The characteristic ideal of a finite, connected, regular graph *

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Abstract

Let $\Phi(x,y) \in \mathbb{C}[x,y]$ be a symmetric polynomial of partial degree d. The graph $G(\Phi)$ is defined by taking $\mathbb{C}$ as set of vertices and the points of $\mathbb{V}(\Phi(x,y))$ as edges. We study the following problem: given a finite, connected, d-regular graph H, find the polynomials $\Phi(x,y)$ such that $G(\Phi)$ has some connected component isomorphic to H and, in this case, if $G(\Phi)$ has (almost) all components isomorphic to H. The problem is solved by associating to H a characteristic ideal which offers a new perspective to the conjecture formulated in a previous paper, and allows to reduce its scope. In the second part, we determine the characteristic ideal for cycles of lengths ≤ 5 and for complete graphs of order ≤ 6. This results provide new evidence for the conjecture.

Key words: Galois graph, polynomial graph, strongly polynomial graph, polynomial digraph, connected component, characteristic ideal, pairing, variety of a pairing, conjecture.

1 Introduction

In the previous papers [2, 1] there are given basic notations and descriptions that will be assumed here and we refer to them for all not defined concepts. In this paper, we only consider symmetrical polynomials. Let us recall two basic definitions restricted to a symmetrical polynomial $\Phi(x,y) \in \mathbb{C}[x,y]$ of partial degree d. The graph $G(\Phi)$ is defined by taking $\mathbb{C}$ as set of vertices and the points of $\mathbb{V}(\Phi(x,y))$ as edges. As shown in [2], for standard symmetrical polynomials of partial degree d (defined in [2]), all the connected components of $G(\Phi)$ but a finite number are d-regular graphs without loops nor multiple arcs nor defective vertices. The graph $G(\Phi)^*$ is obtained by removing from $G(\Phi)$ the finite set of singular components.

The problem studied here is the following: given a finite, connected, d-regular graph H, find the polynomials $\Phi(x,y)$ (if any exists), such that H is isomorphic to some (connected) component of $G(\Phi)^*$. If it is the case, the question of deciding when H is isomorphic to *all* components of $G(\Phi)^*$ is the matter of the conjecture formulated in [2], which, for symmetric polynomials, admits the following formulation: If $\Phi(x,y) \in \mathbb{C}[x,y]$ is a standard symmetric polynomial and $G(\Phi)^*$ has a finite component, then all components are isomorphic.

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In Section 2, we define a system $S(H,\Phi)$ and a variety $W(H,\Phi)$ associated to a *pairing*, which is a pair (H,Φ) formed by a finite, connected, d-regular graph H, and a symmetric polynomial of partial degree d. The points $(u_1,\ldots,u_n)$ of $W(H,\Phi)$ such that $u_1,\ldots,u_n$ induce a component in $G(\Phi)$ form a variety $U(H,\Phi)\subseteq W(H,\Phi)$. The correspondence between points of $U(H,\Phi)$ and components of $G(\Phi)$ is established.

In Section 3 we characterize the finite, connected, d-regular graphs that are isomorphic to a component of some $G(\Phi)^*$ by its associated *characteristic ideal*. This leads to an algebraic formulation of the conjecture, and to the reduction of its scope. It also provides the theoretic frame for constructing an algorithm to determine the characteristic ideal of H.

In Section 4 we show how to improve the initial polynomial system by eliminating undesired solutions in order to determine the characteristic ideal of a graph. The general algorithm is applied to find the characteristic ideals of 3-cycles and 4-cycles.

Because of the complexity of the computations using the general algorithm, specific algorithms are valuable for some kind of graphs. In section 5, we give an algorithm for cycles, providing the characteristic ideal for cycles of length ≤ 5; and, in Section 6 another for complete graphs, providing the characteristic ideal for complete graphs of order ≤ 6. All these results provide further evidence of the conjecture, besides those obtained in [2].

Finally in the conclusions, some open problems are formulated.

Besides [2], for undefined algebraic concepts we refer to [4, 5], and for graph theoretic ones to [3, 8].

2 The variety of a pairing

Let H be a finite, connected, d-regular graph and $\Phi(x,y)$ a symmetric polynomial of partial degree d. The immediate goal is to decide if H is isomorphic to a component of $G(\Phi)^*$. A *pairing* (or *d-pairing* if we wish emphasize the degree of the graph and the partial degree of the polynomial) is a pair (H,Φ) where H is a finite, connected, d-regular graph ($d\geq 1$) and $\Phi(x,y)$ a symmetric polynomial of partial degree d. A pairing (H,Φ) is *standard* (resp. *non standard*) if the polynomial $\Phi(x,y)$ is standard (resp. non standard). We shall assume that, if H is of order n, the set of vertices of H is $[n]=\{1,\ldots,n\}$. Associate to a d-pairing (H,Φ), with $H=([n],E)$, we define the set of polynomials

$$S=S(H,\Phi)=\{\Phi(x_i,x_j)\ :\ ij\in E\},$$

and the variety of $S(H,\Phi)$,

$$W=W(H,\Phi)=\mathbb{V}(S(H,\Phi)).$$

Note that if H is d-regular of order n, then H has $m=dn/2$ edges, and the system S has m polynomials. Moreover $W\subseteq\mathbb{C}^n$. A point $(u_1,\ldots,u_n)\in W$ is called a *proper point* if $u_i\neq u_j$ for $1\leq i<j\leq n$; otherwise it is an *improper* point.

Lemma 1. *Let (H,Φ) be a pairing. Then, the dimension of the variety $W(H,\Phi)$ is at most 1.*

Proof. Let H be of order n and degree d. Let $(u_1,\ldots,u_n)$ be a point in W. If $1j$ is an edge of H, then the number of distinct values of x_j in the points of W with $x_1=u_1$ is at most d, the maximum number of roots of $\Phi(u_1,y)$. By induction, if j is a vertex of H at distance r from the vertex 1, then the number of distinct values of x_j in points of W with $x_1=u_1$ is at most d^r. Therefore, the number of points of W with $x_1=u_1$ is finite. □

The following theorem shows that for standard pairings (H, Φ), the proper points of $W(H, \Phi)$ correspond to components of $G(\Phi)^*$. Given $u_1, \ldots, u_n \in \mathbb{C}$, we denote by $\langle u_1, \ldots u_n \rangle$ the subdigraph of $G(\Phi)$ induced by $u_1, \ldots, u_n$ (the polynomial $\Phi(x, y)$ being implicit).

Proposition 1. *Let (H, Φ) be a standard pairing. Then a point $(u_1, \ldots, u_n) \in W(H, \Phi)$ is a proper point if and only if $\langle u_1, \ldots u_n \rangle$ is a component of $G(\Phi)^*$ isomorphic to H.*

Proof. Let $(u_1, \ldots, u_n) \in W$ be a proper point. Define $f\colon [n] \to \{u_1, \ldots, u_n\}$ by $f(i) = u_i$. For $i \neq j$ we have $u_i \neq u_j$, hence f is injective. As the two sets $[n]$ and $\{u_1, \ldots, u_n\}$ have the same cardinality n, the mapping f is bijective.

If $ij \in E$, then (u_i, u_j) is a zero of the polynomial in S corresponding to the edge ij, that is, u_i is adjacent to u_j in $G(\Phi)$. As H is connected, the subdigraph $\langle u_1, \ldots, u_n \rangle$ is connected. Both graphs are d-regular, so f is an isomorphism. From the fact that H is a d-regular graph, it follows that it has neither loops, nor multiple edges, nor defective vertices. Therefore $\langle u_1, \ldots, u_n \rangle = G(\Phi, u_1)$ is not a singular component of $G(\Phi)$.

