

A Thouless formula and Aubry duality for long-range Schrödinger skew-products

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Abstract

In this paper, we study the dynamical properties of a class of ergodic linear skew-products which includes the linear skew-products defined by quasi-periodic Schrödinger operators and their duals, in Aubry sense, when the potential is a trigonometric polynomial. Notably, these linear skew-products preserve an adapted complex-symplectic structure. We prove a Thouless formula relating the sum of the positive Lyapunov exponents and the logarithmic potential associated with the density of states of the corresponding operator. In particular, for quasi-periodic Schrödinger operators and their duals, we prove an identity for the upper Lyapunov exponent of the skew-product and the sum of the positive Lyapunov exponents of their dual, which generalizes the well-known formula for the Almost Mathieu. We illustrate these identities with some numerical illustrations.

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(Some figures may appear in colour only in the online journal)

1. Introduction, setting and main results

In recent years there have been significant advances in the theory of Schrödinger operators with quasi-periodic coefficients and their associated eigenvalue equations. These display rich and interesting properties and are ubiquitous in many situations, both in dynamical systems and mathematical physics. Two of the main approaches in their study are the possible extension of Floquet reducibility of the associated linear skew-products (after the work of Eliasson [Eli92]) and that of exponential localization (with the nonperturbative results of Jitomirskaya [Jit99]) and

Bourgain and Goldstein [BG00]). In the case of the Almost Mathieu model, the two approaches are intimately related by means of *Aubry duality* (invariance through Fourier transform) because the potential has a single harmonic. This connection has led to quite definitive answers to some long-standing problems [AJ09, AK06, Pui04].

In this paper, we will be interested in extending this duality to other trigonometric potentials and to see which are the consequences for the Lyapunov exponents. Although our setting is more general, let us introduce our main focus of interest now in order to formulate the results. We will consider (discrete) quasi-periodic Schrödinger operators on $\ell^2(\mathbb{Z})$ of the form

$$(hx)_n = x_{n+1} + x_{n-1} + W(\theta_n)x_n, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\theta_n = \theta_0 + n\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, for an irrational frequency $\omega \in \mathbb{R}$, is a quasi-periodic orbit and $W : \mathbb{T} \rightarrow \mathbb{R}$ is the *potential*. When

$$W(\theta) = \sum_{k=-d'}^{d'} W_k e^{2\pi k\theta i}$$

is a trigonometric polynomial, the Aubry dual of the operator above [AA80], which will be reviewed in section 1.5.1, is the following difference operator of order $2d$:

$$(h'x)_n = \sum_{k=-d'}^d W_k x_{n+k} + 2 \cos(2\pi\theta_n)x_n, \quad n \in \mathbb{Z}. \tag{1.2}$$

For any value of the energy α , the eigenvalue equation of (1.1), $hx = \alpha x$, gives rise to a *quasi-periodic Schrödinger skew-product* on \mathbb{R}^2

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha - W(\theta_n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \quad \theta_{n+1} = \theta_n + \omega \tag{1.3}$$

while for its dual, $h'x = \alpha x$, the corresponding linear skew-product is 2-dimensional. The main application of this paper is a relation between the upper Lyapunov exponent of quasi-periodic Schrödinger skew-products with trigonometric potentials and the sum of the positive Lyapunov exponents of the linear skew-product associated with their dual. Although this result will be presented in corollary 1.3, we now anticipate the main application of our paper:

Main Application. *Let $W : \mathbb{T} \rightarrow \mathbb{R}$ be a trigonometric polynomial and ω an irrational frequency. Let us denote by $\gamma^h(\alpha)$ the upper Lyapunov exponent of (1.1) for a given energy $\alpha \in \mathbb{C}$ and $\gamma^{h'}(\alpha)$ the sum of the positive Lyapunov exponents of the linear skew-product generated by the eigenvalue equation of its dual operator (1.2). Then,*

$$\gamma^h(\alpha) = \gamma^{h'}(\alpha) + \log |W_{d'}|. \tag{1.4}$$

The Almost Mathieu case, when $W(\theta) = \beta \cos(2\pi\theta)$, is the only potential when (1.1) and (1.2) are the same, which leads to the above formula, referred to sometimes as the *duality of the Lyapunov exponents*. Of course, this does not happen for other potentials, even simple trigonometric ones, as we see in the following example.

Main example. *As an illustration of the main application, let us consider the following Schrödinger operator with a potential with two harmonics:*

$$(hx)_n = x_{n+1} + x_{n-1} + 2\beta (\cos(2\pi\theta_n) + \cos(4\pi\theta_n)) x_n, \quad \theta_n = \theta_0 + \omega n, \quad n \in \mathbb{Z}, \tag{1.5}$$

being $\beta \neq 0$ a coupling constant. Its dual is a difference operator of order 4:

$$(h'x)_n = \beta (x_{n+2} + x_{n+1} + x_{n-1} + x_{n-2}) + 2 \cos(2\pi\theta_n)x_n, \quad n \in \mathbb{Z}. \tag{1.6}$$

For any α , the eigenvalue equation of the dual operator (1.2), $h'x = \alpha x$, generates a linear skew-product on \mathbb{R}^4 ,

$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \\ x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} -1 & \frac{\alpha}{\beta} - \frac{2}{\beta} \cos(2\pi\theta_n) & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \\ x_{n-1} \\ x_{n-2} \end{pmatrix}, \quad \theta_{n+1} = \theta_n + \omega.$$

As we will see, for any $\alpha \in \mathbb{C}$ and $\beta \neq 0$, this linear skew-product has two Lyapunov exponents which are non-negative out of four Lyapunov exponents, which we denote by $\gamma_1(\alpha, \beta)$ and $\gamma_2(\alpha, \beta)$. Then, formula (1.4) above reads as

$$\gamma^h(\alpha, \beta) = \gamma_1^{h'}(\alpha, \beta) + \gamma_2^{h'}(\alpha, \beta) + \log |\beta|, \tag{1.7}$$

where $\gamma^h(\alpha, \beta)$ is the upper Lyapunov exponent of the Schrödinger skew-product (1.3) with $W(\theta) = 2\beta(\cos(2\pi\theta) + \cos(4\pi\theta))$. One can see a numerical illustration of this identity in figure 1.

Although our initial motivation was to consider applications like above, we have proven most of the results in a more general setting. In the rest of the introduction, which we now briefly outline, we shall present the setting and the results. In section 1.1, we introduce the general framework of the ergodic self-adjoint operators of our interest, using Schrödinger operators and their duals as motivating examples. The corresponding dynamical systems are introduced in section 1.2, again referencing to Schrödinger operators. The main result for these general operators is given in section 1.3 and the consequences for Schrödinger operators, which were our motivating application, in section 1.5. We conclude this introduction with some numerical illustrations in section 1.6.

1.1. Setting

Although our motivation comes from Schrödinger operators with quasi-periodic potentials and their duals, in this paper we will consider the spectral properties of the following class of ergodic long-range operators in ℓ^2 spaces of sequences. These are defined from the following ingredients:

- (HL1) A dynamical system (Θ, τ, μ) given by a homeomorphism $\tau : \Theta \rightarrow \Theta$ on a compact metric space Θ , the *base*, and preserving an ergodic measure μ that is topological (i.e. positive on open sets);
- (HL2) A dense *base orbit* $(\theta_n = \tau^n(\theta_0))_{n \in \mathbb{Z}} \subset \Theta$ with initial phase $\theta_0 \in \Theta$;
- (HL3) A trigonometric polynomial $V : \mathbb{T} = \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$, with Fourier representation

$$V(\theta) = \sum_{k=-d}^d V_k e^{2\pi i k \theta}$$

with average $V_0 = 0$, $V_{-k} = \bar{V}_k$ for $k = 1, \dots, d$ and $V_d \neq 0$. We will refer to it as the *symbol*;

- (HL4) A continuous function $W : \Theta \rightarrow \mathbb{R}$, called the *potential*.

The existence of dense orbits, i.e. that τ is topologically transitive, assumed in (HL2), comes from the fact that τ is regionally recurrent (all the points are nonwandering) because (HL1) holds.

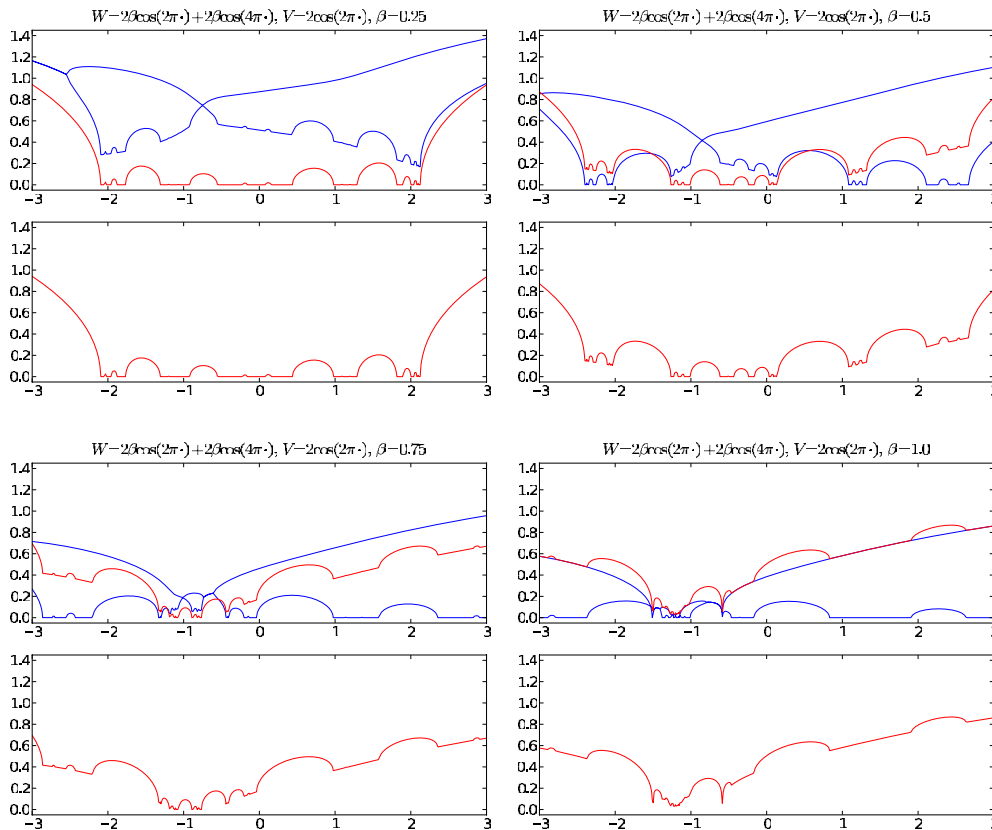


Figure 1. Numerical computation of the Lyapunov exponents and their sums for (1.6) and (1.5) as a function of α for selected values of β and $\omega = (\sqrt{5} - 1)/2$. In each cell the upper plot displays the Lyapunov exponents for the dual (1.6), $\gamma_1^{h'}(\alpha, \beta)$ and $\gamma_2^{h'}(\alpha, \beta)$, in blue, and the sum $\gamma_1^{h'}(\alpha, \beta) + \gamma_2^{h'}(\alpha, \beta) + \log |\beta|$ in red. The lower plot of each cell displays, in red, the upper Lyapunov exponent, $\gamma^h(\alpha, \beta)$, of the original Schrödinger model (1.5). Formula (1.7) implies that the two curves in red agree in each cell. We used 10^4 iterations of the skew-products with 10^3 values on the α -axis.

