_{p,\mu+1,\sigma} \leq C \alpha^2 \|h_1 - h_2\|_* \|\phi\|_{2,p,\mu,\sigma}. \quad (8.17)$$

Now, in the error term

$$\mathcal{R} = -\tilde{S}(u_1) + \alpha^2 \Delta h_1 w',$$

we have that

$$\|\mathcal{R}(h_1) - \mathcal{R}(h_2)\|_{p,3,\sigma} \leq C \alpha^2 \|h_1 - h_2\|_*. \quad (8.18)$$

To see this, again we go term by term in expansion (5.15). For instance the linear term $\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1 w''$. We have

$$|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1| \leq C \alpha^2 (1 + r_\alpha)^{-3} e^{-\sigma|t|} \|h_1\|_*$$

so that

$$\|\alpha^2 a_{ij}^0 \partial_i h_0 \partial_j h_1\|_{p,3,\sigma} \leq C \alpha^2 \|h_1\|_*,$$

the remaining terms are checked similarly.

Combining estimates (8.17), (8.18) and the fixed point characterization (8.5) we obtain the desired Lipschitz dependence (5.27) of Φ .

This concludes the proof.

9. The reduced problem: proof of Proposition 5.4

In this section we prove Proposition 5.4 based on the linear theory provided by Proposition 5.3. Thus, we want to solve the problem

$$\mathcal{J}(h_1) = \Delta_M h_1 + h_1 |A|^2 = G(h_1) + \sum_{i=1}^J \frac{c_i}{1+r^4} \hat{z}_i \quad \text{in } M, \quad (9.1)$$

$$\int_M \frac{h_1 \hat{z}_i}{1+r^4} dV = 0 \quad \text{for all } i = 1, \dots, J,$$

where the linearly independent Jacobi fields \hat{z}_i will be chosen in (10.1) and (10.2) of §8, and $G = G_1 + G_2$ was defined in (5.29), (5.30). We will use contraction mapping principle to determine the existence of a unique solution h_1 for which constraint (5.7), namely

$$\|h_1\|_* := \|h_1\|_{L^\infty(M)} + \|(1+r^2)Dh_1\|_{L^\infty(M)} + \|D^2h_1\|_{p,4-\frac{4}{p}} \leq \mathcal{K}\alpha, \quad (9.2)$$

is satisfied after fixing \mathcal{K} sufficiently large.

We need to analyze the size of the operator G , for which the crucial step is the following estimate.

Lemma 9.1: *Let $\psi(y, t)$ be a function defined in $M_\alpha \times \mathbb{R}$ such that*

$$\|\psi\|_{p,\mu,\sigma} := \sup_{(y,t) \in M_\alpha \times \mathbb{R}} e^{\sigma|t|} (1+r_\alpha^\mu) \|\psi\|_{L^p(B((y,t),1))} < +\infty$$

for $\sigma, \mu \geq 0$. The function defined in M as

$$q(y) := \int_{\mathbb{R}} \psi(y/\alpha, t) w'(t) dt$$

satisfies

$$\|q\|_{p,a} \leq C \|\psi\|_{p,\mu,\sigma} \quad (9.3)$$

provided that

$$\mu > \frac{2}{p} + a.$$

In particular, for any $\tau > 0$,

$$\|q\|_{p,2-\frac{2}{p}-\tau} \leq C \|\psi\|_{p,2,\sigma} \quad (9.4)$$

and

$$\|q\|_{p,4-\frac{4}{p}} \leq C \|\psi\|_{p,4,\sigma}. \quad (9.5)$$

Proof: We have that for $|y| > R_0$

$$\int_{|y|>R_0} |y|^{\alpha p} \left| \int_{\mathbb{R}} \psi(y/\alpha, t) w'(t) dt \right|^p dV \leq C \int_{\mathbb{R}} w'(t) dt \int_{|y|>R_0} |y|^{\alpha p} |\psi(y/\alpha, t)|^p dV.$$