Conversely, let C be a component of $G(\Phi)^*$ isomorphic to H. Let $f\colon i \mapsto u_i$ be the isomorphism from H onto C. Clearly, if $1 \leq i < j \leq n$, then $u_i \neq u_j$. For each polynomial $\Phi(x_i, x_j)$ of S, we have an edge $ij \in E$. As f is an isomorphism, u_i is adjacent to u_j in $G(\Phi)$, which is equivalent to $\Phi(u_i, u_j) = 0$. Therefore, $(u_1, \ldots, u_n)$ is a proper solution of $S(H, \Phi)$. ∎

Now consider improper points of W. Recall that, even if $\Phi(x, y)$ is a symmetric polynomial, the singular components of $G(\Phi)$ can be digraphs with loops or multiple arcs. The following decomposition helps to eliminate solutions of S which do not correspond to components of $G(\Phi)$. For a given pairing (H, Φ), define

$$\begin{aligned} Z = Z(H, \Phi) &= W(H, \Phi) \bigcap \left(\textstyle\bigcup_{i>j} \mathbb{V}(x_i - x_j) \right); \\ U = U(H, \Phi) &= \overline{W(H, \Phi) \setminus Z(H, \Phi)}. \end{aligned}$$

Note that Z is the set of improper points of W, and that the proper points of W are in $W \setminus Z$, so they are in its algebraic closure U.

Proposition 2. *Let (H, Φ) be a standard d-pairing and let J be the set of improper points of $U(H, \Phi)$. Then*

(i) *The set J is finite.*

(ii) *If $J \neq \emptyset$, then* $\dim U(H, \Phi) = 1$.

Proof. (i) If J is empty, the result is trivial. Assume $J \neq \emptyset$. Note that $J = U \cap \left(\bigcup_{j<i} \mathbb{V}(x_i - x_j) \right)$. We have $J \subseteq U \subseteq W$ and, by Lemma 1, $\dim W \leq 1$. Therefore, $\dim J \leq 1$. If J were not finite, then $1 = \dim J \leq \dim U \leq 1$. If U is irreducible, then $J = U$. In this case, $\overline{W \setminus Z} \subseteq U = J$, and the set $W \setminus Z$ has no proper points, so it is empty. Therefore $J = U = \overline{W \setminus Z} = \overline{\emptyset} = \emptyset$, a contradiction. If U is reducible, then decompose it as a union of irreducible varieties $U = \bigcup_{i=1}^{s} U_i$ and set $J_i = J \cap U_i$. For each infinite J_i, we have $J_i = U_i$ and $\overline{U_i \setminus Z} = \emptyset$. Therefore $U = \overline{W \setminus Z} = \bigcup_{i=1}^{s} \overline{U_i \setminus Z}$ is finite, a contradiction.

(ii) Let $\overline{u} \in J$. We have $\overline{u} \notin W \setminus Z$ but $\overline{u} \in \overline{W \setminus Z}$. Thus $\overline{u}$ is not an isolated point of U. Therefore U is infinite. By using Lemma 1, we have $1 \leq \dim U \leq \dim W \leq 1$. Therefore, we conclude $\dim U = 1$. ∎

As a consequence of propositions 1 and 2, we have:

Theorem 1. *Let (H, Φ) be a standard pairing. Then the graph $G(\Phi)^*$ has some component isomorphic to H if and only if $U(H, \Phi) \neq \emptyset$.*

Proof. Proposition 1 ensures that if $G(\Phi)^*$ has a component C isomorphic to H and $i \mapsto u_i$ is the isomorphism from H to C, then $(u_1, \ldots, u_n)$ is a proper solution, i.e., $(u_1, \ldots, u_n) \in W \setminus Z \subseteq U$. Therefore $U \neq \emptyset$.

Conversely, assume that there exists $\overline{u} = (u_1, \ldots, u_n) \in U$. If $\overline{u}$ is a proper point, then Proposition 1 ensures that a component of $G(\Phi)^*$ is isomorphic to H. If $\overline{u}$ is an improper point, then by Proposition 2, $\dim U = 1$. Therefore, $W \setminus Z$ is not empty and there exists a proper point $\overline{u} \in W \setminus Z$. By Proposition 1, there exists a component of $G(\Phi)^*$ isomorphic to H. $\square$

Consider now non standard pairings.

Proposition 3. *Let (H, Φ) be a non standard d-pairing. Put $n = d + 1$. If $U(H, \Phi) \neq \emptyset$ then*

(i) *$\Phi(x, y) = f(x)f(y)\Phi_1(x, y)$ where $\deg f(x) \geq 1$ and $\Phi_1(x, y)$ is a standard polynomial.*

(ii) *$G(\Phi)$ has universal vertices, say $w_1, \ldots, w_r$, and it is connected.*

(iii) *All non singular components of $G(\Phi_1)^*$ are isomorphic to K_{n-r}, and H is isomorphic to K_n.*

Proof. (i) If $L(x) = \Phi(x, x)$ is the zero polynomial or $\Phi(x, y) \neq \operatorname{rad} \Phi(x, y)$ then each point $\overline{u} \in W$ has some repeated coordinates. Hence, $\overline{u} \in Z$. Then, $W \setminus Z = \emptyset$ and $U = \emptyset$. As $\Phi(x, y)$ is non standard, it must be of the form $\Phi(x, y) = f(x)f(y)\Phi_1(x, y)$ with $\deg f(x) \geq 1$ and $\Phi_1(x, y)$ standard.

(ii) The roots $w_1, \ldots, w_r$ of $f(x)$ are the universal vertices. The existence of universal vertices implies that $G(\Phi)$ is connected.

(iii) A point $(u_1, \ldots, u_n) \in U$ has r coordinates which are the r universal vertices. The remaining coordinates induce a subgraph $(n-1-r)$-regular. The partial degree of $\Phi_1(x, y)$ is $d - r$. Therefore $n - 1 = d$ and the components of $G(\Phi_1)^*$ are isomorphic to K_{n-r}. Moreover $H \simeq \langle u_1, \ldots, u_n \rangle = K_n$. $\square$

In the context of graphs, the conjecture stated in [2] is the following:

Conjecture 1. *Let (H, Φ) be a standard pairing. If H is isomorphic to a component of $G(\Phi)^*$, then H is isomorphic to all components of $G(\Phi)^*$.*

Let (H, Φ) be a standard pairing. The graph H is said to be Φ-*polynomial* if it is isomorphic to a component of $G(\Phi)^*$; H is said to be *strongly* Φ-*polynomial* if it is isomorphic to all components of $G(\Phi)^*$. Conjecture 1 states that if H is Φ-polynomial, then H is strongly Φ-polynomial.

A finite, connected, d-regular graph H is *polynomial* (resp. *strongly polynomial*) if it is Φ-polynomial (resp. *strongly* Φ-*polynomial*) for some standard polynomial $\Phi(x, y)$.

The condition of being strongly polynomial graph is quite restrictive. Indeed, only vertex-transitive graphs can be strongly polynomial, as shown in the following theorem.

Theorem 2. *Let H be a strongly Φ-polynomial graph. Then $G(\Phi)^*$ is vertex-transitive. In particular, H is vertex transitive.*

Proof. Each component of $G(\Phi)^*$ is isomorphic to H and, by Proposition 1, each component provides a proper point of W. The number of components of $G(\Phi)^*$ is uncountable so, by Lemma 1, $\dim W = 1$. Therefore, in the system S, one indeterminate, say x_1 is free. For each vertex u_1 of $G(\Phi)^*$, we have some proper point of W of the form $(u_1, \ldots, u_n)$ and an isomorphism f_{u_1} from H to $G(\Phi, u_1)$ given by $i \mapsto u_i$. Let $u_1, v_1 \in \mathbb{C}$ be two vertices in $G(\Phi)^*$. Then $f = f_{v_1} f_{u_1}^{-1}$ is an isomorphism from $G(\Phi, u_1)$ onto $G(\Phi, v_1)$ which applies u_1 in v_1. This implies that $G(\Phi)^*$ is vertex transitive. In particular, each component of $G(\Phi)^*$, which is isomorphic to H, is vertex transitive. $\square$

Thus, only finite, connected, d-regular, vertex symmetric graphs can be strongly polynomial. On the other side we cannot ensure that every finite, connected, d-regular, vertex symmetrical graph is strongly polynomial. Petersen's graph is the smallest d-regular vertex transitive graph for which we do not know if it is polynomial. Our guess is that it is not, but the question is not yet settled. All the strongly polynomial graphs given in [2] are Cayley graphs. The fact that Petersen's graph is a well-known example of a vertex transitive graph which is not a Cayley graph suggests that it is possible that every strongly polynomial graph is not only vertex transitive, as Theorem 2 ensures, but also a Cayley graph.