Assuming (HL1–HL4), we define the *long-range operator* (of range d) on $\ell^2(\mathbb{Z}, \mathbb{C})$, $h = h_{V,W,\theta_0}$ as

$$(hx)_n = \sum_{k=-d}^d V_k x_{n+k} + W(\theta_n)x_n, \quad n \in \mathbb{Z},$$

where $x = (x_n)_n \in \ell^2(\mathbb{Z}, \mathbb{C})$. Our main examples of quasi-periodic Schrödinger operators (1.1) and their duals (1.2) are included in this setting when the dynamical system (Θ, τ, μ) is the quasi-periodic rotation on the base $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with frequency ω , $\tau(\theta) = \theta + \omega$. For the Schrödinger operator with potential W we take $V = 2 \cos 2\pi \cdot$ while for the duals, we exchange V and W when the latter is a trigonometric polynomial.

1.2. From spectrum to dynamics

The long-range operator is self-adjoint and the spectrum $\text{Spec}(h, \ell^2(\mathbb{Z}, \mathbb{C}))$ is a compact subset of the real line which consists of approximate eigenvalues. It turns out that the spectrum of

the long-range operator can be characterized by dynamical and spectral properties of some associated linear skew-products, their Lyapunov exponents and the density of states, which we now introduce.

1.2.1. Long-range linear skew-products. Since the symbol V is a trigonometric polynomial, one can express the eigenvalue equation of h as

$$\sum_{k=-d}^d V_k x_{n+k} + W(\theta_n) x_n = \alpha x_n, \quad n \in \mathbb{Z}, \tag{1.8}$$

for $\alpha \in \mathbb{C}$, as the following recursion of order $2d$:

$$x_{n+d} = - \sum_{k=-d}^{d-1} \frac{V_k}{V_d} x_{n+k} + \frac{1}{V_d} (\alpha - W(\theta_n)) x_n, \quad n \in \mathbb{Z}.$$

As usual, instead of considering the previous finite difference equation of order $2d$, we consider a linear system of dimension $2d$,

$$\underbrace{\begin{pmatrix} x_{n+d} \\ \vdots \\ x_{n+2} \\ x_{n+1} \\ x_n \\ \vdots \\ x_{n-d+2} \\ x_{n-d+1} \end{pmatrix}}_{u_{n+1}} = \frac{1}{V_d} \underbrace{\begin{pmatrix} -V_{d-1} \cdots -V_1 & \alpha - W(\theta_n) & -V_{-1} \cdots -V_{-d+1} - V_{-d} \\ & V_d & \\ & \ddots & \\ & & V_d \\ \hline & & & V_d & \\ & & & & \ddots & \\ & & & & & V_d \end{pmatrix}}_{A_\alpha^h(\theta_n)} \underbrace{\begin{pmatrix} x_{n+d-1} \\ \vdots \\ x_{n+1} \\ x_n \\ x_{n-1} \\ \vdots \\ x_{n-d+1} \\ x_{n-d} \end{pmatrix}}_{u_n}$$

$$\theta_{n+1} = \tau(\theta_n), \tag{1.9}$$

defining a *long-range linear skew-product* $(A_\alpha^h, \tau) : \mathbb{C}^{2d} \times \Theta \rightarrow \mathbb{C}^{2d} \times \Theta$. As we will see, the dynamical properties of these linear skew-product systems are intimately related to the spectral properties of the corresponding long-range operator.

1.2.2. Lyapunov exponents and entropy. In general, for a linear skew-product (A, τ) in \mathbb{C}^m over τ , generated by a continuous map $A : \Theta \rightarrow \text{GL}(m, \mathbb{C})$, a fundamental matrix is given by the composition

$$A(n; \theta) = \begin{cases} A(\tau^{n-1}(\theta)) \cdots A(\theta), & n \geq 1, \\ I & n = 0, \\ A(\tau^n(\theta))^{-1} \cdots A(\tau^{-1}(\theta))^{-1} & n \leq -1. \end{cases} \tag{1.10}$$

An important tool when studying the dynamics of linear skew-products are the Lyapunov exponents, which measure the rates of growth of orbits. As a consequence of Oseledec's theory [Ose68, BP02], there exist m real numbers (the Lyapunov exponents) $\gamma_1 \geq \dots \geq \gamma_m$ such that for any $j = 1, \dots, m$

$$\gamma_1 + \dots + \gamma_j = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Theta} \log \|\wedge^j A(n; \theta)\| \, d\mu,$$

and, for μ -a.e. $\theta \in \Theta$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\wedge^j A(n; \theta)\| = \gamma_1 + \dots + \gamma_j.$$

To measure the total expansion rate of a linear skew-product one can introduce the (*fibred*) *entropy* as the sum of the positive Lyapunov exponents, $\gamma^h(\alpha)$.

For a long-range linear skew-product of dimension $2d$, (A_α^h, τ) , we will see that this entropy is precisely the sum

$$\gamma^h(\alpha) = \gamma_1^h(\alpha) + \dots + \gamma_d^h(\alpha),$$

where $\gamma_1^h(\alpha), \dots, \gamma_d^h(\alpha)$ are the first d Lyapunov exponents of (A_α^h, τ) . Indeed, when α is not real we will see in section 2.4 that long-range linear skew-products are uniformly hyperbolic and the dimensions of the unstable and stable subbundles agree, so that there are exactly d positive and d negative Lyapunov exponents. For α real, the skew-products are complex symplectic and the Lyapunov exponents come into positive-negative pairs, even if these can vanish, see section 2.3. For convenience, for long-range skew-products, we introduce the *normalized (fibred) entropy* as

$$\bar{\gamma}^h(\alpha) = \gamma^h(\alpha) + \log |V_d|.$$

In the Schrödinger case, the entropy and the normalized entropy agree and are equal to the upper Lyapunov exponent of the corresponding Schrödinger skew-product.

1.2.3. The integrated density of states. An essential ingredient for the description of the spectrum of these long-range operators is the integrated density of states, IDS for short, which we now consider shortly. For any long-range operator h satisfying (HL1–HL4) and any integer $N > 0$, consider $h^{[-N, N]}$, its restriction to the interval $[-N, N]$ with zero boundary conditions. Since θ_0 generates a dense orbit in Θ , the quantity

$$k_h^{[-N, N]}(a) = \frac{1}{2N + 1} \# \{ \text{eigenvalues} \leq a \text{ of } h^{[-N, N]} \}.$$

converges to a continuous, non-decreasing, function $a \mapsto k_h(a)$, which is independent of the chosen dense orbit, and is called the *integrated density of states* of the operator h . The IDS k_h is constant exactly at the open intervals in the resolvent set of the spectrum of h and it is the distribution function of a Borel measure κ_h , called the *density of states* measure of the operator [AS83, GJLS97, BJ02a, Pui06].

1.3. Thouless formula for long-range operators

The fundamental result of this paper is that the normalized entropy of a long-range linear skew-product is the logarithmic potential [Tho95] associated with the density of states measure of the corresponding long-range operator:

Theorem 1.1 (Thouless formula). *Let (Θ, τ, μ) , $\theta_0 \in \Theta$, V and W satisfy (HL1–HL4) above, and let h be the associated long-range operator. Then, the following integral formula holds:*

$$\int_{\mathbb{R}} \log |\alpha - a| d\kappa_h(a) = \bar{\gamma}^h(\alpha), \tag{1.11}$$

where κ_h is the density of states of the long-range operator h , and $\bar{\gamma}^h(\alpha) = \gamma^h(\alpha) + \log(V_d)$ is the normalized entropy.

This result is analogous to the case of Schrödinger operators on a strip, which was already considered by Craig and Simon [CS83a], and Kotani and Simon [KS88]. Nevertheless, our methods are closer to the dynamical and geometrical approach of Johnson [Joh87].

1.4. Application to log-Hölder continuity of the IDS

As a first application of theorem 1.11 we borrow some arguments from [CS83a] to prove the log-Hölder continuity of the IDS.

Theorem 1.2 (Log-Hölder continuity of the IDS). *Let (Θ, τ, μ) , $\theta_0 \in \Theta$, V and W satisfy (HL1–HL4) above, and let h be the associated long-range operator. Let ρ_h be its spectral radius and κ_h be its IDS. Then, for $\alpha_1, \alpha_2 \in \mathbb{R}$ with $|\alpha_2 - \alpha_1| < 1$,*

$$|k_h(\alpha_2) - k_h(\alpha_1)| \leq \frac{\log(|V_d|^{-1}(|\alpha_1| + \rho_h))}{\log(|\alpha_2 - \alpha_1|^{-1})}. \tag{1.12}$$

Proof. Let us assume $\alpha_1 < \alpha_2$. From Thouless formula (1.11), and since the entropy is non-negative,

$$\log |V_d| \leq \gamma^h(\alpha_1) + \log |V_d| = \int_{\mathbb{R}} \log |\alpha_1 - a| d\kappa_h(a).$$

We split the domain of the integral, \mathbb{R} , in three subdomains

$$\begin{aligned} R_1 &= [\alpha_1, \alpha_2], \\ R_2 &= \{a \in \mathbb{R} \mid |a - \alpha_1| \leq 1, a < \alpha_1 \text{ or } a > \alpha_2\}, \\ R_3 &= \{a \in \mathbb{R} \mid |a - \alpha_1| \geq 1\}. \end{aligned}$$

Then, since

$$\begin{aligned} \int_{R_1} \log |\alpha_1 - a| d\kappa_h(a) &\leq \log(\alpha_2 - \alpha_1) (\kappa_h(\alpha_2) - \kappa_h(\alpha_1)), \\ \int_{R_2} \log |\alpha_1 - a| d\kappa_h(a) &\leq 0 \end{aligned}$$

and

$$\int_{R_3} \log |\alpha_1 - a| d\kappa_h(a) \leq \log(|\alpha_1| + \rho_h),$$

we obtain the bound (1.12). □

1.5. Applications to quasi-periodic operators

We are now ready to state in general the main application of this paper, which is the relationship between the Lyapunov exponents of ‘dual’ quasi-periodic models through Aubry duality that we outlined in the introduction. Assuming that V and W are two real-analytic potentials and that the dynamics of the base $\Theta = \mathbb{T}$ is given by an irrational rotation of frequency ω we will consider both the long-range operators

$$(h_{\theta_0} x)_n = (h_{V,W,\theta_0} x)_n = \sum_{k \in \mathbb{Z}} V_k x_{n+k} + W(\theta_0 + n\omega)x_n, \quad n \in \mathbb{Z}, \tag{1.13}$$

and their Aubry *duals*

$$(h'_{\theta_1} x)_n = (h_{W,V,-,\theta_1} x)_n = \sum_{k \in \mathbb{Z}} W_k x_{n+k} + V(-\theta_1 - n\omega)x_n, \quad n \in \mathbb{Z}. \tag{1.14}$$

Aubry duality comes from the observation that if a is a point eigenvalue of h_{θ_0} whose eigenfunction $(x_n)_n$ decays sufficiently fast (e.g. exponentially), then the eigenvalue equation $h'_{\theta_1} - aI = 0$ has a quasi-periodic Bloch wave solution $(y_n)_n$, with $y_n = e^{2\pi i n \theta_0} \psi(\theta_1 + n\omega)$ where $\psi(\theta) = \sum_{k \in \mathbb{Z}} x_k e^{2\pi i k \theta}$.