Now

$$\int_{|y|>R_0} |y|^{\alpha p} |\psi(y/\alpha, t)|^p dV = \alpha^{\alpha p+2} \int_{|y|>R_0/\alpha} |y|^{\alpha p} |\psi(y, t)|^p dV_\alpha$$

and

$$\int_{|y|>R_0/\alpha} |y|^{\alpha p} |\psi(y, t)|^p dV_\alpha \leq C \sum_{i \geq [R_0/\alpha]} i^{\alpha p} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha.$$

Now, $i < |y| < i + 1$ is contained in $O(i)$ balls with radius one centered at points of the annulus, hence

$$\begin{aligned} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha &\leq C e^{-\sigma p|t|} i^{1-\mu p} \|\psi\|_{p,\mu}^p \\ &\leq C e^{-\sigma p|t|} \|\psi\|_{p,\mu}^p \int_{i < |y| < i+1} (1+r_\alpha)^{-\mu p} dV_\alpha \\ &\leq C e^{-\sigma p|t|} \|\psi\|_{p,\mu}^p \int_{i < |y| < i+1} |\alpha y|^{-\mu p} dV_\alpha \\ &\leq C e^{-\sigma p|t|} \|\psi\|_{p,\mu}^p \alpha^{-\mu p} i^{1-\mu p}. \end{aligned}$$

Then we find

$$\| |y|^a q \|_{L^p(|y|>R_0)}^p \leq C \alpha^{ap-\mu p+2} \|\psi\|_{p,\mu}^p \sum_{i \geq [R_0/\alpha]} i^{ap-\mu p+1}.$$

The sum converges if $\mu > \frac{2}{p} + a$ and in this case

$$\| |y|^a q \|_{L^p(|y|>R_0)}^p \leq C \alpha^{ap-\mu p+2} \alpha^{-ap+\mu p-2} \|\psi\|_{p,\mu}^p = C \|\psi\|_{p,\mu}^p$$

so that

$$\| |y|^a q \|_{L^p(|y|>R_0)} \leq C \|\psi\|_{p,\mu}.$$

Now, for the inner part $|y| < R_0$ in M , the weights play no role. We have

$$\begin{aligned} \int_{|y| < R_0} |\psi(y/\alpha, t)|^p dV &= \alpha^2 \int_{|y| < R_0/\alpha} |\psi(y, t)|^p dV_\alpha \leq \\ C \alpha^2 \sum_{i \leq R_0/\alpha} \int_{i < |y| < i+1} |\psi(y, t)|^p dV_\alpha &\leq C \alpha^2 \|\psi\|_{p,\mu}^p e^{-\sigma p|t|} \sum_{i \leq R_0/\alpha} i \\ &\leq C \|\psi\|_{p,\mu}^p e^{-\sigma p|t|}. \end{aligned}$$

Hence if $\mu > \frac{2}{p} + a$ we finally get

$$\|q\|_{p,a} \leq C \|\psi\|_{p,\mu}$$

and the proof of (9.3) is concluded. Letting $(\mu, a) = (2, 2 - \frac{2}{p} - \tau)$, $(\mu, a) = (4, 4 - \frac{4}{p})$ respectively in (9.3), we obtain (9.4) and (9.5).

□

Let us apply this result to $\psi(y, t) = \mathbf{N}(\Phi(h_1))$ to estimate the size of the operator G_2 in (5.30). For $\phi = \Phi(h_1)$ we have that

$$G_2(h_1)(y) := c_*^{-1} \alpha^{-2} \int_{\mathbb{R}} \mathbf{N}(\phi)(y/\alpha, t) w' dt$$

satisfies

$$\|G_2(h_1)\|_{p, 4 - \frac{4}{p}} \leq C \alpha^{-2} \|\mathbf{N}(\phi)\|_{p, 4, \sigma} \leq C \alpha^2.$$