3 The characteristic ideal of a graph

Fix a finite, connected, d-regular, graph $H = ([n], E)$. If the goal is to find polynomials $\Phi(x, y)$ such that H is isomorphic to one or all components of $G(\Phi)^*$, the coefficients of $\Phi(x, y)$ must be unknowns. Then we define $S(H)$, $W(H)$, $Z(H)$ and $U(H)$ in a similar way as in the previous section, but considering the coefficients of the polynomials also as unknowns. Let $m = (d+1)(d+2)/2$. For each $\overline{a} = (a_{d\,d}, a_{d\,d-1}, \ldots, a_{d\,0}, a_{d-1\,d-1}, \ldots, a_{0\,0}) \in \mathbb{C}^m$ let

$$\Phi_{\overline{a}}(x, y) = \sum_{i,j=1}^{d} a_{i\,j} x^i y^j, \text{ where } a_{i\,j} = a_{j\,i}.$$

As before, define

$$\begin{array}{rcl} S(H) & = & \{\Phi_{\overline{a}}(x_i, x_j) \; : \; ij \in E\}, \\ W(H) & = & \mathbb{V}(S(H)) \subseteq \mathbb{C}^{m+n}, \\ Z(H) & = & W(H) \bigcap \left(\bigcup_{i>j} \mathbb{V}(x_i - x_j) \right), \\ U(H) & = & \overline{W(H) \setminus Z(H)}. \end{array}$$

A point $(\overline{c}, \overline{u}) = (c_{d\,d}, \ldots, c_{0\,0}, u_1, \ldots, u_n)$ of $W(H)$ is said to be a *proper point* if $c_{d\,j} \neq 0$ for some j and $\overline{u}$ is a proper point of $S(H, \Phi_{\overline{c}})$; otherwise it is an *improper point*.

In order to decide wether H is polynomial or not, the following ideals are the key. Define

$$\begin{array}{rcl} \mathcal{I}(H) & = & \mathbb{I}(U(H)), \\ \mathcal{I}_{\overline{a}}(H) & = & \mathcal{I}(H) \cap \mathbb{C}[\overline{a}], \\ \mathcal{I}_{\overline{a}, x_1}(H) & = & \mathcal{I}(H) \cap \mathbb{C}[\overline{a}, x_1]. \end{array}$$

These three ideals satisfy $\mathcal{I}_{\overline{a}}(H) \subseteq \mathcal{I}_{\overline{a}, x_1}(H) \subseteq \mathcal{I}(H)$. The ideal $\mathcal{I}_{\overline{a}}(H)$ is called the *characteristic ideal* of H, its name being justified by Theorem 3. First, let us put aside a special case.

If $H = K_n$, then H is circulant, hence strongly polynomial, see [2]. On the other hand, Proposition 3 shows that there exists a non standard polynomial $\Phi(x, y)$ and a point $(u_1, \ldots, u_n) \in U(H, \Phi)$ such that $G(\Phi)$ is connected and $\langle u_1, \ldots, u_n \rangle$ is isomorphic to H. Thus, we may consider only the case $H \neq K_n$.

Theorem 3. *Let H, $(H \neq K_{d+1})$, be a finite, connected, d-regular graph. Then, one of the three following statements holds.*

(i) $\mathcal{I}(H) = \mathcal{I}_{\overline{a}(H)} = \langle 1 \rangle$. *In this case H is not a polynomial graph.*

(ii) $\mathcal{I}(H) \neq \langle 1 \rangle$ *and* $\mathcal{I}_{\overline{a},x_1}(H) = \mathcal{I}_{\overline{a}}(H)$. *In this case, for all $\overline{c} \in \mathbb{V}(I_{\overline{a}})$ the polynomial $\Phi_{\overline{c}}(x,y)$ is standard, and H is a strongly $\Phi_{\overline{c}}$-polynomial graph.*

(iii) $\mathcal{I}(H) \neq \langle 1 \rangle$ *and* $\mathcal{I}_{\overline{a},x_1}(H) \neq \mathcal{I}_{\overline{a}}(H)$. *In this case H is polynomial but not strongly polynomial.*

Proof. First, assume $\mathcal{I}(H) = \langle 1 \rangle$. In this case, $\mathcal{I}_{\overline{a}}(H) = \langle 1 \rangle$, too. By the Nullstellensatz, $U(H) = \emptyset$. Then, for all standard polynomial $\Phi(x,y)$, we have $U(H,\Phi) = \emptyset$. Applying Theorem 1, we conclude that $G(\Phi)^*$ has no component isomorphic to H. Therefore, H is not a polynomial graph.

Now, assume $\mathcal{I}(H) \neq \langle 1 \rangle$. By the Nullstellensatz we have $\mathbb{V}(\mathcal{I}_{\overline{a}}(H)) \neq \emptyset$. Note that no graph H is Φ-polynomial for all $\Phi(x,y)$. Then, by Proposition 1, there exists $\Phi(x,y)$ such that $U(H,\Phi) = \emptyset$. This implies $\mathcal{I}_{\overline{a}} \neq \{0\}$.

It is convenient to label vertices $1,\dots,n$ of H in such a way that each vertex $i \geq 2$ is adjacent to some vertex $j < i$. This can be done, for example by putting the labels on the vertices following the generation of a spanning tree by the Depth First Search (DFS) algorithm [6].

In the second case, $\mathcal{I}_{\overline{a},x_1}(H) = \mathcal{I}_{\overline{a}}(H)$. Let $\overline{c} \in \mathbb{V}(\mathcal{I}_{\overline{a}}(H))$. If $\Phi_{\overline{c}}$ is non standard, Proposition 3 implies that $H = K_n$, a contradiction. Hence, $\Phi_{\overline{c}}(x,y)$ is standard.

Let u_1 be a vertex of $G(\Phi_{\overline{c}})^*$. Write $\Phi_{\overline{c}}(x,y)$ in the form $\Phi_{\overline{a}}(x,y) = \sum_{i=0}^{d} a_i(x)y^i$. The hypothesis $\mathcal{I}_{\overline{a},x_1}(H) = \mathcal{I}_{\overline{a}}(H)$ implies $(\overline{c},u_1) \in \mathbb{V}(\mathcal{I}_{\overline{a},x_1}(H))$. By induction, suppose that we have a partial solution $(\overline{c},u_1,\dots,u_k) \in \mathbb{V}(\mathcal{I}_{\overline{a},x_1,\dots,x_k}(H))$. Because of the labelling of the vertices of H, for some $j < k+1$, the vertex u_j is adjacent to the vertex u_{k+1}. Then, the polynomial $\Phi(x_j,x_{k+1})$ belongs to $I_{\overline{a},x_1,\dots,x_{k+1}}(H)$. Moreover, $a_d(u_j) \neq 0$ because u_j is a vertex of the non singular component $G(\Phi_{\overline{c}},u_1)$. By the Extension Theorem [4], the partial solution extends to a solution $(\overline{c},u_1,\dots,u_{k+1})$. Therefore, $(u_1,\dots,u_n)$ is a point of $U(H,\Phi_{\overline{c}})$. By Proposition 1, $\langle u_1,\dots,u_n \rangle$ is a component of $G(\Phi_{\overline{c}})^*$ isomorphic to H. Therefore, H is strongly $\Phi_{\overline{c}}$-polynomial for all $\overline{c} \in \mathbb{V}(\mathcal{I}_{\overline{a}}(H))$.