Since we are considering a class of quasi-periodic operators larger than Schrödinger operators (where the symbol V is simply $2 \cos 2\pi \cdot$), we can also consider other transformations of the operator, like taking *mirror* symbols:

$$(h_{\theta_0}^- x)_n = (h_{V^-,W,\theta_0} x)_n = \sum_{k \in \mathbb{Z}} V_{-k} x_{n+k} + W(\theta_0 + n\omega)x_n, \quad n \in \mathbb{Z}. \tag{1.15}$$

We will devote some time to discuss these operators in section 1.5.1 before stating the applications of Thouless formula in section 1.5.2.

1.5.1. Aubry duality and dihedrality. In order to establish relationships between the operators h, h' and h^- on $\ell^2(\mathbb{Z}, \mathbb{C})$ above, it is convenient to consider their actions on the Hilbert space $L^2(\mathbb{T} \times \mathbb{Z}, \mathbb{C})$, which consists of functions $\Psi = \Psi(\theta, n)$ satisfying

$$\sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} |\Psi(\theta, n)|^2 d\theta < \infty.$$

These actions are given by their *direct integrals* [GJLS97, BJ02a], which are, respectively, the operators

$$\begin{aligned}(\tilde{h}\Psi)(\theta, n) &= \sum_{k \in \mathbb{Z}} V_k \Psi(\theta, n+k) + W(\theta+n\omega) \Psi(\theta, n), \\ (\tilde{h}'\Psi)(\theta, n) &= \sum_{k \in \mathbb{Z}} W_k \Psi(\theta, n+k) + V(-\theta-n\omega) \Psi(\theta, n)\end{aligned}$$

and

$$(\tilde{h}^-\Psi)(\theta, n) = \sum_{k \in \mathbb{Z}} V_{-k} \Psi(\theta, n+k) + W(\theta+n\omega) \Psi(\theta, n).$$

In this setting, Aubry duality corresponds to the unitary equivalence

$$\tilde{h}U = U\tilde{h}',$$

where U is the unitary operator

$$(U\Psi)(\theta, n) = \hat{\Psi}(\theta+n\omega, n),$$

and $\hat{\Psi}$ the Fourier transform. At a formal level, this operator is given by

$$(U\Psi)(\theta, n) = \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{T}} \Psi(\varphi, \ell) e^{-2\pi i n(\varphi+\ell\omega)} d\varphi e^{-2\pi i \ell \theta},$$

and its inverse by

$$(U^{-1}\Psi)(\theta, n) = \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{T}} \Psi(\varphi, \ell) e^{2\pi i n(\varphi+\ell\omega)} d\varphi e^{2\pi i \ell \theta}.$$

A computation shows that $U^4 = I$.

In addition to Aubry duality, what might be called *mirror duality* corresponds to the anti-unitary equivalence

$$\tilde{h}R = R\tilde{h}^-,$$

where R is the anti-unitary operator

$$(R\Psi)(\theta, n) = \overline{\Psi(\theta, n)},$$

which is involutive, $R^2 = I$. Moreover, U is reversible with R as reversion, since $UR = RU^{-1}$.

The combination of the operators U, R generates a group of order 8, namely

$$\text{Dih}_4 = \{I, U, U^2, U^3, R, UR = RU^3, U^2R = RU^2, U^3R = RU\},$$

which is the dihedral group of order 8, i.e. the group of symmetries of the square. Applying each of these elements to a quasi-periodic long-range operator h_{V, W, θ_0} we obtain the following eight quasi-periodic long-range operators

$$h_{V, W, \theta_0}, \quad h_{W, V, \theta_0}, \quad h_{V, W, \theta_0}, \quad h_{W, V, \theta_0} \quad (1.16)$$

and

$$h_{V, W, \theta_0}, \quad h_{W, V, \theta_0}, \quad h_{V, W, \theta_0}, \quad h_{W, V, \theta_0}. \quad (1.17)$$

Following the arguments of [GJLS97, BJ02a, Pui06] the IDS of these operators, and in particular the spectra, agree. See figure 2 for an illustration of the elements of this group.

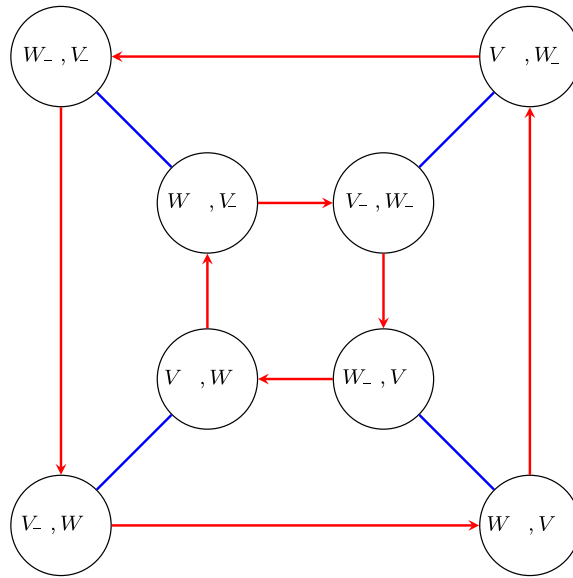


Figure 2. Dihedrality of the action of U (red) and R (blue) on long-range operators.

1.5.2. Duality and Dihedrality of Lyapunov exponents. In view of the equality between the IDS of the operators $h = h_{V,W,\theta_0}$ and $h' = h_{W,V,-,\theta_0}$ (the Aubry duals), as well as any of the operators obtained by the dihedrality in (1.16) and (1.17), the Thouless formula in theorem 1.1 implies that their entropies are related as well whenever V and W are trigonometric polynomials. This is the content of the following result:

Theorem 1.3 (Duality for trigonometric polynomials). *For any real trigonometric polynomials V and W of degrees d and d' , respectively, and an irrational frequency ω , let $\gamma^h(\alpha)$ and $\gamma^{h'}(\alpha)$ be the entropy of the operator (1.13), $h = h_{V,W,\theta_0}$, and its Aubry dual (1.14), $h' = h_{W,V,-,\theta_1}$, for an energy $\alpha \in \mathbb{C}$. Then the entropies (sum of the positive Lyapunov exponents) are related by the formula*

$$\gamma^h(\alpha) + \log |V_d| = \gamma^{h'}(\alpha) + \log |W_{d'}|. \tag{1.18}$$

Remark 1.4. As a consequence of the dihedrality of section 1.5.1, the same relation holds for any of the operators in (1.16) and (1.17).

Since the entropy is always positive (it is the sum of the positive Lyapunov exponents of a skew-product), the above result yields a lower bound on the sum of the positive Lyapunov exponents of the skew-products, which is finer than the one that would be obtained by applying Herman’s subharmonicity trick [Her83] to the d th iteration of the linear skew-products (see section 2.1 for more details):

Corollary 1.5 (Lower bounds for the entropy). *For any real trigonometric polynomials V and W of degrees d and d' respectively and an irrational frequency ω , let $\gamma^h(\alpha)$ the entropy (i.e. the sum of positive Lyapunov exponents) of the operator (1.13), $h = h_{V,W,\theta_0}$ for an energy $\alpha \in \mathbb{C}$. Then the lower bound*

$$\gamma^h(\alpha) \geq \log \left(\frac{|W_{d'}|}{|V_d|} \right) \tag{1.19}$$

holds for any $\alpha \in \mathbb{C}$. In particular, one has the bound on the upper Lyapunov exponent:

$$\gamma_1^h(\alpha) \geq \frac{1}{d} \log \left(\frac{|W_{d'}|}{|V_d|} \right).$$

There are several situations when theorem 1.3 can be refined. The first one is for our motivating example of Schrödinger operators where $V(\theta) = 2 \cos 2\pi\theta$. Indeed, for operators of the form (1.1) and their duals (1.2), the entropy of the former is simply the upper Lyapunov exponent and so that one has:

Corollary 1.6 (Duality for Schrödinger operators). *For any real trigonometric polynomial W of degree d' and an irrational frequency ω , the Lyapunov exponent of the Schrödinger operator (1.1), $\gamma^h(\alpha)$, satisfies the equality*

$$\gamma^h(\alpha) = \gamma^{h'}(\alpha) + \log |W_{d'}|, \quad \alpha \in \mathbb{C}$$

where $\gamma^{h'}$ is the entropy of the dual operator (1.2).

The second interesting situation is what we will call *self-dual* models, among which the Almost Mathieu is the most prominent example (see section 1.6.1 for a numerical illustration), which correspond to trigonometric functions V and W related by a coupling constant $W = \beta V$:

Corollary 1.7 (Self-dual models). *Let V be a trigonometric polynomial of degree d and an irrational frequency ω . For any nonzero coupling constant $\beta \in \mathbb{R}$, let $h = h_{V, \beta V, \theta_0}$ be a self-dual operator and $\gamma^h(\alpha, \beta)$ its entropy. Then the following formula holds*

$$\gamma^h(\alpha, \beta) = \gamma^h\left(\frac{\alpha}{\beta}, \frac{1}{\beta}\right) + \log |\beta|$$

for any $\alpha, \beta \neq 0$. In particular $\gamma^h(\alpha, \beta) \geq \log |\beta|$.

Proof. It suffices to note that $W = \beta V$ and

$$\gamma^{h'}(\alpha, \beta) = \gamma^h\left(\frac{\alpha}{\beta}, \frac{1}{\beta}\right)$$

as a consequence of the eigenvalue equation. □

Note that the only example of a self-dual Schrödinger operator is the Almost Mathieu operator (where the entropy equals the upper Lyapunov exponent). In this case, the above corollary implies that the upper Lyapunov exponent of

$$(h_{2 \cos 2\pi \cdot, 2\beta \cos 2\pi \cdot, \theta_0} x)_n = x_{n+1} + x_{n-1} + 2\beta \cos(2\pi\theta_0 + \omega n) x_n = \alpha x_n, \quad n \in \mathbb{Z}$$

equals that of

$$(h_{\frac{\beta}{2} \cos 2\pi \cdot, \cos 2\pi \cdot, \theta_0} x)_n = \beta (x_{n+1} + x_{n-1}) + 2 \cos(2\pi\theta_0 + \omega n) x_n = \alpha x_n, \quad n \in \mathbb{Z}$$

plus the term $\log |\beta|$. This leads to the well-known formula for the upper Lyapunov exponent, taking $b = 2\beta$ as the new coupling parameter.

To end this section, we state a result for Schrödinger operators with analytic potentials. Note that when W is a real-analytic potential with infinitely many harmonics, the dual does not give rise to a finite-dimensional skew-product. Nevertheless, one can use the continuity in the potential for the upper Lyapunov exponent for quasi-periodic Schrödinger operators with one frequency (as is the case), which was proved by Bourgain and Jitomirskaya [BJ02b], to obtain the following approximation result (see section 1.6.3 for a numerical illustration):

Corollary 1.8 (An approximation result for Schrödinger operators). *Let $W : \mathbb{T} \rightarrow \mathbb{R}$ a real-analytic function and W_k a sequence of trigonometric polynomials of degree d_k , with $d_k \rightarrow \infty$, converging to W in some complex strip of \mathbb{T} . Then the upper Lyapunov exponent of the Schrödinger operator (1.1), $\gamma^h(\alpha)$, is the limit*

$$\gamma^h(\alpha) = \lim_{k \rightarrow \infty} \bar{\gamma}^{h'_k}(\alpha),$$

where $\bar{\gamma}^{h'_k}$ is the normalized entropy of the dual operator (1.2) with symbol W_k .