On the other hand, we have that, similarly, for $\phi_l = \Phi(h_l)$, $l = 1, 2$,

$$\|G_2(h_1) - G_2(h_2)\|_{p, 4 - \frac{4}{p}} \leq C \alpha^{-2} \|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_2, h_2)\|_{p, 4, \sigma}.$$

Now,

$$\|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_1, h_2)\|_{p, 4, \sigma} \leq C \alpha^2 \|h_1 - h_2\|_* \|\phi_1\|_{2, p, 3, \sigma} \leq C \alpha^5 \|h_1 - h_2\|_*,$$

according to inequality (8.17), and

$$\|\mathbf{N}(\phi_1, h_1) - \mathbf{N}(\phi_2, h_1)\|_{p, 4, \sigma} \leq C \alpha^2 \|\phi_1 - \phi_2\|_{p, 3, \sigma} \leq C \alpha^4 \|h_1 - h_2\|_*.$$

We conclude then that

$$\|G_2(h_1) - G_2(h_2)\|_{p, 4 - \frac{4}{p}} \leq C \alpha^2 \|h_1 - h_2\|_*.$$

In addition, we also have that

$$\|G_2(0)\|_{p, 4 - \frac{4}{p}} \leq C \alpha^2.$$

for some $C > 0$ possibly dependent of \mathcal{K} . On the other hand, it is similarly checked that the remaining small operator $G_1(h_1)$ in (5.29) satisfies

$$\|G_1(h_1) - G_1(h_2)\|_{p, 4 - \frac{4}{p}} \leq C_1 \alpha \|h_1 - h_2\|_*.$$

A simple but crucial observation we make is that

$$c_* G_1(0) = \alpha \partial_i h_0 \partial_j h_0 \int_{\mathbb{R}} \zeta_4(t+h_0) a_{ij}^1 w'' w' dt + \alpha^{-2} \int_{\mathbb{R}} \zeta_4 R_1(y, t, 0, 0) w' dt$$

so that for a constant C_2 independent of \mathcal{K} in (9.2) we have

$$\|G_1(0)\|_{p, 4 - \frac{4}{p}} \leq C_2 \alpha.$$

In all we have that the operator $G(h_1)$ has an $O(\alpha)$ Lipschitz constant, and in addition satisfies

$$\|G(0)\|_{p, 4 - \frac{4}{p}} \leq 2C_2 \alpha.$$

Let $h = T(g)$ be the linear operator defined by Proposition 5.3. Then we consider the problem (9.1) written as the fixed point problem

$$h_1 = T(G(h_1)), \quad \|h\|_* \leq \mathcal{K}\alpha. \tag{9.6}$$

We have

$$\|T(G(h_1))\|_* \leq \|T\| \|G(0)\|_{p, 4-\frac{4}{p}} + C\alpha \|h_1\|_*.$$

Hence fixing $\mathcal{K} > 2C_2\|T\|$, we find that for all α sufficiently small, the operator TG is a contraction mapping of the ball $\|h\|_* \leq \mathcal{K}\alpha$ into itself. We thus have the existence of a unique solution of the fixed problem (9.6), namely a unique solution h_1 to problem (9.1) satisfying (9.2) and the proof of Proposition 5.4 is concluded.

10. Conclusion of the proof of Theorem 6

We denote in what follows

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad \hat{r} = \frac{1}{r}(x_1, x_2, 0), \quad \hat{\theta} = \frac{1}{r}(-x_2, x_1, 0).$$

We consider the four Jacobi fields associated to rigid motions, z_1, \dots, z_4 introduced in (2.13). Let J be the number of bounded, linearly independent Jacobi fields of \mathcal{J} . By our assumption and the asymptotic expansion of the ends (2.11), $3 \leq J \leq 4$. (Note that when M is a catenoid, $z_4 = 0$ and $J = 3$.) Let us choose

$$\hat{z}_j = \sum_{l=1}^4 d_{jl} z_{0l}, \quad j = 1, \dots, J \tag{10.1}$$

be normalized such that

$$\int_M q(y) \hat{z}_i \hat{z}_j = 0, \text{ for } i \neq j, \quad \int_M q(y) \hat{z}_i^2 = 1, \quad i, j = 1, \dots, J. \tag{10.2}$$