Finally, assume $\mathcal{I}_{\overline{a},x_1}(H) \neq \mathcal{I}_{\overline{a}}(H)$. As before, $U(H) \neq \emptyset$, and, for any proper point $(\overline{c},u_1,\dots,u_n) \in U(H)$ the graph $\langle u_1,\dots,u_n \rangle$ is a component of $G(\Phi_{\overline{c}})^*$. Therefore H is $\Phi_{\overline{c}}$-polynomial. But as the indeterminate x_1 is not free, H is not strongly $\Phi_{\overline{c}}$-polynomial. □

If $H = K_n$, then $\mathcal{I}(K_n) \neq \langle 1 \rangle$ and $\mathcal{I}_{\overline{a},x_1}(K_n) = \mathcal{I}_{\overline{a}}(K_n)$, as in (ii). In this case, besides the standard polynomials $\Phi_{\overline{c}}(x,y)$ such that K_n is strongly $\Phi_{\overline{c}}$-polynomial, there are points in $\mathbb{V}(\mathcal{I}_{\overline{a}})$ corresponding to non standard polynomials as described in Proposition 3.

The proof of case (ii) provides also some insight about the singular components:

Proposition 4. *Let H be a strongly Φ-polynomial graph. If C is a singular component of $G(\Phi)$ without defective vertices, then there exists an improper point $(u_1,\dots,u_n)$ in $U(H,\Phi)$ such that $C = \langle u_1,\dots,u_n \rangle$. Moreover there exists a graph morphism from H onto C.*

Proof. In order to apply the Extension Theorem to Theorem 3 (ii), the crucial point is the condition $a_d(u_j) \neq 0$, that means that u_j is not a defective vertex. Therefore if u_1 is taken in a singular component C without defective vertices, then $(u_1,\dots,u_n)$ is an improper point in $U(H,\Phi)$. It is easily checked that $i \mapsto u_i$ is an exhaustive morphism from H to C. □

Conjecture 1 is equivalent to saying that case (iii) in Theorem 3 never occurs. The following proposition reduces the scope of the conjecture.

Proposition 5. *Let $\Phi(x, y)$ be a standard symmetric polynomial of partial degree d. Then,*

(i) *If $G(\Phi)^*$ has an uncountable number of finite components, then all components of $G(\Phi)^*$ are finite and isomorphic.*

(ii) *If $G(\Phi)^*$ has a countable number of finite components, then each finite component of $G(\Phi)^*$ is not a strongly polynomial graph.*

Proof. (i) There exist a countable number of non isomorphic finite graphs. As the number of components of $G(\Phi)^*$ is uncountable, a finite graph H exists such that an uncountable number of components of $G(\Phi)^*$ are isomorphic to H. Then $U(H, \Phi)$ is 1-dimensional and, for each vertex u_1 of $G(\Phi)^*$, we have a proper point of $U(H, \Phi)$ with $x_1 = u_1$, i.e. a component isomorphic to H.

(ii) Let H be a finite component of $G(\Phi)^*$. Consider the three cases of Theorem 3. As H is a Φ-polynomial graph, we are not in case (i). As there are only a countable number of finite components, we are not in case (ii). Then, case (iii) applies. □

Thus, Conjecture 1 is reduced to the following: there exists no standard symmetric polynomial $\Phi(x, y)$ such that $G(\Phi)^*$ has a countable number of finite components any of which is isomorphic to a strongly polynomial graph.

On the other side, computational evidence suggests that if $\Phi(x, y)$ is a standard symmetric polynomial and $G(\Phi)^*$ has infinite graphs as components, then it is not true that all components of $G(\Phi)^*$ are isomorphic. For instance, this seems to be the case with the polynomial $\Phi(x, y) = x^3 + y^3 + xy - 1$.

4 General algorithm

Given a finite, connected, d-regular graph H, we want to determine its characteristic ideal $\mathcal{I}_{\overline{a}}(H) = \mathbb{I}(U(H)) \cap \mathbb{C}[\overline{a}]$. We start with the system of polynomials $S(H)$, choose the monomial order $\text{lex}(x_n, \ldots, x_1, a_{0\,0}, a_{1\,0}, a_{1\,1}, \ldots, a_{d\,d})$, and use the generalized gaussian elimination algorithm `gge` in the *Maple* library `dpgb` [7] in order to simplify $S(H)$. At any step before launching Buchberger's algorithm, we must eliminate factors of the form $x_i - x_j$ in every new polynomial generated. Reductions and Buchberger's algorithm can be combined, to obtain the Gröbner basis of the ideal $\mathcal{I}(H)$. The polynomials in this basis depending only on the variables $\overline{a}$, are the Gröbner basis of the characteristic ideal $\mathcal{I}_{\overline{a}}(H)$. Then we can also test if $\mathcal{I}_{\overline{a},x_1}(H) = \mathcal{I}_{\overline{a}}(H)$ to decide, by Theorem 3, if H is strongly polynomial.

To make the computation effective it is strictly needed to add to $S(H)$ as many polynomials in $\mathbb{I}(U(H)) \setminus \mathbb{I}(Z(H))$ as possible. Before giving a method for obtaining polynomials of this kind, let us consider an example. Let H be the 4-cycle C_4, and consider the system

$$S(C_4) = \{\Phi(x_1, x_2), \Phi(x_2, x_3, \Phi(x_3, x_4), \Phi(x_4, x_1)\},$$

where $\Phi = \Phi_{\overline{a}}$. For a given $\overline{a} \in \mathbb{C}^m$ and $u \in \mathbb{C}$ let λ_1, λ_2 be the two roots of $\Phi(u, y)$. Then $(\overline{a}, u, \lambda_1, u, \lambda_2)$, $(\overline{a}, u, \lambda_2, u, \lambda_1)$, $(\overline{a}, u, \lambda_1, u, \lambda_1)$ and $(\overline{a}, u, \lambda_2, u, \lambda_2)$ are improper points in $W(C_4)$. Let now μ_1, μ_2 be such that u, μ_i are the two distinct roots of $\Phi(\lambda_i, y)$, for $i = 1, 2$. If $\mu_1 = \mu_2$, then $(\overline{a}, u, \lambda_1, \mu_1, \lambda_2)$ and $(\overline{a}, u, \lambda_2, \mu_1, \lambda_1)$ are proper points in $U(C_4)$. Thus, for any $\overline{a}$ and u there exist a finite number of solutions in $Z(C_4, \Phi_{\overline{a}})$, and this variety is of dimension 1 for any $\overline{a}$. But the condition $\mu_1 = \mu_2$ will be satisfied only if C_4 is a $\Phi_{\overline{a}}$-polynomial graph, and there are proper points in $U(C_4, \Phi_{\overline{a}})$. If $\mu_1 = \mu_2$ for any u, then C_4 is strongly $\Phi_{\overline{a}}$ -polynomial. The undesired solutions in $Z(C_4)$ appear owing to the fact that no distinction is made in $S(C_4)$ between the y-roots of $\Phi(u, y)$.

Let H be a finite, connected, d-regular graph. The following method allows to obtain a set of polynomials in $\mathbb{I}(U(H))$ (depending on a vertex of H) that separates roots. Consider a vertex i_0 in H and let $x_{i_1}, \dots x_{i_d}$ be the indeterminates corresponding to the vertices adjacent to i_0. In the following discussion we write x_j instead of x_{i_j} to avoiding subscripts. Consider the polynomials $\Phi(x_0, x_j)$, $1 \le j \le d$, in $S(H)$. Set

$$\Phi_0(x_0; x_1) = \Phi(x_0, x_1),$$

and define recursively

$$\Phi_{\ell-1}(x_0; x_1, \dots, x_\ell) = \frac{\Phi_{\ell-2}(x_0; x_1 \dots, x_{\ell-1}) - \Phi_{\ell-2}(x_0; x_1 \dots, x_{\ell-2}, x_\ell)}{x_\ell - x_{\ell-1}}.$$

Proposition 6. *The polynomials Φ_ℓ have the following properties:*

(i) $\Phi_{\ell-1}$ *is a polynomial of* $U(H)$ *for* $1 \le \ell \le d$.