1.6. Numerical illustrations

In order to illustrate the results on the duality (and dihedrality) of Lyapunov exponents, we selected few models in which we computed numerically the Lyapunov exponents and the entropy in the same way that we presented the main example in the introduction. In all the examples we approximated the Lyapunov exponents computing the products (1.10) for a certain number of iterates, after a transient, and computing their eigenvalues. See [BS98, PS11b, PS11a] for more detailed numerical computations on similar quasi-periodic Schrödinger operators.

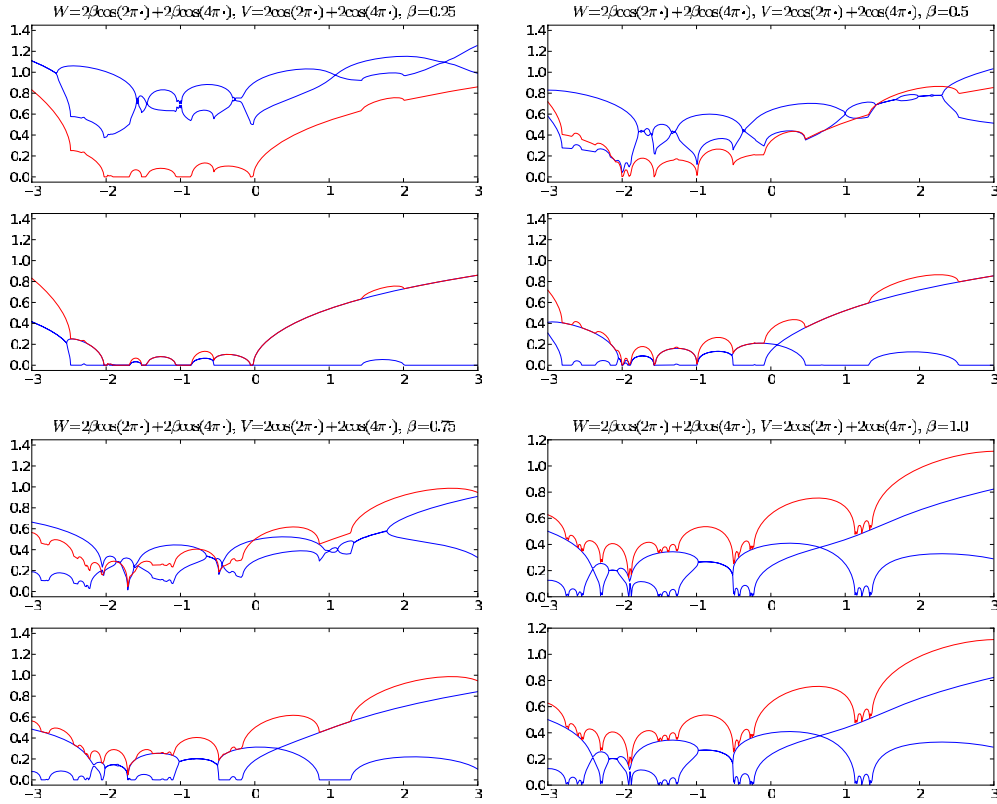


Figure 3. Numerical computation of the Lyapunov exponents of (1.21) and (1.20) as a function of α for selected values of β . In each cell the upper plot displays the Lyapunov exponents for the dual (1.21), $\gamma_1^{h'}$ and $\gamma_2^{h'}$ in blue, as well as the value of its normalized entropy (in red). The lower plot of each cell displays the same for γ_1^h and γ_2^h . We used 10^4 iterations of the skew-products with 10^3 values on the a -axis.

1.6.1. *A self-dual model.* To illustrate corollary 1.7, let us consider a self-dual model, probably the simplest one after the Almost Mathieu. This is given by

$$(hx)_n = x_{n+2} + x_{n+1} + x_{n-1} + x_{n-2} + \beta V(\theta_0 + \omega n)x_n, \quad n \in \mathbb{Z}, \tag{1.20}$$

when $V(\theta) = 2 \cos(2\pi\theta) + 2 \cos(4\pi\theta)$, $W(\theta) = \beta V(\theta)$, $\omega = (\sqrt{5} - 1)/2$ and β is a nonzero real coupling constant. Its Aubry dual is the operator

$$(h'x)_n = \beta(x_{n+2} + x_{n+1} + x_{n-1} + x_{n-2}) + V(\theta_0 + \omega n)x_n, \quad n \in \mathbb{Z}. \tag{1.21}$$

For both h and its dual h' , and for any $\alpha \in \mathbb{C}$, the entropy is always the sum of two Lyapunov exponents which satisfy

$$\gamma_1^h(\alpha, \beta) + \gamma_2^h(\alpha, \beta) = \gamma_1^{h'}(\alpha, \beta) + \gamma_2^{h'}(\alpha, \beta) + \log |\beta|.$$

A numerical illustration for selected values of β can be seen in figure 3. A very special case occurs when $\beta = 1$ and the two operators are the same, thus having equal Lyapunov exponents as well.

1.6.2. *Dihedrality of the Lyapunov exponents.* In order to illustrate the dihedrality of the normalized entropy of the eight operators (1.16) and (1.17), we selected a model with symbol

$$V(\theta) = 2 \cos(2\pi\theta) + 3 \sin(2\pi\theta) + \cos(4\pi\theta) + \sin(4\pi\theta)$$

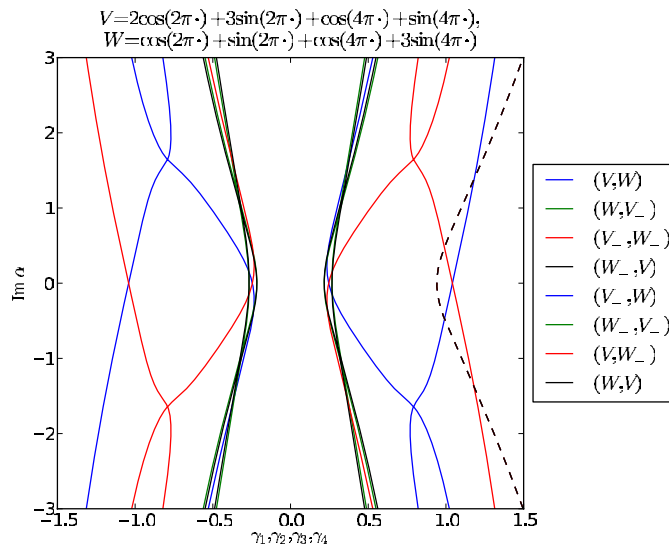


Figure 4. Lyapunov exponents for the operators (1.16) and (1.17) for non-even functions V and W (horizontal axis) and complex values of α (vertical axis) with real part equal to one. In continuous lines, we plot all the Lyapunov exponents, whereas the dashed line, which is the same for all eight combinations, is the normalized entropy. We used 10^4 iterations of the skew-products with 200 values of the imaginary part of a equally spaced in $[-3, 3]$.

and potential

$$W(\theta) = \cos(2\pi\theta) + \sin(2\pi\theta) + \cos(4\pi\theta) + 3\sin(4\pi\theta).$$

We computed the Lyapunov exponents for complex energies α in a segment with real part equal to one and imaginary part in $[-3, 3]$, see figure 4.

1.6.3. An analytic example. We conclude this section on numerical illustrations with an example of a Schrödinger operator (thus $V = 2\cos(2\pi\theta)$) with an analytic potential which is not a trigonometric polynomial. We took

$$W(\theta) = \sum_{j \geq 1} \frac{1}{10^j} \cos(2\pi j\theta) = \frac{10\cos 2\pi\theta - 1}{101 - 20\cos 2\pi\theta} \tag{1.22}$$

and the sequence of approximating trigonometric polynomials

$$W_k(\theta) = \sum_{j=1}^k \frac{1}{10^j} \cos(2\pi j\theta). \tag{1.23}$$

We approximated numerically the Lyapunov exponent of the Schrödinger operator h_{V,W,θ_0} for $\alpha = 1$ and the entropy of the dual operators h_{W_k,V,θ_0} for increasing values of k . Each of these operators defines $2k$ -dimensional skew-products which have k non-negative Lyapunov exponents. Results are summarized in figure 5, where a convergence to the original Lyapunov exponent is observed.

1.7. Organization of the paper

The rest of the paper is devoted to the proof of theorem 1.1, although an important part of this proof is the discussion of relationship between spectral properties of long-range operators and the dynamical properties of the associated skew-products, which is done in section 2. Finally, in section 3, the Thouless formula of theorem 1.1 is proven.

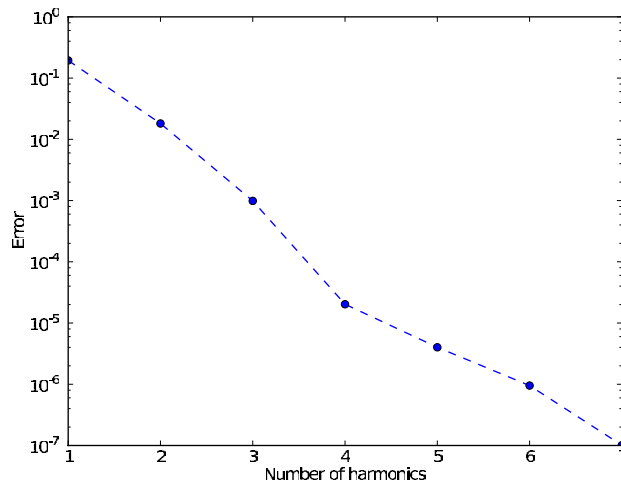


Figure 5. Plot of the differences between the normalized entropies of the dual model of the Schrödinger operator with analytic potential (1.22) for an increasing number of harmonics (1.23) and the original operator, whose value, with 10⁸ iterations, is 0.403 738 759 209. The results correspond to $\alpha = 1$.

2. Spectrum of generalized Schrödinger skew-products

This section contains the main tools which are used to prove theorem 1.1 and to understand the dynamics of the long-range skew-products. It is convenient to consider a more general class of operators, which is done in section 2.1 and the class defined in 2.2. Similarly to Schrödinger operators, a key idea is the combination of the spectral properties with the dynamics of the linear skew-products that their eigenvalue equations define. The dynamics of these skew-products, whose complex-symplectic structure is considered in section 2.3, is uniformly hyperbolic in the resolvent set, as is proven in theorem 2.8 of section 2.4.