In what follows we fix the function q as

$$q(y) := \frac{1}{1 + r(y)^4}. \tag{10.3}$$

So far we have built, for certain constants \tilde{c}_i a solution u of equation (5.36), namely

$$\Delta u + f(u) = \sum_{j=1}^J \tilde{c}_j \hat{z}_j(\alpha y) w'(t) q(\alpha y) \zeta_2$$

where u , defined in (5.35) satisfies the following properties

$$u(x) = w(t) + \phi(y, t) \tag{10.4}$$

near the manifold, meaning this $x = y + (t + h(\alpha y)) \nu(\alpha y)$ with

$$y \in M_\alpha, \quad |t| \leq \frac{\delta}{\alpha} + \gamma \log(2 + r(\alpha y)).$$

The function ϕ satisfies in this region the estimate

$$|\phi| + |\nabla\phi| \leq C\alpha^2 \frac{1}{1 + r^2(\alpha y)} e^{-\sigma|t|}. \tag{10.5}$$

Moreover, we have the validity of the global estimate

$$|\nabla u(x)| \leq \frac{C}{1 + r^3(\alpha x)} e^{-\sigma \frac{\delta}{\alpha}}.$$

We introduce the functions

$$Z_i(x) = \partial_{x_i} u(x), \quad i = 1, 2, 3, \quad Z_4(x) = -\alpha x_2 \partial_{x_2} u + \alpha x_1 \partial_{x_2} u.$$

From the expansion (10.4) we see that

$$\nabla u(x) = w'(t) \nabla t + \nabla \phi.$$

Now, $t = z - h(\alpha y)$ where z designates normal coordinate to M_α . Since $\nabla z = \nu = \nu(\alpha y)$ we then get

$$\nabla t = \nu(\alpha y) - \alpha \nabla h(\alpha y).$$

Let us recall that h satisfies $h = (-1)^k \beta_k \log r + O(1)$ along the k -th end, and

$$\nabla h = (-1)^k \frac{\beta_k}{r} \hat{r} + O(r^{-2}).$$

From estimate (10.5) we we find that

$$\nabla u(x) = w'(t) (\nu - \alpha (-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}). \tag{10.6}$$

From here we get that near the manifold,

$$Z_i(x) = w'(t) (z_i(\alpha y) - \alpha (-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_i) + O(\alpha r_\alpha^{-2} e^{-\sigma|t|}), \quad i = 1, 2, 3, \tag{10.7}$$

$$Z_4(x) = w'(t) z_{04}(\alpha y) + O(\alpha r_\alpha^{-1} e^{-\sigma|t|}). \tag{10.8}$$

Using the characterization (5.36) of the solution u and barriers (in exactly the same way as in Lemma ?? below which estimates eigenfunctions of the linearized operator), we find the following estimate for $r_\alpha(x) > R_0$:

$$|\nabla u(x)| \leq C \sum_{k=1}^m e^{-\sigma|x_3 - \alpha^{-1}(F_k(\alpha x') + \beta_j \alpha \log |\alpha x'|)|} . \tag{10.9}$$

We claim that

$$\int_{\mathbb{R}^3} (\Delta u + f(u)) Z_i(x) dx = 0 \quad \text{for all } i = 1, \dots, 4 \tag{10.10}$$

so that

$$\sum_{j=1}^J \tilde{c}_j \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) Z_i(x) \zeta_2 dx = 0 \quad \text{for all } i = 1, \dots, 4. \tag{10.11}$$

Let us accept this fact for the moment. Let us observe that from estimates (10.7) and (10.8),

$$\alpha^2 \int_{\mathbb{R}^3} q(\alpha x) \hat{z}_j(\alpha y) w'(t) \sum_{l=1}^4 d_{il} Z_l(x) \zeta_2 dx = \int_{-\infty}^{\infty} w'(t)^2 dt \int_M q \hat{z}_j \hat{z}_i dV + o(1)$$

with $o(1)$ is small with α . Since the functions \hat{z}_i are linearly independent on any open set because they solve an homogeneous elliptic PDE, we conclude that the matrix with the above coefficients is invertible. Hence from (10.11) and (10.2), all \tilde{c}_i 's are necessarily zero. We have thus found a solution to the Allen Cahn equation (2.1) with the properties required in Theorem 6.