(ii) $\Phi_{\ell-1}(x_0; x_1, \dots, x_\ell)$ *is symmetrical in the set of variables* $\{x_1, \dots, x_\ell\}$, *and its total degree as a polynomial in the variables* $\overline{x}$ *is* $2d - \ell + 1$.

(iii) *If* $\Phi(u_0, y)$ *has* d *different roots* $u_1, \dots, u_d$, *then the set of solutions of the system formed by the* d *polynomials* $\Phi_0(u_0; x_1)$, $\Phi_1(u_0; x_1, x_2)$, $\dots$, $\Phi_{d-1}(u_0; x_1, \dots, x_d)$ *is exactly the set of all permutations of the solution* $\{x_1 = u_1, \dots, x_d = u_d\}$.

Proof. (i) For $\ell = 2$ we have

$$\Phi_0(x_0; x_1) - \Phi_0(x_0; x_2) = \sum_{i,j=0}^{d} a_{ij} x_0^i (x_1^j - x_2^j) \quad = \sum_{i,j=0}^{d} a_{ij} x_0^i (x_1 - x_2) \sum_{k=1}^{j} x_1^{j-k} x_2^{j-1}.$$

Thus $\Phi_1(x_0; x_1, x_2)$ belongs to $U(H)$ and we have

$$\Phi_1(x_0; x_1, x_2) = \sum_{i,j=0}^{d} a_{ij} x_0^i \sum_{k=1}^{j} x_1^{j-k} x_2^{j-1}.$$

Iterating, we obtain an explicit formula for $\Phi_{\ell-1}$:

$$\Phi_{\ell-1}(x_0; x_1, \dots, x_\ell) = \sum_{i,j=0}^{d} a_{ij} x_0^i \sum_{k_1=1}^{j} x_1^{j-k_1} \sum_{k_2=1}^{k_1} x_2^{k_1-k_2} \cdots \sum_{k_{\ell-1}=1}^{k_{\ell-2}} x_{\ell-1}^{k_{\ell-2}-k_{\ell-1}} x_\ell^{k_{\ell-1}-1}.$$

showing that it belongs to $U(H)$.

(ii) It can be proved by induction that formula (4) is equivalent to

$$\Phi_{\ell-1}(x_0; x_1, \dots, x_\ell) = \sum_{i,j=0}^{d} a_{ij} x_0^i \sum_{\overline{k}} x_1^{k_1} \cdots x_\ell^{k_\ell},$$

where the sum over $\overline{k}$ is extended to all $\overline{k} = (k_1, \ldots, k_\ell)$ verifying $k_i \geq 0$ and $\sum_{s=1}^{\ell} k_s = j-\ell+1$. This formula is explicitly symmetric in the set of variables $\{x_1, \ldots, x_\ell\}$, and its degree in $\overline{x}$ is obviously $2d - \ell + 1$.

(iii) $\Phi(u_0, x_1)$ has exactly the d solutions $\{x_1 = u_1, \ldots, x_1 = u_d\}$. Then $\Phi_0(u_1; u_i) = 0$ and $\Phi_1(u_0; u_i, x_2) = 0$ imply $\Phi_0(u_i; x_2) = 0$ and thus $\Phi_1(u_0; u_i, x_2)$ has the same roots as $\Phi(u_0, x_2)$ except for u_i. Similarly, we can prove that $\Phi_{\ell-1}(u_0; u_{i_1}, \ldots, u_{i_{\ell-1}}, x_\ell)$ has the same roots as $\Phi(u_0, x_\ell)$ except for $\{u_{i_1}, \ldots, u_{i_{\ell-1}}\}$. Thus the set of solutions of $S(u_0)$ is the set of all permutations of $\{x_1 = u_1, \ldots, x_d = u_d\}$. □

Let V_i be the set of vertices of H adjacent to the vertex i. The *completed system* $S'(H)$ is formed by all the polynomials $\Phi_{\ell-1}(x_0; x_{i_1}, \ldots, x_{i_\ell})$, where $\{i_1, \ldots, i_\ell\}$ is a ℓ-subset of V_i, for all $\ell \in [d]$ and $i \in [n]$. Note that for $\ell = 1$ we obtain the polynomials in $S(H)$. The number of polynomials in $S'(H)$ is

$$n \sum_{\ell=1}^{d} \binom{d}{\ell} = n(2^d - 1).$$

Nevertheless, in this account there are repeated polynomials. For instance $\Phi(x_i, x_j) = \Phi_0(x_i; x_j)$ appear twice. The system $S'(H)$ being a set, repetitions have to be crossed out.

In general, the solutions of the completed system $S'(H)$ are not exactly the points in $U(H)$. Factors $x_i - x_j$ can appear in the computing of a Gröbner basis. Often, it is possible to take into account the symmetry of the graph in order to eliminate these extraneous solutions, by introducing a new set of reduced polynomials. For instance, for graphs with cliques of order $\ell + 1$ the following polynomials are helpful. Set

$$\Phi_{\ell-1,0}(x_0; x_1, \ldots, x_\ell) = \Phi_{\ell-1}(x_0; x_1, \ldots, x_\ell),$$

and, recursively,

$$\begin{aligned}\Phi_{\ell-1,k}(x_0, \ldots, x_k; x_{k+1}, \ldots, x_\ell) &= (\Phi_{\ell-1,k-1}(x_0, \ldots, x_{k-2}, x_k; x_{k-1}, x_{k+1} \ldots, x_\ell) \\ &\quad - \Phi_{\ell-1,k-1}(x_0, \ldots, x_{k-1}; x_k, \ldots, x_\ell))\,/(x_{k-1} - x_k).\end{aligned}$$

Proposition 7. *Suppose that the vertices $\{0, 1, \ldots, \ell\}$ is a clique of H. Then,*

(i) $\Phi_{\ell-1,k}$ *are polynomials in* $\mathbb{I}(U(H))$ *for* $0 \leq k \leq \ell - 1$.

(ii) $\Phi_{\ell-1,k}$ *are symmetric in the second set of variables.*

(iii) *The degree of* $\Phi_{\ell-1,k}$ *in* $\overline{x}$ *is* $2d - \ell + 1 - k$.

Proof. The fact that each two vertices in $0, \ldots, \ell$ are adjacent implies that $\Phi_{\ell-1,k}$ are polynomials of $\mathcal{I}(H)$. The symmetry of the $\Phi_{\ell-1}$ in the second set of variables produces the symmetry of the $\Phi_{\ell-1,k}$ in the second set of variables and their degree in $\overline{x}$ is deduced directly from the degree of the $\Phi_{\ell-1}$. □

4.1 Application

We apply the general algorithm to determine the characteristic ideal of a 3-cycle and of a 4-cycle. Let

$$\begin{aligned}\Phi(x, y) &= a_{00} + a_{10}(x + y) + a_{11}xy + a_{20}(x^2 + y^2) + a_{21}xy(x + y) + a_{22}x^2y^2, \\ \Phi_1(x; y, z) &= a_{10} + a_{11}x + a_{20}(y + z) + a_{21}x(x + y + z) + a_{22}x^2(y + z), \\ \Phi_{11}(x, y; z) &= a_{11} - a_{20} + a_{21}(x + y + z) + a_{22}(xy + yz + zx).\end{aligned}$$

The complete system for the 3-cycle is

$$S'(C_3) = \{\Phi(x_1, x_2), \Phi(x_2, x_3), \Phi(x_3, x_1), \Phi_1(x_1; x_2, x_3), \Phi_1(x_2; x_3, x_1), \Phi_1(x_3; x_1, x_2)\}.$$

As C_3 is a complete graph, we add the polynomial $\Phi_{11}(x_1, x_2; x_3)$. Let $S''(H) = S'(H) \cup \{\Phi_{11}(x_1, x_2; x_3)\}$. We take the monomial order

$$\mathrm{lex}(x_3, x_2, x_1, a_{00}, a_{10}, a_{11}, a_{20}, a_{21}, a_{22})$$

and calculate the Gröbner basis of $S''(C_3)$, which is easily computed and contains 9 polynomials. The quick computation is owing to the inclusion of the polynomial Φ_{11}, that reflects the symmetry. The Gröbner basis provides the following elimination ideal:

$$\mathcal{I}_{\overline{a}}(C_3) = \langle \mathbf{a}_{00}\, a_{22} + a_{20}\, a_{11} - a_{20}^2 - a_{21}\, a_{10} \rangle$$

and $\mathcal{I}_{\overline{a}\, x_1}(C_3) = \mathcal{I}_{\overline{a}}(C_3)$.