2.1. From long-range operators to generalized Schrödinger operators

Given a long-range operator $h = h_{V,W,\theta_0}$, instead of writing the eigenvalue equation $hx = \alpha x$ in $\ell^2(\mathbb{Z}, \mathbb{C})$ for $\alpha \in \mathbb{C}$ as a $2d$ th order 1-dimensional difference equation (1.9), we can write it as a second-order d -dimensional difference equation by introducing the auxiliary variables

$$X_k = (x_{dk+d-1} \ \dots \ x_{dk+1} \ x_{dk})^\top$$

for $k \in \mathbb{Z}$. Hence, it is easy to check that $(X_k)_k$ satisfies

$$CX_{k+1} + B(\theta_{dk})X_k + C^*X_{k-1} = \alpha X_k \tag{2.1}$$

where

$$C = \begin{pmatrix} V_d & \dots & V_1 \\ 0 & \ddots & \vdots \\ 0 & 0 & V_d \end{pmatrix}, \tag{2.2}$$

C^* is its adjoint (the conjugate transpose) and $B(\theta)$ is the Hermitian matrix

$$B(\theta) = \begin{pmatrix} W(\theta_{d-1}) & V_{-1} & \dots & V_{-d+1} \\ V_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & W(\theta_1) & V_{-1} \\ V_{d-1} & \dots & V_1 & W(\theta_0) \end{pmatrix}, \tag{2.3}$$

where $\theta_j = \tau^j(\theta_0)$. Note that equation (2.1) is an eigenvalue equation of the following generalized Schrödinger operator

$$(H_{C,B,\theta_0}X)_k = CX_{k+1} + B(\vartheta_k)X_k + C^*X_{k-1},$$

acting on $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, over the orbit $\vartheta_k = \tau^{dk}(\vartheta_0)$ where $\vartheta_0 = \theta_0$.

To obtain a first-order system and the corresponding linear skew-product we use the fact that C is invertible (since $V_d \neq 0$ because the degree of V is exactly d) and write

$$\underbrace{\begin{pmatrix} X_{k+1} \\ X_k \end{pmatrix}}_{U_{k+1}} = \underbrace{\begin{pmatrix} C^{-1}(\alpha I - B(\vartheta_k)) & -C^{-1}C^* \\ I_d & O_d \end{pmatrix}}_{A_\alpha^H(\vartheta_k)} \underbrace{\begin{pmatrix} X_k \\ X_{k-1} \end{pmatrix}}_{U_k}$$

$$(\vartheta_{k+1}) = \tau^d(\vartheta_k),$$

where $\vartheta_0 = \theta_0$. Here, and in the following, I_d and O_d are the d -dimensional identity and zero matrices, respectively.

Note that the matrices of the skew-products associated with the eigenvalue equations (1.8) and (2.1) are related by

$$(A_\alpha^H, \tau^d) = (A_\alpha^h, \tau)^d$$

or, equivalently, $A_\alpha^H(\vartheta) = A_\alpha^h(d; \vartheta)$.

2.2. Spectrum of generalized Schrödinger operators and their transfer operators

In the following, we will generalize the setting of the previous section with the following ingredients:

- (HS1) A base dynamical system (Θ, ν, μ) given by a homeomorphism $\nu : \Theta \rightarrow \Theta$ on a compact metric space Θ , and preserving an ergodic measure μ that is topological (i.e. positive on open sets);
- (HS2) A dense base orbit $(\vartheta_k = \nu^k(\vartheta_0))_{k \in \mathbb{Z}} \subset \Theta$ with initial phase $\vartheta_0 \in \Theta$;
- (HS3) An invertible complex matrix $C \in M_d(\mathbb{C})$;
- (HS4) A continuous matrix valued map $B : \Theta \rightarrow M_d(\mathbb{C})$ such that for each $\vartheta \in \Theta$, $B(\vartheta)$ is Hermitian (i.e. $B(\vartheta)^* = B(\vartheta)$).

We consider the following (generalized) Schrödinger operator acting on $\ell^2(\mathbb{Z}, \mathbb{C}^d)$

$$(HX)_k = CX_{k+1} + B(\vartheta_k)X_k + C^*X_{k-1}, \tag{2.4}$$

where $X = (X_k)_k \in \ell^2(\mathbb{Z}, \mathbb{C}^d)$. In the setting of the previous section, ν , which is simply τ^d , and C and B , given by (2.2) and (2.3), respectively, satisfy the above hypotheses.

Hence, the associate eigenvalue problems are of the form

$$HX = \alpha X,$$

for $\alpha \in \mathbb{C}$. As in the previous section, the (generalized) Schrödinger skew-product arising from this eigenvalue equation is $(A_\alpha, \nu) : \mathbb{C}^{2d} \times \Theta \rightarrow \mathbb{C}^{2d} \times \Theta$ where

$$A_\alpha(\vartheta) = \begin{pmatrix} C^{-1}(\alpha I - B(\vartheta)) & -C^{-1}C^* \\ I_d & O_d \end{pmatrix}.$$

As it happens with many families of $SL(2, \mathbb{R})$ linear skew-products [Joh86], it turns out that the spectrum of H is linked to the uniform hyperbolicity of the linear skew-products (A_α, ν) .

Let us recall that a linear skew-product (A, ν) in $\mathbb{C}^m \times \Theta$ is uniformly hyperbolic, or it has an exponential dichotomy, if and only if there exist positive constants $\rho < 1$ and $C > 0$ and an invariant

Whitney splitting $\mathbb{C}^m \times \Theta = E^s \oplus E^u$ characterized by the following rates of growth:

- $v^s \in E^s(\vartheta) \Leftrightarrow \|A(k; \vartheta)v^s\| \leq C\rho^k \|v^s\|$ for all $k \geq 0$;
- $v^u \in E^u(\vartheta) \Leftrightarrow \|A(k; \vartheta)v^u\| \leq C\rho^{-k} \|v^u\|$ for all and $k \leq 0$.

Here, and in the following, by $E^s(\vartheta)$ and $E^u(\vartheta)$ we mean the fibres of the continuous bundles E^s and E^u , respectively, at the point ϑ .

The key result that links the spectral theory of Schrödinger operators and the dynamics of the corresponding linear skew-products is the following.

Theorem 2.1. *Let H be a generalized Schrödinger operator of the form (2.4) satisfying (HS1-4) above. Then, the spectrum of H acting in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ is the set $\alpha \in \mathbb{C}$ for which (A_α, ν) is not uniformly hyperbolic:*

$$\text{Spec}(H, \ell^2(\mathbb{Z}, \mathbb{C}^d)) = \{\alpha \in \mathbb{C} \mid (A_\alpha, \nu) \text{ is not uniformly hyperbolic}\}.$$

In particular, the spectrum of H does not depend on the dense base orbit chosen.

Proof. In the present proof, we use several results relating dynamical properties of linear skew-products and spectral properties of associated transfer operators, a point of view that was initiated in [Mat68], and that has been fruitfully explored by many authors, i.e. [CL99, HPS77, dIL93, Mañ78, Swa81]. Some results in [HdlL07] are also used.

The linear skew-product (A_α, ν) defines a transfer operator acting on sections $U : \Theta \rightarrow \mathbb{C}^{2d}$ of the trivial bundle $\mathbb{C}^{2d} \times \Theta$, by

$$A_\alpha U(\vartheta) = A_\alpha(\nu^{-1}(\vartheta)) U(\nu^{-1}(\vartheta)). \tag{2.5}$$

Note that, in principle, the spectrum of the operator depends on the space of sections it is acting on. However, it is well known that the spectra of the transfer operator acting on the space of bounded sections, $B(\Theta, \mathbb{C}^{2d})$, and on the space of continuous sections, $C^0(\Theta, \mathbb{C}^{2d})$, are the same:

$$\text{Spec}((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})) = \text{Spec}((A_\alpha, \nu), C^0(\Theta, \mathbb{C}^{2d})).$$

An ingredient of the proof is that the spectral subbundles produced by the spectral gaps in $\text{Spec}((A_\alpha, \mu), B(\Theta, \mathbb{C}^{2d}))$ are continuous. Moreover, one has the following characterization of uniform hyperbolicity (see also [HPS77]):

$$(A_\alpha, \nu) \text{ is uniformly hyperbolic} \Leftrightarrow (A_\alpha, \nu) \text{ is a hyperbolic operator in } B(\Theta, \mathbb{C}^{2d}).$$

In general, a bounded linear operator is hyperbolic if its spectrum does not intersect the unit circle of the complex plane. Using again Mather’s approach [Mat68], and the fact that there is a dense orbit, one can see that the Weyl spectrum (also known as approximate point spectrum) is rotationally invariant, and hence the full spectrum is rotationally invariant. That is, $\text{Spec}((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d}))$ consists of finitely many annuli of the complex plane centred at 0, and, hence,

$$(A_\alpha, \nu) \text{ is not uniformly hyperbolic} \Leftrightarrow 1 \in \text{Spec}((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})).$$

A crucial fact is that, since the base dynamics is regionally recurrent (and hence chain recurrent), then the whole spectrum is Weyl spectrum:

$$\text{Spec}((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})) = \text{Spec}_W((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})).$$

Here, one uses fundamental results of Sacker and Sell [SS76], and Selgrade [Sel75], see [Swa81, HdlL07].

One can also consider (A_α, ν) as a transfer operator acting on sequences of vectors $U = (U_k)_{k \in \mathbb{Z}}$ supported on the dense orbit $(\vartheta_k = \nu^k(\vartheta_0))_k$, by

$$(A_\alpha U)_k = A_\alpha(\vartheta_{k-1}) U_{k-1}.$$

We consider its action on the space of bounded sequences, $b(\mathbb{Z}, \mathbb{C}^{2d})$, and on the space of ℓ^2 sequences, $\ell^2(\mathbb{Z}, \mathbb{C}^d)$. Using localization arguments, in [HdlL07] it is proved that the corresponding Weyl spectra are equal:

$$\text{Spec}_W((A_\alpha, \nu), b(\mathbb{Z}, \mathbb{C}^{2d})) = \text{Spec}_W((A_\alpha, \nu), \ell^2(\mathbb{Z}, \mathbb{C}^{2d})).$$

Moreover, since the orbit $(\vartheta_k = v^k(\vartheta_0))_k$ is dense, the spectra of the transfer operator acting on bounded sections and on bounded sequences are the same, and the spectra are in fact Weyl spectra:

$$\begin{aligned} \text{Spec}((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})) &= \text{Spec}((A_\alpha, \nu), b(\mathbb{Z}, \mathbb{C}^{2d})), \\ &\parallel \\ \text{Spec}_W((A_\alpha, \nu), B(\Theta, \mathbb{C}^{2d})) &= \text{Spec}_W((A_\alpha, \nu), b(\mathbb{Z}, \mathbb{C}^{2d})). \end{aligned}$$

In order to complete the proof, first note that

$$\alpha \in \text{Spec}_W(H_{\vartheta_0}, \ell^2(\mathbb{Z}, \mathbb{C}^d)) \Leftrightarrow 1 \in \text{Spec}_W(A_\alpha, \ell^2(\mathbb{Z}, \mathbb{C}^{2d})) = \text{Spec}(A_\alpha, B(\Theta, \mathbb{C}^{2d})).$$

Then, since H_{ϑ_0} is self-adjoint in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, Weyl criterium applies and

$$\text{Spec}(H_{\vartheta_0}, \ell^2(\mathbb{Z}, \mathbb{C}^d)) = \text{Spec}_W(H_{\vartheta_0}, \ell^2(\mathbb{Z}, \mathbb{C}^d)).$$

With this spectral identity we are done with the proof of theorem 2.1. □

2.3. Geometry of Schrödinger skew-products

An important ingredient for our results is the complex-symplectic structure

$$\Omega = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}, \tag{2.6}$$

which satisfies $\Omega^* = -\Omega$, and the fact that the our Schrödinger skew-products (A_α, ν) , are complex symplectic for real α , as we will now see. Following lemma, which is a straightforward computation, summarizes the main complex-symplectic geometrical properties of our Schrödinger skew-products.