It remains to prove identities (10.10). The idea is to use the invariance of $\Delta + f(u)$ under rigid translations and rotations. This type of Pohozaev identity argument has been used in a number of places, see for instance [33].

In order to prove that the identity (10.10) holds for $i = 3$, we consider a large number $R \gg \frac{1}{\alpha}$ and the infinite cylinder

$$C_R = \{x / x_1^2 + x_2^2 < R^2\}.$$

Since in C_R the quantities involved in the integration approach zero at exponential rate as $|x_3| \rightarrow +\infty$ uniformly in (x_1, x_2) , we have that

$$\int_{C_R} (\Delta u + f(u)) \partial_{x_3} u - \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_3} u = \int_{C_R} \partial_{x_3} (F(u) - \frac{1}{2} |\nabla u|^2) = 0.$$

We claim that

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_3} u = 0.$$

Using estimate (10.6) we have that near the manifold,

$$\partial_{x_3} u \nabla u(x) \cdot \hat{r} = w'(t)^2 \left((\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r}) \cdot \hat{r} \right) \nu_3 + O(\alpha e^{-\sigma|t|} \frac{1}{r^2}).$$

Let us consider the k -th end, which for large r is expanded as

$$x_3 = F_{k,\alpha}(x_1, x_2) = \alpha^{-1}(a_k \log \alpha r + b_k + O(r^{-1}))$$

so that

$$(-1)^k \nu = \frac{1}{\sqrt{1 + |\nabla F_{k,\alpha}|^2}} (\nabla F_{k,\alpha}, -1) = \frac{a_k}{\alpha} \frac{\hat{r}}{r} - e_3 + O(r^{-2}). \quad (10.12)$$

Then on the portion of C_R near this end we have that

$$\left(\nu - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r} \right) \cdot \hat{r} \nu_3 = -\alpha^{-1} \frac{a_k + \alpha \beta_k}{R} + O(R^{-2}). \quad (10.13)$$

In addition, also, for $x_1^2 + x_2^2 = R^2$ we have the expansion

$$t = (x_3 - F_{k,\alpha}(x_1, x_2) - \beta_k \log \alpha r + O(1))(1 + O(R^{-2}))$$

with the same order valid after differentiation in x_3 , uniformly in such (x_1, x_2) . Let us choose $\rho = \gamma \log R$ for a large, fixed γ . Observe that on ∂C_R the distance between ends is greater than 2ρ whenever α is sufficiently small. We get,

$$\int_{F_{k,\alpha}(x_1, x_2) + \beta_k \log \alpha r - \rho}^{F_{k,\alpha}(x_1, x_2) + \beta_k \log \alpha r + \rho} w'(t)^2 dx_3 = \int_{-\infty}^{\infty} w'(t)^2 dt + O(R^{-2}).$$

Because of estimate (10.9) we conclude, fixing appropriately γ , that

$$\int_{\bigcap_k \{|x_3 - F_{k,\alpha}| > \rho\}} \partial_{x_3} u \nabla u(x) \cdot \hat{r} dx_3 = O(R^{-2}).$$

As a conclusion

$$\int_{-\infty}^{\infty} \partial_{x_3} u \nabla u \cdot \hat{r} dx_3 = -\frac{1}{\alpha R} \sum_{k=1}^m (a_k + \alpha \beta_k) \int_{-\infty}^{\infty} w'(t)^2 dt + O(R^{-2})$$

and hence

$$\int_{\partial C_R} \partial_{x_3} u \nabla u(x) \cdot \hat{r} = -\frac{2\pi}{\alpha} \sum_{k=1}^m (a_k + \alpha \beta_k) + O(R^{-1}).$$

But $\sum_{k=1}^m a_k = \sum_{k=1}^m \beta_k = 0$ and hence (10.10) for $i = 3$ follows after letting $R \rightarrow \infty$.