Consider now 4-cycles. The complete system is:

$$\begin{aligned} S'(C_4) \quad = \quad & \{\Phi(x_1, x_2), \Phi(x_2, x_3), \Phi(x_3, x_4), \Phi(x_4, x_1), \\ & \Phi_1(x_1; x_4, x_2), \Phi_1(x_2; x_1, x_3), \Phi_1(x_3; x_2, x_4), \Phi_1(x_4; x_3, x_1)\}. \end{aligned}$$

The computations become only effective when we add a new reduced polynomial that reflects the symmetry, and eliminates the extraneous solution $x_1 = x_3$, namely:

$$\begin{aligned} \Psi(x_1, x_3; x_2, x_4) &= \frac{\Phi_1(x_1; x_2, x_4) - \Phi_1(x_3; x_2, x_4)}{x_1 - x_3} \\ &= a_{11} + a_{21}(x_1, x_2, x_3, x_4) + a_{22}(x_1 + x_3)(x_2 + x_4). \end{aligned}$$

Take $S''(C_4) = S'(C_4) \cup \{\Psi(x_1, x_3; x_2, x_4)\}$. The direct computation of the Gröbner basis, when using an automatic method, becomes difficult. We use the technique of stopping the computation when a high number of polynomials have been computed and then use `gge` routine in the `dpgb` library to reduce the basis. The result is a basis of 24 polynomials, which provides the following characteristic ideal:

$$\mathcal{I}_{\overline{a}}(C_4) = \langle \mathbf{a_{00} a_{11} a_{22}} + 2a_{21}a_{20}a_{10} - a_{20}^2 a_{11} - a_{21}^2 a_{00} - a_{10}^2 a_{22} \rangle.$$

and, as before, $\mathcal{I}_{\overline{a}x_1}(C_4) = \mathcal{I}_{\overline{a}}(C_4)$.

As shown in [2], the polynomial of partial degree two $\Phi_{\overline{a}}(x, y)$ can be reduced by a translation to a polynomial with $a_{21} = 0$. By performing the above computations in this case, the number of polynomials in the basis reduces to 8 polynomials for $\mathbb{I}(S''(C_3))$ and 19 for $\mathbb{I}(S''(C_4))$.

5 Cycles

In [2] a complete study of the components of $G(\Phi)$ when $\Phi(x, y)$ is a symmetric polynomial of total degree two is given. The method can be used to determine conditions on the coefficients of a polynomial $\Phi(x, y) = a(x)y^2 + b(x)y + c(x)$ of partial degree 2 for obtaining cycles of length n as components of $G(\Phi)^*$. Let

$$\begin{aligned} a(x) &= a_{22}x^2 + a_{21}x + a_{20}, \\ b(x) &= a_{21}x^2 + a_{11}x + a_{10}, \\ c(x) &= a_{20}x^2 + a_{10}x + a_{00}. \end{aligned}$$

As a polynomial in y, the sum of the two roots of $\Phi(x,y)$ equals $-b(x)/a(x)$. Then, we have the recurrence:

$$v_n = -v_{n-2} - \frac{b(v_{n-1})}{a(v_{n-1})} = \frac{p_n}{q_n}.$$

By iterating the recurrence with free initial values v_0 and v_1, we obtain, by substitution and simplification, expressions for p_n and q_n, in terms of v_0, v_1 and of the coefficients $\overline{a}$. To obtain n-cycles we must impose $K_n = p_n - v_0 q_n = 0$ and $\Phi(v_0, v_1) = 0$. We use the above conditions, dividing K_n by $[\Phi(v_0, v_1)]$ using a convenient monomial order. The result is a polynomial that has one factor depending only on the parameters $\overline{a}$. Consequently the polynomial produces n-cycles for any initial point v_0, when the factor containing only the parameters vanishes. In this way we obtain the characteristic ideals for 3, 4 and 5 cycles, which are principal ideals. These are:

$$\begin{aligned}
\Delta_3 &= \mathbf{a_{22}\,a_{00}} + a_{11}\,a_{20} - a_{20}^2 - a_{21}a_{10},\\
\Delta_4 &= \mathbf{a_{22}\,a_{11}\,a_{00}} - a_{22}\,a_{10}^2 - a_{11}\,a_{20}^2 + 2a_{21}\,a_{20}\,a_{10} - a_{21}^2\,a_{00},\\
\Delta_5 &= \mathbf{a_{22}^3\,a_{00}^3} - a_{21}^3\,a_{10}^3 - 4a_{22}\,a_{20}^3\,a_{10}^2 + 5\,a_{21}\,a_{20}^4\,a_{10} + a_{20}^2\,a_{10}^2\,a_{21}^2\\
&\quad + a_{10}^2\,a_{21}^2\,a_{20}\,a_{11} - 4\,a_{10}\,a_{21}\,a_{20}^3\,a_{11} - a_{22}^2\,a_{10}^4 - a_{11}\,a_{20}^5 - a_{20}^6\\
&\quad + 3\,a_{22}\,a_{11}\,a_{10}^2\,a_{20}^2 + a_{22}\,a_{21}\,a_{11}\,a_{10}^3 - a_{22}\,a_{11}^2\,a_{20}\,a_{10}^2 + a_{11}^2\,a_{20}^4\\
&\quad + 4\,a_{22}^2\,a_{20}\,a_{10}^2\,a_{00} + 3\,a_{11}\,a_{21}^2\,a_{20}^2\,a_{00} - 2\,a_{22}\,a_{21}\,a_{20}^2\,a_{10}\,a_{00}\\
&\quad - a_{20}\,a_{11}^2\,a_{21}^2\,a_{00} - 4\,a_{20}^3\,a_{21}^2\,a_{00} - 3\,a_{22}^2\,a_{20}^2\,a_{00}^2 + a_{22}^2\,a_{10}^2\,a_{11}\,a_{00}\\
&\quad + a_{11}\,a_{21}^2\,a_{22}\,a_{00}^2 - 3\,a_{22}^2\,a_{10}\,a_{21}\,a_{00}^2 + a_{10}\,a_{11}\,a_{21}^3\,a_{00} - a_{22}\,a_{20}^2\,a_{11}^2\,a_{00}\\
&\quad - 4\,a_{22}\,a_{10}\,a_{11}\,a_{20}\,a_{21}\,a_{00} + a_{22}\,a_{20}\,a_{11}^3\,a_{00} + 2\,a_{22}\,a_{11}\,a_{20}^3\,a_{00}\\
&\quad + 4\,a_{22}\,a_{21}^2\,a_{20}\,a_{00}^2 - a_{11}\,a_{22}^2\,a_{20}\,a_{00}^2 + a_{22}\,a_{10}^2\,a_{21}^2\,a_{00}\\
&\quad - a_{22}\,a_{10}\,a_{21}\,a_{11}^2\,a_{00} - a_{21}^4\,a_{00}^2 + 3a_{22}\,a_{20}^4\,a_{00}.
\end{aligned}$$

Using the above characteristics ideals, it is easy to obtain examples of polynomials producing cycles:

Graph	Polynomial
C_3	$x^2y^2 + x^2 + y^2 - xy + 2$
C_4	$x^2y^2 + x^2 + y^2 + xy + 1$
C_5	$x^2y^2 + x^2 + y^2 - 2xy + x + y - 2.$

6 Complete graphs

For complete graphs K_{d+1} we use a specific technique that takes into account the symmetry of the graph. We start writing the system $S(H)$ of polynomials corresponding to K_{d+1}. Then, as the number of parameters $\overline{a}$ is $(d+2)(d+1)/2$, and the number of edges (= equations) is $d(d-1)/2$ we can solve the linear system considering the $\overline{a}$ as variables. This provides some of the $\overline{a}$ in terms of the rest. In order to obtain the correct result, it is important to choose the coefficients with greatest indices as parameters and to express the $\overline{a}$ with smaller indexes in terms of them. Being careful we can obtain an expression for some of the $\overline{a}$ linearly dependent in the rest of the $\overline{a}$, and polynomial in the $\overline{x}$. Owing to the symmetry of the complete graph in the vertices, we can now transform the dependence of these expressions in the $\overline{x}$ in terms of the elementary symmetrical polynomials of the $\overline{x}$ say $s_1, s_2, \ldots, s_d$.