Lemma 2.2. $A_\alpha(\vartheta) = A_0(\vartheta) + \alpha\Gamma$ where:

(SP1) $A_0(\vartheta)$ is complex symplectic with respect to Ω : $A_0(\vartheta)^*\Omega A_0(\vartheta) = \Omega$;

(SP2) Γ is constant and zero-symplectic: $\Gamma^*\Omega\Gamma = 0$;

(SP3) $G = A_0(\vartheta)^*\Omega\Gamma$ is constant, Hermitian and positive-semidefinite;

with

$$A_0(\vartheta) = \begin{pmatrix} -C^{-1}B(\vartheta_k) & -C^{-1}C^* \\ I_d & O_d \end{pmatrix}, \quad \Gamma = \begin{pmatrix} C^{-1} & O_d \\ O_d & O_d \end{pmatrix}, \quad G = \begin{pmatrix} I_d & O_d \\ O_d & O_d \end{pmatrix}.$$

An immediate consequence is the following:

Lemma 2.3. For $\alpha_1, \alpha_2 \in \mathbb{C}$,

$$A_{\alpha_1}^*\Omega A_{\alpha_2} = \Omega + (\alpha_2 - \bar{\alpha}_1)G.$$

In particular, for $\alpha \in \mathbb{C}$, $A_\alpha^*\Omega A_\alpha = \Omega + 2 \text{Im } \alpha G$ i. Hence, if $\alpha \in \mathbb{R}$, then A_α is complex symplectic.

2.4. Dynamics of Schrödinger skew-products

In view of theorem 2.1, the Schrödinger skew-product (A_α, ν) is uniformly hyperbolic if α is non-real or sufficiently big, since the spectrum of the Schrödinger operator H_{ϑ_0} is a compact set in the real line. In this section, we review this statement, proving at the same time that the stable and unstable bundles of (A_α, ν) are trivial, and constructing the so-called M-matrices (see [Joh87]) to parametrize them.

The following Atkinson’s condition is a key property for a direct proof of uniform hyperbolicity of the Schrödinger skew-products (A_α, ν) for $\alpha \in \mathbb{C} \setminus \mathbb{R}$, which we will see later in lemma 2.5.

Lemma 2.4. The Schrödinger skew-products (A_α, ν) satisfy:

(SP4) for any orbit $(U_k, \vartheta_k)_{k \in \mathbb{Z}}$ of (A_α, ν) ,

$$\sum_{k \in \mathbb{Z}} U_k^* U_k \leq c \sum_{k \in \mathbb{Z}} U_k^* G U_k,$$

with $c = 2$.

Lemma 2.5. For any $\alpha \in \mathbb{C} \setminus \mathbb{R}$, the Schrödinger skew-product (A_α, ν) is uniformly hyperbolic, and its stable and unstable bundles are d -dimensional and trivial. That is, the bundles E_α^s, E_α^u are parametrized by global frames $V_\alpha^s, V_\alpha^u : \Theta \rightarrow M_{2d \times d}(\mathbb{C})$ of the form

$$V_\alpha^s(\vartheta) = \begin{pmatrix} M_\alpha^s(\vartheta) \\ I_d \end{pmatrix}, \quad V_\alpha^u(\vartheta) = \begin{pmatrix} I_d \\ M_\alpha^u(\vartheta) \end{pmatrix},$$

where $M_\alpha^s, M_\alpha^u : \Theta \rightarrow M_d(\mathbb{C})$ are the so-called M -matrices, in such a way that

$$E_\alpha^s = \{(V_\alpha^s(\theta)v^s, \vartheta) \mid v^s \in \mathbb{C}^d, \vartheta \in \Theta\}, \quad E_\alpha^u = \{(V_\alpha^u(\theta)v^u, \vartheta) \mid v^u \in \mathbb{C}^d, \vartheta \in \Theta\}.$$

Moreover, $M_\alpha^s(\vartheta)$ and $M_\alpha^u(\vartheta)$ are invertible matrices for any $\vartheta \in \Theta$.

Proof. Since base dynamics is chain recurrent, in order to prove the uniform hyperbolicity of the Schrödinger skew-product (A_α, ν) it suffices to prove that the only bounded orbits are the zero orbits [SS76, Sel75]. Hence, let $(U_k, \vartheta_k)_{k \in \mathbb{Z}}$ be a bounded orbit. Using (SP1–SP3) above, for any couple of indices $m < n$,

$$U_n^* \Omega U_n - U_m^* \Omega U_m = 2 \operatorname{Im} \alpha \sum_{k=m}^{n-1} U_k^* G U_k \text{ i.} \tag{2.7}$$

The left-hand side of (2.7) is uniformly bounded in both m, n and, since $\operatorname{Im} \alpha \neq 0$,

$$\infty > c \sum_{k=-\infty}^{\infty} U_k^* G U_k \geq \sum_{k=-\infty}^{\infty} U_k^* U_k,$$

where in the second inequality we apply (SP4). Since the last infinite sum of non-negative numbers is convergent, then $\lim_{k \rightarrow \pm\infty} U_k^* U_k = 0$, and $\lim_{k \rightarrow \pm\infty} U_k = 0$. Taking limits $m \rightarrow -\infty$ and $n \rightarrow +\infty$ in (2.7), we reach

$$0 = c \sum_{k=-\infty}^{\infty} U_k^* G U_k \geq \sum_{k=-\infty}^{\infty} U_k^* U_k,$$

and hence $U_k = 0$ for all $k \in \mathbb{Z}$. That is, the only bounded orbit is the trivial one, and the Schrödinger skew-product (A_α, ν) is uniformly hyperbolic.

In order to prove that the stable bundle E_α^s can be parametrized by a global frame we proceed as follows. First, for any $\vartheta \in \Theta$ and $U_0 = (X_0, X_{-1}) \in E^s(\vartheta) \setminus \{0\}$, consider the corresponding orbit $(U_k, \vartheta_k)_{k \in \mathbb{Z}}$. Then, taking $m = 0$ and $\lim_{n \rightarrow \infty}$ in (2.7), we obtain

$$-U_0^* \Omega U_0 = 2 \operatorname{Im} \alpha \sum_{k=0}^{\infty} U_k^* G U_k \text{ i.}$$

and, using the definitions of Ω and G ,

$$\operatorname{Im} (X_0^* C^* X_{-1}) = \sum_{k=0}^{\infty} X_k^* X_k \operatorname{Im} \alpha.$$

Hence, $X_0 \neq 0 \neq X_{-1}$, because otherwise $\sum_{k=0}^{\infty} X_k^* X_k = 0$, that implies $U_j = 0$ for all $j \geq 1$ and, hence, $U_j = 0$ for all $j \in \mathbb{Z}$, in contradiction with the fact that $U_0 \neq 0$. As a consequence, for any $\vartheta \in \Theta$ and for any $X_{-1} \in \mathbb{C}^d$, there exists a unique $X_0 \in \mathbb{C}^d$ such that $U_0 = (X_0, X_{-1}) \in E_\alpha^s(\vartheta)$. This defines a ϑ -depending invertible linear mapping $M_\alpha^s(\vartheta) : \mathbb{C}^d \rightarrow \mathbb{C}^d$ such that $U_0 = (M^s(\vartheta)X_{-1}, X_{-1}) \in E_\alpha^s(\vartheta)$. The dependence of ϑ is continuous because the bundle E^s is continuous.

Similar arguments lead to finding a parametrization of the unstable bundle E_α^u , as stated in the lemma. □

In the following lemma, we consider the uniform hyperbolicity of (A_α, ν) when the modulus of α is sufficiently large.

Lemma 2.6. *There exists $\rho > 0$ such that, for any $\alpha \in \mathbb{C}$ with $|\alpha| > \rho$, the Schrödinger skew-product (A_α, ν) is uniformly hyperbolic, and its stable and unstable bundles are d -dimensional and trivial. That is, the bundles E_α^s, E_α^u can be parametrized by global frames $V_\alpha^s, V_\alpha^u : \Theta \rightarrow M_{2d \times d}(\mathbb{C})$ of the form*

$$V_\alpha^s(\vartheta) = \begin{pmatrix} M_\alpha^s(\vartheta) \\ I_d \end{pmatrix}, \quad V_\alpha^u(\vartheta) = \begin{pmatrix} I_d \\ M_\alpha^u(\vartheta) \end{pmatrix},$$

where $M_\alpha^s, M_\alpha^u : \Theta \rightarrow M_d(\mathbb{C})$ are the so-called M -matrices, in such a way that

$$E_\alpha^s = \{(V_\alpha^s(\vartheta)v^s, \vartheta) \mid v^s \in \mathbb{C}^d, \vartheta \in \Theta\}, \quad E_\alpha^u = \{(V_\alpha^u(\vartheta)v^u, \vartheta) \mid v^u \in \mathbb{C}^d, \vartheta \in \Theta\}.$$

Moreover, $M_\alpha^s(\vartheta)$ and $M_\alpha^u(\vartheta)$ are invertible matrices for any $\vartheta \in \Theta$.

Proof. We write the invariance equations of the stable and unstable bundles as

$$A_\alpha(\vartheta)V^s(\vartheta) = V^s(\nu(\vartheta))\Lambda^s(\vartheta), \tag{2.8}$$

$$A_\alpha(\vartheta)V^u(\vartheta) = V^u(\nu(\vartheta))\Lambda^u(\vartheta), \tag{2.9}$$

where the unknowns are the M -matrices $M^s, M^u : \Theta \rightarrow M_{d \times d}(\mathbb{C})$, and the corresponding dynamics given by the Λ -matrices $\Lambda^s, \Lambda^u : \Theta \rightarrow M_{d \times d}(\mathbb{C})$.

Let us start by analysing (2.8). This is equivalent to solving the Riccati equation

$$C^{-1}(\alpha - B(\vartheta))M^s(\vartheta) - C^{-1}C^* = M^s(\nu(\vartheta))M^s(\vartheta), \tag{2.10}$$

and taking $\Lambda^s(\vartheta) = M^s(\vartheta)$ or, equivalently, the fixed point equation

$$M^s(\vartheta) = \frac{1}{\alpha}(C^* + B(\vartheta))M^s(\vartheta) + CM^s(\nu(\vartheta))M^s(\vartheta). \tag{2.11}$$

Note that a solution $M^s(\vartheta)$ of the Riccati equation is invertible for all $\vartheta \in \Theta$, since C is invertible. We consider the operator $\Gamma^s : C^0(\Theta, M_{d \times d}(\mathbb{C})) \rightarrow C^0(\Theta, M_{d \times d}(\mathbb{C}))$ defined by the right-hand side of (2.11) in the Banach space $C^0(\Theta, M_{d \times d}(\mathbb{C}))$ endowed with the sup-norm $\|\cdot\|$. Let us define $a_s = |C^*|, b_s = \|B\|, c_s = |C|$. Then, for any closed ball of radius R in $C^0(\Theta, M_{d \times d}(\mathbb{C}))$, and for any $M_1^s, M_2^s \in \bar{B}(0, R)$, then

$$\|\Gamma^s M_1^s\| \leq \frac{1}{|\alpha|}(a_s + b_s R + c_s R^2)$$

and

$$\|\Gamma^s M_2^s - \Gamma^s M_1^s\| \leq \frac{1}{|\alpha|}(b_s + 2c_s R)\|M_2^s - M_1^s\|.$$

Hence, for $R_s^2 \geq \frac{a_s}{c_s} = \frac{|C^*|}{|C|}$ and $|\alpha| > b_s + 2c_s R_s$, the operator Γ^s is contracting in $\bar{B}(0, R_s)$, with contraction rate $K_s = \frac{1}{|\alpha|}(b_s + 2c_s R_s)$. Hence, there is a unique $M_\alpha^s \in \bar{B}(0, R_s)$ solving (2.11), and $M_\alpha^s, \Lambda^s = M^s$ solve (2.8). Moreover, taking $M_0^s(\vartheta) = 0$, then

$$\|M_\alpha^s\| \leq \frac{1}{1 - K_s}\|\Gamma^s M_0^s - M_0^s\| = \frac{1}{1 - K_s} \frac{1}{|\alpha|} a_s.$$

As a result, if $|\alpha| > a_s + b_s + 2c_s R_s$, then $\|\Lambda_\alpha^s\| = \|M_\alpha^s\| < 1$ and the forward dynamics on the bundle parametrized by V_α^s is uniformly contracting.