Let us prove the identity for $i = 2$. We need to carry out now the integration against $\partial_{x_2} u$. In this case we get

$$\int_{C_R} (\Delta u + f(u)) \partial_{x_2} u = \int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_2} u + \int_{C_R} \partial_{x_2} (F(u) - \frac{1}{2} |\nabla u|^2).$$

We have that

$$\int_{C_R} \partial_{x_2} (F(u) - \frac{1}{2} |\nabla u|^2) = \int_{\partial C_R} (F(u) - \frac{1}{2} |\nabla u|^2) n_2$$

where $n_2 = x_2/r$. Now, near the ends estimate (10.6) yields

$$|\nabla u|^2 = |w'(t)|^2 + O(e^{-\sigma|t|} \frac{1}{r^2})$$

and arguing as before, we get

$$\int_{-\infty}^{\infty} |\nabla u|^2 dx_3 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt + O(R^{-2}).$$

Hence

$$\int_{\partial C_R} |\nabla u|^2 n_2 = m \int_{-\infty}^{\infty} |w'(t)|^2 dt \int_{[r=R]} n_2 + O(R^{-1}).$$

Since $\int_{[r=R]} n_2 = 0$ we conclude that

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} |\nabla u|^2 n_2 = 0.$$

In a similar way we get

$$\lim_{R \rightarrow +\infty} \int_{\partial C_R} F(u) n_2 = 0.$$

Since near the ends we have

$$\partial_{x_2} u = w'(t)(\nu_2 - \alpha(-1)^k \frac{\beta_k}{r_\alpha} \hat{r} e_2) + O(\alpha r^{-2} e^{-\sigma|t|})$$

and from (10.12) $\nu_2 = O(R^{-1})$, completing the computation as previously done yields

$$\int_{\partial C_R} \nabla u \cdot \hat{r} \partial_{x_2} u = O(R^{-1}).$$

As a conclusion of the previous estimates, letting $R \rightarrow +\infty$ we finally find the validity of (10.10) for $i = 2$. Of course the same argument holds for $i = 1$.

Finally, for $i = 4$ it is convenient to compute the integral over C_R using cylindrical coordinates. Let us write $u = u(r, \theta, z)$. Then

$$\begin{aligned} & \int_{C_R} (\Delta u + f(u)) (x_2 \partial_{x_1} u - x_1 \partial_{x_2} u) = \\ & \int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} [u_{zz} + r^{-1}(ru_r)_r + f(u)] u_{\theta} r \, d\theta \, dr \, dz = \\ & -\frac{1}{2} \int_0^{2\pi} \int_0^R \int_{-\infty}^{\infty} \partial_{\theta} [u_z^2 + u_r^2 - 2F(u)] r \, d\theta \, dr \, dz + R \int_{-\infty}^{\infty} \int_0^{2\pi} u_r u_{\theta}(R, \theta, z) \, d\theta \, dz = \\ & 0 + \int_{\partial C_R} u_r u_{\theta} . \end{aligned}$$

On the other hand, on the portion of ∂C_R near the ends we have

$$u_r u_{\theta} = w'(t)^2 R(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) + O(R^{-2} e^{-\sigma|t|}).$$

From (10.12) we find

$$(\nu \cdot \hat{r})(\nu \cdot \hat{\theta}) = O(R^{-3}),$$

hence

$$u_r u_{\theta} = w'(t)^2 O(R^{-2}) + O(R^{-2} e^{-\sigma|t|})$$

and finally

$$\int_{\partial C_R} u_r u_{\theta} = O(R^{-1}).$$

Letting $R \rightarrow +\infty$ we obtain relation (10.10) for $i = 4$. The proof is concluded.

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