The resulting system of equations turns out to be linear in the s_i and very simple. For K_3, K_4 and K_5 the corresponding set of polynomials defining the systems are:

$$S(K_3) = \begin{cases} a_{00} - a_{20}\, s_2 + a_{21}\,(-1+s_1), \\ a_{10} + a_{20}\, s_1 + a_{22}\,(-1+s_1), \\ a_{11} - a_{20} + a_{21}\, s_1 + a_{22}\, s_2. \end{cases}$$

$$S(K_4) = \begin{cases} a_{00} + a_{30}\, s_3 + a_{31}\, s_1, \\ a_{10} - a_{30}\, s_2 + a_{32}\, s_1, \\ a_{11} + a_{30}\, s_1 - a_{31}\, s_2 - a_{32}\, s_3 + a_{33}\, s_1, \\ a_{21} - a_{30} + a_{31}\, s_1 - a_{33}\, s_3, \\ a_{22} - a_{31} + a_{32}\, s_1 + a_{33}\, s_2, \\ a_{20} + a_{30}\, s_1 + a_{33}\, s_1. \end{cases}$$

$$S(K_5) = \begin{cases} a_{00} - a_{40}\, s_4 + a_{41}\,(-1+s_1), \\ a_{10} + a_{40}\, s_3 + a_{42}\,(-1+s_1), \\ a_{11} - a_{40}\, s_2 + a_{41}\, s_3 + a_{42}\, s_4 + a_{43}\,(-1+s_1), \\ a_{20} - a_{40}\, s_2 + a_{43}\,(-1+s_1), \\ a_{21} + a_{44}\,(-1+s_1) + a_{40}\, s_1 - a_{41}\, s_2 + a_{43}\, s_4, \\ a_{22} + a_{44}\, s_4 - a_{40} + a_{41}\, s_1 - a_{42}\, s_2 - a_{43}\, s_3, \\ a_{30} + a_{44}\,(-1+s_1) + a_{40}\, s_1, \\ a_{31} + a_{44}\, s_4 - a_{40} + a_{41}\, s_1, \\ a_{32} - a_{44}\, s_3 - a_{41} + a_{42}\, s_1, \\ a_{33} + a_{44}\, s_2 - a_{42} + a_{43}\, s_1. \end{cases}$$

As we see, the equations do not depend on s_d, the latest elementary symmetrical polynomial. This proves directly the conjecture, namely the ideal $\mathcal{I}$ in the variables $s_1, \dots, s_d, \overline{a}$ has one degree of freedom more than the elimination ideal in the variables $\overline{a}$, and the variable s_d is free.

Now we apply the standard method with the new variables $\overline{s}$, using the order $\succ_s = \mathrm{lex}(s_1, s_2, \dots, s_{d-1}, a_{00}, a_{10}, \dots, a_{dd})$, and determine the Gröbner basis of the ideals $\mathcal{I}_{\overline{a}}(K_n)$. In this way, we obtain the characteristic ideals for K_3, K_4, K_5 and K_6. These are

$$\mathcal{I}_{\overline{a}}(K_3) = \langle \mathbf{a_{00}\, a_{22}} + a_{20}\, a_{11} - a_{20}^2 - a_{21}\, a_{10} \rangle.$$

$$\begin{aligned} \mathcal{I}_{\overline{a}}(K_4) = \langle & \mathbf{a_{11}\, a_{33}} - a_{32}\, a_{21} + a_{32}\, a_{30} - a_{31}^2 + a_{31}\, a_{22} - a_{33}\, a_{20}, \\ & \mathbf{a_{10}\, a_{33}} - a_{32}\, a_{20} - a_{30}\, a_{31} + a_{30}\, a_{22}, \\ & \mathbf{a_{10}\, a_{21}\, a_{32}} - a_{10}\, a_{31}\, a_{22} - a_{10}\, a_{32}\, a_{30} + a_{10}\, a_{31}^2 - a_{11}\, a_{32}\, a_{20} \\ & + a_{11}\, a_{30}\, a_{22} - a_{11}\, a_{30}\, a_{31} + a_{32}\, a_{20}^2 - a_{20}\, a_{30}\, a_{22} + a_{20}\, a_{30}\, a_{31}, \\ & \mathbf{a_{00}\, a_{33}} - a_{31}\, a_{20} + a_{30}\, a_{21} - a_{30}^2, \\ & \mathbf{a_{00}\, a_{32}} - a_{31}\, a_{10} + a_{30}\, a_{11} - a_{30}\, a_{20}, \\ & \mathbf{a_{00}\, a_{22}} - a_{10}\, a_{21} + a_{10}\, a_{30} - a_{20}^2 + a_{20}\, a_{11} - a_{31}\, a_{00} \rangle. \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{\overline{a}}(K_5) = \langle & \mathbf{a_{22}\, a_{44}} - a_{44}\, a_{31} + a_{41}\, a_{43} - a_{43}\, a_{32} - a_{42}^2 + a_{42}\, a_{33}, \\ & \mathbf{a_{21}\, a_{44}} - a_{44}\, a_{30} + a_{43}\, a_{40} - a_{43}\, a_{31} - a_{41}\, a_{42} + a_{41}\, a_{33}, \\ & \mathbf{a_{21}\, a_{32}\, a_{43}} + a_{43}\, a_{41}\, a_{30} - a_{30}\, a_{43}\, a_{32} - a_{30}\, a_{42}^2 + a_{30}\, a_{42}\, a_{33} - a_{43}\, a_{40}\, a_{31} \\ & + a_{43}\, a_{31}^2 + a_{31}\, a_{41}\, a_{42} - a_{31}\, a_{41}\, a_{33} - a_{21}\, a_{41}\, a_{43} + a_{21}\, a_{42}^2 - a_{21}\, a_{42}\, a_{33} \\ & + a_{22}\, a_{43}\, a_{40} - a_{22}\, a_{43}\, a_{31} - a_{22}\, a_{41}\, a_{42} + a_{22}\, a_{41}\, a_{33}, \\ & \mathbf{a_{20}\, a_{44}} - a_{43}\, a_{30} - a_{42}\, a_{40} + a_{40}\, a_{33}, \end{aligned}$$