The analysis of (2.9) is analogous. The Riccati equation is

$$M^u(\nu(\vartheta))C^{-1}(\alpha - B(\vartheta)) - M^u(\nu(\vartheta))C^{-1}C^*M^u(\vartheta) = I, \tag{2.12}$$

and, then $\Lambda^u(\vartheta) = C^{-1}(\alpha - B(\vartheta)) - C^{-1}C^*M^u(\vartheta)$. Note that, *a posteriori*, both $\Lambda^u(\vartheta)$ and $M^u(\nu(\vartheta))$ are invertible, and $\Lambda^u(\vartheta) = (M^u(\nu(\vartheta)))^{-1}$. The Riccati equation is equivalent to the fixed point equation

$$M^u(\vartheta) = \frac{1}{\alpha}(C + M^u(\vartheta)C^{-1}B(\nu^{-1}(\vartheta))C + M^u(\vartheta)C^{-1}C^*M^u(\nu^{-1}(\vartheta))C). \tag{2.13}$$

We consider the operator $\Gamma^u : C^0(\Theta, M_{d \times d}(\mathbb{C})) \rightarrow C^0(\Theta, M_{d \times d}(\mathbb{C}))$ defined by the right-hand side of (2.13). Let us define $a_u = |C|, b_u = \|C^{-1}BC\|, c_u = |C^{-1}C^*| |C|$. Hence, for $R_u^2 \geq \frac{a_u}{c_u} = \frac{1}{|C^{-1}C^*|}$ and $|\alpha| > b_u + 2c_u R_u$, the operator Γ^u is contracting in $\bar{B}(0, R_u)$, and there is a unique $M_\alpha^u \in \bar{B}(0, R_u)$ solving (2.13), and $M_\alpha^u, \Lambda_\alpha^u = (M_\alpha^u \circ \nu)^{-1}$ solve (2.9). If, moreover, $|\alpha| > a_u + b_u + 2c_u R_u$, then

$\|(\Lambda_\alpha^u)^{-1}\| = \|M_\alpha^u\| < 1$, and the backward dynamics on the bundle parametrized by V_α^u is uniformly contracting.

In summary, if $|\alpha|$ is sufficiently large, the skew-product (A_α, ν) is hyperbolic, and the stable and unstable bundles are d -dimensional. \square

Remark 2.7. Since the linear skew-product (A_α, ν) is hyperbolic for any α in the resolvent set $\mathbb{C} \setminus \text{Spec}(H, \ell^2(\mathbb{Z}, \mathbb{C}^d))$, the corresponding stable and unstable bundles are d -dimensional, since they depend analytically on α in the resolvent set. In fact, for $\alpha \in \mathbb{R} \setminus \text{Spec}(H, \ell^2(\mathbb{Z}, \mathbb{C}^d))$ the stable and unstable bundles are Lagrangian (and hence d -dimensional), since the linear skew-product is complex symplectic.

Lemmas 2.5 and 2.6 are statements about the existence of a *conjugacy* (P_α, id) between the Schrödinger skew-product (A_α, ν) and a block-diagonal skew-product (Λ_α, ν) , for $\alpha \in \mathbb{C} \setminus \mathbb{R}$ or sufficiently large $|\alpha|$, respectively. In the following theorem, we restate these results, for future reference.

Theorem 2.8. *There exists $\rho > 0$ such that, for any $\alpha \in \mathbb{C} \setminus [-\rho, \rho]$, there exist an invertible linear skew-product (P_α, id) , of the form*

$$P_\alpha(\vartheta) = \begin{pmatrix} M_\alpha^s(\vartheta) & I \\ I & M_\alpha^u(\vartheta) \end{pmatrix}$$

and a block-diagonal skew-product $(\Lambda_\alpha, \text{id})$, of the form

$$\Lambda_\alpha(\vartheta) = \begin{pmatrix} \Lambda_\alpha^s(\vartheta) & 0 \\ 0 & \Lambda_\alpha^u(\vartheta) \end{pmatrix}$$

such that:

(1) (P_α, id) conjugates (A_α, ν) with $(\Lambda_\alpha, \text{id})$, that is:

$$A_\alpha(\vartheta)P_\alpha(\vartheta) = P_\alpha(\nu(\vartheta))\Lambda_\alpha(\vartheta); \tag{2.14}$$

(2) the stable and unstable bundles of the uniformly hyperbolic linear skew-product (A_α, ν) are

$$E_\alpha^s = \{(V_\alpha^s(\vartheta)v^s, \vartheta) \mid v^s \in \mathbb{C}^d, \vartheta \in \Theta\},$$

$$E_\alpha^u = \{(V_\alpha^u(\vartheta)v^u, \vartheta) \mid v^u \in \mathbb{C}^d, \vartheta \in \Theta\},$$

where

$$V_\alpha^s(\vartheta) = \begin{pmatrix} M_\alpha^s(\vartheta) \\ I_d \end{pmatrix}, \quad V_\alpha^u(\vartheta) = \begin{pmatrix} I_d \\ M_\alpha^u(\vartheta) \end{pmatrix};$$

(3) For any $\vartheta \in \Theta$, the $d \times d$ matrices $M_\alpha^s(\vartheta)$, $M_\alpha^u(\vartheta)$, $\Lambda_\alpha^s(\vartheta)$, $\Lambda_\alpha^u(\vartheta)$ are invertible, and $\Lambda_\alpha^s(\vartheta) = M_\alpha^s(\vartheta)$, $\Lambda_\alpha^u(\vartheta) = (M_\alpha^u(\nu(\vartheta)))^{-1}$;

(4) For any $\vartheta \in \Theta$, the $d \times d$ matrices $I - M_\alpha^s(\vartheta)M_\alpha^u(\vartheta)$ and $I - M_\alpha^u(\vartheta)M_\alpha^s(\vartheta)$ are invertible and

$$P_\alpha(\vartheta)^{-1} = \begin{pmatrix} -M_\alpha^u(\vartheta) & I \\ I & -M_\alpha^s(\vartheta) \end{pmatrix} \begin{pmatrix} I - M_\alpha^s(\vartheta)M_\alpha^u(\vartheta) & 0 \\ 0 & I - M_\alpha^u(\vartheta)M_\alpha^s(\vartheta) \end{pmatrix}^{-1} \tag{2.15}$$

Proof. Statements (1), (2) and (3) of the theorem follow from lemmas 2.5 and 2.6, for $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $|\alpha| > \rho$, respectively.

We have to prove now that matrices $I - M_\alpha^s(\vartheta)M_\alpha^u(\vartheta)$ and $I - M_\alpha^u(\vartheta)M_\alpha^s(\vartheta)$ are invertible, and then the formula for $P_\alpha(\vartheta)^{-1}$ follows immediately. Suppose that $I - M_\alpha^s(\vartheta)M_\alpha^u(\vartheta)$ is not invertible. Hence, there exists $v \in \mathbb{C}^d \setminus \{0\}$ such that $M_\alpha^s(\vartheta)M_\alpha^u(\vartheta)v = v$. Note that $(v, M_\alpha^u(\vartheta)v) \in E_\alpha^u(\vartheta) \setminus \{0\}$, and $w = M_\alpha^u(\vartheta)v \in \mathbb{C}^d \setminus \{0\}$. Now, $(M_\alpha^s(\vartheta)w, w) \in E_\alpha^s(\vartheta) \setminus \{0\}$. But, then, $(M_\alpha^s(\vartheta)w, w) = (v, M_\alpha^u(\vartheta)v) \in E_\alpha^u(\vartheta) \setminus \{0\}$, which is incompatible. The proof that $I - M_\alpha^u(\vartheta)M_\alpha^s(\vartheta)$ is invertible is similar. \square

As a consequence of the previous result we can derive, for $\alpha \in \mathbb{C} \setminus [-\rho, \rho]$, a formula for the entropy of the uniformly hyperbolic linear skew-product (A_α, ν) , given by the sum of the d positive Lyapunov exponents.

Corollary 2.9. For $\alpha \in \mathbb{C} \setminus [-\rho, \rho]$, and for μ -a.e. $\vartheta \in \Theta$,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log |\det A_{11}(k; \vartheta, \alpha)| = \gamma(\alpha),$$

where

$$A_\alpha(k; \vartheta) = \begin{pmatrix} A_{11}(k; \vartheta, \alpha) & A_{12}(k; \vartheta, \alpha) \\ A_{21}(k; \vartheta, \alpha) & A_{22}(k; \vartheta, \alpha) \end{pmatrix}$$

and

$$\gamma(\alpha) = \gamma_1(\alpha) + \dots + \gamma_d(\alpha)$$

is the sum of the d positive Lyapunov exponents of the uniformly hyperbolic linear skew-product (A_α, ν) .

Proof. By iterating the conjugacy identity $A_\alpha(\vartheta)P_\alpha(\vartheta) = P_\alpha(\nu(\vartheta))\Lambda_\alpha(\theta)$, we obtain

$$A_\alpha(k; \vartheta) = P_\alpha(\nu^k(\vartheta))\Lambda_\alpha(k; \vartheta)P_\alpha(\vartheta)^{-1},$$

from where

$$A_{11}(k; \vartheta, \alpha) = (I - \Lambda_\alpha^s(k+1, \vartheta)\Lambda_\alpha^u(-(k+1), \nu^k(\vartheta))) \Lambda_\alpha^u(k; \theta)(I - \Lambda_\alpha^s(1, \vartheta)\Lambda_\alpha^u(-1, \vartheta))^{-1}.$$

Hence, since both forward dynamics of (Λ_α^s, ν) and backward dynamics of (Λ_α^u, ν) are contracting,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log |\det A_{11}(k; \vartheta, \alpha)| = \lim_{k \rightarrow \infty} \frac{1}{k} \log |\det \Lambda_\alpha^u(k; \vartheta)|.$$

Oseledec’s theorem [Ose68, BP02] applies to the linear skew-product (Λ_α^u, ν) , so that its Lyapunov exponents, $\gamma_1(\alpha) \geq \dots \geq \gamma_d(\alpha)$, which are defined μ -almost everywhere and are positive, satisfy

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log |\det \Lambda_\alpha^u(k; \vartheta)| = \gamma_1(\alpha) + \dots + \gamma_d(\alpha).$$

The proof of the corollary is finished by noting that these Lyapunov exponents are in fact the first d Lyapunov exponents of (A_α, ν) . □

3. Proof of Thouless formula

To prove theorem 1.1 we will proceed as in [KS88] and [CS83b] but replacing the Schrödinger skew-product on a strip by the generalized Schrödinger skew-products. We must show that

$$\bar{\gamma}^h(\alpha) = \gamma^h(\alpha) + \log |V_d| = \int_{\mathbb{R}} \log |\alpha - a| d\kappa_h(a), \tag{3.1}$$

where the integration is in terms of the density of states for the long-range operator

$$(hx)_n = \sum_{k=-d}^d V_k x_{n+k} + W(\theta_n)x_n,$$

and $\gamma^h(\alpha) = \gamma_1^h(\alpha) + \dots + \gamma_d^h(\alpha)$ is the entropy of the long-range skew-product (A_α^h, τ) . Recall that the relation between the long-range skew-product (A_α^h, τ) and the generalized Schrödinger skew-product (A_α^H, τ^d) is given by the composition formula $(A_\alpha^H, \tau^d) = (A_\alpha^h, \tau)^d$. Hence, the corresponding entropies satisfy the identity

$$\gamma^H(\alpha) = d\gamma^h(\alpha) = d(\bar{\gamma}^h(\alpha) - \log |V_d|). \tag{3.2}$$

3.1. Subharmonicity of the entropy

A key point in the proof of a Thouless formula is that the two sides of (3.1) are subharmonic functions of α in \mathbb{C} (see [CS83a, CS83b, Joh87, KS88]). Indeed, the right-hand side is clearly a subharmonic function [CS83b] and, as we will see in lemma 3.1, one can use a classical argument to prove that the left-hand side is subharmonic as well. Since two subharmonic functions which agree on a full-measure set of \mathbb{C} (as, for example, $\mathbb{C} \setminus \mathbb{R}$), must agree on all the complex plane, it suffices to prove the formula (3.1) for non-real α .

Lemma 3.1. *The map $\alpha \in \mathbb{C} \mapsto \gamma^h(\alpha)$ is subharmonic.*

Proof. For any $\theta \in \mathbb{T}$ the map

$$\alpha \mapsto \wedge^p A_\alpha^h(n; \theta)$$

is analytic in \mathbb{C} so

$$\alpha \mapsto \gamma^{h,n}(\alpha) = \frac{1}{n} \int_{\Theta} \log \|\wedge^d A_\alpha^h(n; \theta)\| \, d\mu$$

is subharmonic for $n > 0$. Since, for any $\alpha \in \mathbb{C}$, the sequence $(\gamma^{h,n}(\alpha))_n$ is subadditive, then $\lim_{n \rightarrow \infty} \gamma^{h,n}(\alpha) = \inf_{n > 0} \gamma^{h,n}(\alpha)$. Moreover, the subsequence $(\gamma^{h,2^k}(\alpha))_k$ is decreasing. Since the pointwise limit of a decreasing family of subharmonic functions is subharmonic, then the map

$$\alpha \mapsto \gamma^h(\alpha) = \lim_{k \rightarrow +\infty} \gamma^{h,2^k}(\alpha)$$

is subharmonic. □

3.2. Spectral properties of the restriction of the long-range operator

A connection between the dynamics and the spectral problem for *finite* sequences is given by the following relationship. This will be useful later on, since we will pass to the limit.

Lemma 3.2. *Let (A_α^h, τ) be the long-range skew-product of the long-range operator $h_{\theta_0} = h_{V,W,\theta_0}$. Denote*

$$A_\alpha^h(n; \theta_0) = \begin{pmatrix} A_{11}^h(n; \theta_0, \alpha) & A_{12}^h(n; \theta_0, \alpha) \\ A_{21}^h(n; \theta_0, \alpha) & A_{22}^h(n; \theta_0, \alpha) \end{pmatrix}.$$

For $k \geq 1$:

$$\det A_{11}^h(kd; \theta_0, \alpha) = V_d^{-kd} \det \left(\alpha I_{kd} - h_{\theta_0}^{[0, kd[} \right),$$

where $h_{\theta_0}^{[0, n[}$ is the restriction of the operator h_{θ_0} to $[0, n[$ with zero boundary conditions.

Proof. From the definition of $A_\alpha^h(n; \theta_0)$

$$\begin{pmatrix} x_{n+d-1} \\ \vdots \\ x_n \\ x_{n-1} \\ \vdots \\ x_{n-d} \end{pmatrix} = A_\alpha^h(n; \theta_0) \begin{pmatrix} x_{d-1} \\ \vdots \\ x_0 \\ x_{-1} \\ \vdots \\ x_{-d} \end{pmatrix}.$$

In addition, the restriction of the operator $\alpha I - h_{\theta_0}$ to $[0, n[$ with zero boundary conditions, i.e.

$$x_{n+d-1} = \dots = x_n = 0 \quad \text{and} \quad x_{-1} = \dots = x_{-d} = 0,$$

is given by the n -dimensional self-adjoint matrix $\alpha I_n - h_{\theta_0}^{[0,n]}$ where

$$h_{\theta_0}^{[0,n]} = \begin{pmatrix} w_{n-1} & V_1 & V_2 & \dots & \\ V_{-1} & w_{n-2} & V_1 & \ddots & \\ V_{-2} & V_{-1} & \ddots & \ddots & V_2 \\ \vdots & \ddots & \ddots & & V_1 \\ & & V_{-2} & V_{-1} & w_0 \end{pmatrix},$$

where $w_j = W(\tau^j(\theta_0))$.

Therefore, a vector

$$x = (x_{n-1} \dots x_0)^\top \in \mathbb{C}^n$$

is a nonzero solution of the eigenvalue equation $h_{\theta_0}^{[0,n]}x = \alpha x$ if and only if the vector of the last d components, $x_{[0,d]} = (x_{d-1}, \dots, x_0)$, is nonzero and moreover

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{n-1} \\ \vdots \\ x_{n-d} \end{pmatrix} = A_\alpha^h(n; \theta) \begin{pmatrix} x_{d-1} \\ \vdots \\ x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

In terms of the block decomposition of A_α^h , this condition reads as

$$0 = A_{11}^h(n; \theta, \alpha)x_{[0,d]}.$$

This implies that the determinant of $A_{11}^h(n; \theta_0, \alpha)$ vanishes if and only if the determinant of $(\alpha I_n - h_{\theta_0}^{[0,n]})$ vanishes. As functions of α both are polynomials and the latter is monic of degree n and divides $\det A_{11}^h(n; \theta_0, \alpha)$. Indeed, since $h_{\theta_0}^{[0,n]}$ is self-adjoint, it diagonalizes and the geometric and algebraic multiplicities of its eigenvalues agree. Moreover, $\dim \ker(\alpha I_n - h_{\theta_0}^{[0,n]}) \leq \dim \ker A_{11}^h(n; \theta_0, \alpha)$. We will show now that the degree of $A_{11}^h(n; \theta_0, \alpha)$ is also n and therefore, the two polynomials differ in a multiplicative constant. We will prove that this multiplicative constant is V_d^{-n} , for n multiple of d (although this is true for any n).

The 11 block of the fundamental matrix of the linear skew-product (A_α^H, τ) , given by $A_\alpha^H(k; \theta_0) = A_\alpha^h(kd; \theta_0)$, satisfies the recurrence

$$CA_{11}^H(k+1; \theta_0, \alpha) + B_k(\theta_0)A_{11}^H(k; \theta_0, \alpha) + C^*A_{11}^H(k-1; \theta_0, \alpha) = \alpha A_{11}^H(k; \theta_0, \alpha),$$

with $A_{11}^H(0; \theta_0, \alpha) = I$ and $A_{11}^H(1; \theta_0, \alpha) = C^{-1}(\alpha - B_0(\theta_0))$. Here $B_k(\theta_0) = B(\tau^{dk}(\theta_0))$. By induction, $A_{11}^H(k; \theta_0, \alpha)$ is of the form

$$A_{11}^H(k; \theta_0, \alpha) = C^{-k}\alpha^k + \sum_{\ell=0}^{k-1} D_{k\ell}(\theta_0)\alpha^\ell,$$

where $D_{k,0}(\theta_0), \dots, D_{k,k-1}(\theta_0)$ are matrices that can be computed recursively (in particular $D_{1,0} = -C^{-1}B_0(\theta_0)$). Hence, $\det A_{11}^h(kd; \theta_0, \alpha) = \det A_{11}^H(k; \theta_0, \alpha)$ is a polynomial of degree kd in α , whose main coefficient is V_d^{-kd} . □

3.3. Proof of Thouless formula for $\text{Im } \alpha \neq 0$

With the preliminaries of the previous section, the proof is now as for Schrödinger operators on a strip. First of all we show the following:

Proposition 3.3. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left| \det \left(\alpha I_n - h_{\theta_0}^{[0,n]} \right) \right| = \int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h(a),$$

where κ_h is the density of states of the long-range operator h .

Proof. Let $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ the eigenvalues of $h_{\theta_0}^{[0,n]}$ (which are real) counted with multiplicities. That is

$$\det \left(\alpha I - h_{\theta_0}^{[0,n]} \right) = (\alpha - \lambda_1^{(n)}) \cdots (\alpha - \lambda_n^{(n)}).$$

Since $\text{Im } \alpha \neq 0$, we can write

$$\begin{aligned} \frac{1}{n} \log \left| \det \left(\alpha I - h_{\theta_0}^{[0,n]} \right) \right| &= \frac{1}{n} \sum_{k=1}^n \log \left| \alpha - \lambda_k^{(n)} \right| \\ &= \int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h^{[0,n]}(a), \end{aligned}$$

where

$$\kappa_h^{[0,n]} = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k^{(n)}}$$

is the spectral measure of $h_{\theta_0}^{[0,n]}$, which is a discrete uniform probability measure supported on the eigenvalues $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$. Since the density of states is the weak limit of the measures $\kappa_h^{[0,n]}$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h^{[0,n]}(a) = \int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h(a),$$

because $\log |\alpha - a|$ is continuous for $(\alpha, a) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{R}$ (note that the support of the measures is contained in \mathbb{R}). □

Corollary 3.4. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h(a) = \gamma^h(\alpha) + \log |V_d|,$$

where $\gamma^h(\alpha) = \gamma_1^h(\alpha) + \dots + \gamma_d^h(\alpha)$ is the sum of the d positive Lyapunov exponents of the long-range linear skew-product (A_α^h, τ) .

Proof. The result is a consequence of lemma 3.2 and corollary 2.9, since

$$\begin{aligned} \int_{\mathbb{R}} \log |\alpha - a| \, d\kappa_h(a) &= \lim_{k \rightarrow \infty} \frac{1}{kd} \log \left| \det \left(\alpha I_{kd} - h_{\theta_0}^{[0,kd]} \right) \right| \\ &= \lim_{k \rightarrow \infty} \frac{1}{kd} \log \left| V_d^{kd} \det A_{11}^h(kd; \theta_0, \alpha) \right| \\ &= \lim_{k \rightarrow \infty} \frac{1}{kd} \log \left| \det A_{11}^h(kd; \theta, \alpha) \right| + \log |V_d| \\ &= \frac{1}{d} \lim_{k \rightarrow \infty} \frac{1}{k} \log \left| \det A_{11}^H(k; \theta, \alpha) \right| + \log |V_d| \\ &= \frac{1}{d} \gamma^H(\alpha) + \log |V_d| \\ &= \gamma^h(\alpha) + \log |V_d|. \end{aligned} \quad \square$$

With the proof of this corollary we end the proof of theorem 1.1.

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