$\mathbf{a_{20}\,a_{32}\,a_{43}} - a_{42}\,a_{33}\,a_{20} - a_{41}\,a_{43}\,a_{20} + a_{42}^2\,a_{20} - a_{22}\,a_{43}\,a_{30} + a_{22}\,a_{40}\,a_{33}$
$-a_{42}\,a_{40}\,a_{22} + a_{31}\,a_{43}\,a_{30} - a_{31}\,a_{40}\,a_{33} + a_{42}\,a_{40}\,a_{31},$
$\mathbf{a_{20}\,a_{31}\,a_{43}} - a_{41}\,a_{33}\,a_{20} + a_{41}\,a_{42}\,a_{20} - a_{21}\,a_{43}\,a_{30} + a_{21}\,a_{40}\,a_{33} + a_{43}\,a_{30}^2$
$-a_{30}\,a_{40}\,a_{33} + a_{40}\,a_{42}\,a_{30} - a_{40}\,a_{42}\,a_{21} - a_{40}\,a_{43}\,a_{20},$
$\mathbf{a_{20}\,a_{31}\,a_{42}} + a_{22}\,a_{41}\,a_{30} - a_{42}\,a_{30}\,a_{21} - a_{32}\,a_{41}\,a_{20} - a_{31}\,a_{41}\,a_{30} + a_{41}^2\,a_{20}$
$+a_{42}\,a_{30}^2 - a_{40}\,a_{32}\,a_{30} + a_{40}\,a_{32}\,a_{21} + a_{40}\,a_{31}^2 - a_{22}\,a_{40}\,a_{31} - a_{40}\,a_{42}\,a_{20}$
$-\,a_{40}^2\,a_{31} + a_{40}^2\,a_{22} + a_{40}\,a_{41}\,a_{30} - a_{40}\,a_{41}\,a_{21},$
$\mathbf{a_{11}\,a_{44}} - a_{43}\,a_{30} - a_{42}\,a_{31} - a_{41}^2 + a_{41}\,a_{32} + a_{40}\,a_{33},$
$\mathbf{a_{11}\,a_{43}} - a_{41}\,a_{31} + a_{41}\,a_{22} + a_{42}\,a_{30} - a_{42}\,a_{21} - a_{43}\,a_{20},$
$\mathbf{a_{11}\,a_{33}} + a_{32}\,a_{30} - a_{32}\,a_{21} - a_{31}^2 + a_{31}\,a_{22} - a_{42}\,a_{11} - a_{33}\,a_{20} + a_{42}\,a_{20}$
$+a_{40}\,a_{31} - a_{40}\,a_{22} - a_{41}\,a_{30} + a_{41}\,a_{21},$
$\mathbf{a_{10}\,a_{44}} - a_{42}\,a_{30} + a_{40}\,a_{32} - a_{41}\,a_{40},$
$\mathbf{a_{10}\,a_{43}} - a_{42}\,a_{20} + a_{40}\,a_{22} - a_{40}\,a_{31},$
$\mathbf{a_{10}\,a_{33}} + a_{41}\,a_{20} - a_{10}\,a_{42} - a_{32}\,a_{20} - a_{30}\,a_{31} + a_{30}\,a_{22},$
$\mathbf{a_{10}\,a_{31}\,a_{42}} - a_{32}\,a_{41}\,a_{10} - a_{40}\,a_{10}\,a_{42} + a_{41}^2\,a_{10} - a_{30}\,a_{42}\,a_{11} + a_{32}\,a_{40}\,a_{11}$
$-a_{41}\,a_{40}\,a_{11} + a_{30}\,a_{42}\,a_{20} - a_{40}\,a_{32}\,a_{20} + a_{40}\,a_{41}\,a_{20},$
$\mathbf{a_{10}\,a_{21}\,a_{42}} - a_{22}\,a_{41}\,a_{10} - a_{30}\,a_{10}\,a_{42} + a_{31}\,a_{41}\,a_{10} - a_{20}\,a_{42}\,a_{11} + a_{22}\,a_{40}\,a_{11}$
$-a_{31}\,a_{40}\,a_{11} + a_{42}\,a_{20}^2 - a_{20}\,a_{40}\,a_{22} + a_{20}\,a_{40}\,a_{31},$
$\mathbf{a_{10}\,a_{21}\,a_{32}} - a_{10}\,a_{40}\,a_{31} + a_{22}\,a_{40}\,a_{10} + a_{11}\,a_{41}\,a_{20} - a_{11}\,a_{32}\,a_{20} - a_{11}\,a_{30}\,a_{31}$
$+a_{11}\,a_{30}\,a_{22} - a_{20}\,a_{30}\,a_{22} - a_{10}\,a_{31}\,a_{22} - a_{10}\,a_{32}\,a_{30} - a_{21}\,a_{41}\,a_{10} - a_{41}\,a_{20}^2$
$+\,a_{10}\,a_{31}^2 + a_{32}\,a_{20}^2 + a_{30}\,a_{41}\,a_{10} + a_{20}\,a_{30}\,a_{31},$
$\mathbf{a_{00}\,a_{44}} - a_{41}\,a_{30} + a_{40}\,a_{31} - a_{40}^2,$
$\mathbf{a_{00}\,a_{43}}, - a_{41}\,a_{20} - a_{40}\,a_{30} + a_{40}\,a_{21},$
$\mathbf{a_{00}\,a_{42}} - a_{40}\,a_{20} - a_{41}\,a_{10} + a_{40}\,a_{11},$
$\mathbf{a_{00}\,a_{33}} - a_{41}\,a_{10} - a_{30}^2 + a_{30}\,a_{21} + a_{40}\,a_{11} - a_{31}\,a_{20},$
$\mathbf{a_{00}\,a_{32}} - a_{31}\,a_{10} + a_{40}\,a_{10} - a_{00}\,a_{41} - a_{30}\,a_{20} + a_{30}\,a_{11},$
$\mathbf{a_{00}\,a_{22}} - a_{10}\,a_{21} + a_{10}\,a_{30} - a_{20}^2 + a_{20}\,a_{11} - a_{31}\,a_{00}\rangle.$

We do not write the characteristic ideal $\mathcal{I}_{\overline{a}}(K_6)$, because it contains 48 polynomials using the reduced polynomial with $a_{54} = 0$. The Gröbner basis of $\mathcal{I}(K_6)$ contains 104 polynomials. The following are examples of polynomials $\Phi(x, y)$ such that K_n is strongly Φ-polynomial.

Graph	Polynomial
K_3	$x^2y^2 + x^2 + y^2 + 3x + 3y + 1$
K_4	$x^3y + xy^3 + x^2y^2 + 1$
K_5	$2x^4y^4 + 2x^4 + 2y^4 + x^3y + xy^3 + x^2y^2 + 1$
K_6	$x^5y^5 + x^5 + y^5 - x^4y^2 - x^2y^4 - x^3y^3 + x + y + 1.$

To finish this section, two remarks. First, note that the complete graph K_{d+1} has K_d as induced subgraph. Thus, a polynomial with coefficients in $\mathbb{V}(\mathcal{I}_{\overline{a}}(K_{d+1}))$ having all the coefficients with some subindex n equal to zero must be in $\mathbb{V}(\mathcal{I}_{\overline{a}}(K_d))$. In terms of ideals,

$$\mathcal{I}_{\overline{a}}(K_{d+1})(a_{00}, a_{10}, \ldots, a_{d-1,d-1}, 0, \ldots, 0) = \mathcal{I}_{\overline{a}}(K_d).$$

Second. As a consequence of Proposition 3, if K_d is strongly Φ-polynomial, then the polynomial $xy\,\Phi(x, y)$ satisfies the conditions of K_{d+1}. Therefore if we substitute a_{ij} by $a_{i+1\,j+1}$ in $\mathcal{I}_{\overline{a}}(K_d)$ the resulting ideal is contained in $\mathcal{I}_{\overline{a}}(K_{d+1})$.

The above relations between the ideals K_3, K_4, K_5 and K_6 can be checked.

7 Concluding remarks

We have presented and solved a number of questions concerning polynomial graphs. Nevertheless, we are aware that there are many open questions. Let us remark at least three of them.

The first one is, obviously, to prove or disprove the conjecture: Either to prove that if (H, Φ) is a standard pairing and H is Φ-polynomial, then H is strongly Φ-polynomial or to find a standard pairing (H, Φ) such that H is Φ-polynomial but not strongly Φ-polynomial.

Second. We have seen that any strongly Φ-polynomial graph is vertex transitive. But all examples we have are Cayley graphs. Therefore, it is a natural question to ask if every strongly Φ-polynomial graph is a Cayley graph. In particular it would be interesting to know if Petersen's graph, which is vertex transitive but is not a Cayley graph, is polynomial (our guess is that it is not).

Third. The discussions in this paper are depending on the finiteness of H. It would be interesting to develop methods for d-regular graphs not necessarily finite, and generalize the conjecture.

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