ON POLYNOMIAL DIGRAPHS

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ABSTRACT. Let $\Phi(x,y)$ be a bivariate polynomial with complex coefficients. The zeroes of $\Phi(x,y)$ are given a combinatorial structure by considering them as arcs of a directed graph $G(\Phi)$. This paper studies some relationship between the polynomial $\Phi(x,y)$ and the structure of $G(\Phi)$.

Key words: polynomial digraph, polynomial graph, Galois graph, Cayley digraph, algebraic variety.

1. INTRODUCTION

Let $\Phi(x,y) \in \mathbb{C}[x,y]$ be a bivariate polynomial with complex coefficients and let I be the ideal generated by $\Phi(x,y)$. The variety $\mathbb{V}(I)$ of I is the set of ordered pairs $(u,v) \in \mathbb{C}^2$ such that $\Phi(u,v) = 0$. We give a combinatorial structure to $\mathbb{V}(I)$ by taking its elements as arcs of a digraph $G(\Phi)$, and we explore the relationship between the polynomial $\Phi(x,y)$ and the digraph $G(\Phi)$.

The digraphs $G(\Phi)$ were introduced in [2] for symmetric polynomials $\Phi(x,y)$ under the name of *Galois graphs*. Lengths of walks, distances and cycles were described in terms of $\Phi(x,y)$. Also, when the coefficients of $\Phi(x,y)$ belong to a field k and $\overline{k}$ is the algebraic closure of k, the action of the Galois group $G(\overline{k}/k)$ on $G(\Phi)$ was studied (and this was the motivation for the name of Galois graphs).

Here we give a general overview of the topic for non necessarily symmetric polynomials and some concrete families of polynomials are studied. In [3] we adopt an algebraic approach to the problem of deciding which polynomials produce a given graph as a connected component of $G(\Phi)$.

Let us fix some notation. In this paper digraphs are allowed to be infinite, and to have multiple arcs and loops. A *graph* is a digraph without loops nor multiple arcs such that for each arc (u,v) there exists the arc (v,u). The arcs $(u,v),(v,u)$ form the *edge* uv of the graph. Let u be a vertex of a digraph D. The *strong (connected) component* of u is the subdigraph of D induced by u and the set of vertices v such that there exist a directed path from u to v and a directed path from v to u. The *underlying graph* of a digraph D is the graph obtained by taking as edges the set of uv with (u,v) an arc of D and $u \neq v$. The *(weakly connected) component* of a vertex u in a digraph D is the subdigraph of D induced by u and all vertices of the strong component of u in the underlying graph of D. Note that, in a graph, components and strong components coincide. For undefined concepts about graph theory, we refer to [5, 16]. The component of u in $G(\Phi)$ is denoted by $G(\Phi,u)$ and the strong component by $\vec{G}(\Phi,u)$.

In a monomial $cx^i y^j$, $c \neq 0$, the non negative integer i is called the *partial degree* respect to x (analogously for j and y). The *total degree* or *degree* of the monomial is the integer $i+j$. The *partial degree* respect to x of a bivariate polynomial $\Phi(x,y)$ is the maximum of

the partial degrees of its monomials; and the *total degree* or *degree* of $\Phi(x, y)$ is the maximum of the degrees of its monomials. If $\Phi(x, y) = \Phi_1(x, y)^{n_1} \cdots \Phi_k(x, y)^{n_k}$ is the expression of $\Phi(x, y)$ as a product of irreducible polynomials over $\mathbb{C}$, the *radical* of $\Phi(x, y)$ is the polynomial $\operatorname{rad} \Phi(x, y) = \Phi_1(x, y) \cdots \Phi_k(x, y)$. A polynomial is *radical* if $\operatorname{rad} \Phi(x, y) = \Phi(x, y)$. We refer to [6, 7] for any other undefined concept about polynomials.

The paper is organized as follows. In the rest of this section we give an informal description of its contents. In the next section we define the digraph $G(\Phi)$. We shall see that some natural conditions on $\Phi(x, y)$ (such as $\Phi(x, y)$ to be radical, $\Phi(u, y) \neq 0$ for all $u \in \mathbb{C}$, etc.) can be assumed. Polynomials satisfying such conditions are called *standard polynomials*. If $\Phi(x, y)$ is a standard polynomial, then $G(\Phi)$ has only a finite number of loops and multiple arcs. Moreover, all vertices have finite indegree and all of them, but a finite number, have the same indegree; analogously for outdegrees. The components (resp. strong components) of $G(\Phi)$ containing these vertices, multiple arcs and loops are called *singular components* (resp. *singular strong components*).

In Section 3 we show that every finite strongly connected d-regular digraph is isomorphic to a strong component of $G(\Phi)$ for an appropriate $\Phi(x, y)$. Nevertheless, the construction produces an infinite component. A digraph is called *polynomial* if it is isomorphic to $G(\Phi)$ or to a non singular component of $G(\Phi)$.

In Section 4 we will see that Cayley digraphs on the additive and multiplicative groups of $\mathbb{C}$ are polynomial and we give the corresponding polynomial $\Phi(x, y)$. This implies that directed and undirected cycles, finite complete graphs K_d, finite bipartite complete graphs $K_{d,d}$ and, in general, circulant digraphs are polynomial.

In Section 5 we study polynomials of partial degree one in each indeterminate. It is shown that all non-singular components are isomorphic, and a characterization of polynomials of partial degree one which give directed n-cycles as non-singular components is given, as well as those giving infinite directed paths. By relating these polynomials to the group of linear fractional transformations, we prove that Cayley digraphs on dihedral groups and on the groups of symmetries of regular polyhedra are polynomial.

In Section 6 we consider symmetric polynomials of total degree 2. In this case the structure of $G(\Phi)$ is also completely determined. In particular, all non-singular components are isomorphic.

Finally, the results obtained here and the discussions in [3] give support to the conjecture stated in the last section.

2. The digraph of a polynomial

Let

$$\Phi(x, y) = \sum_{i=0}^{d} a_i(x) y^i = \sum_{j=0}^{e} b_j(y) x^j$$

be a polynomial with complex coefficients. The digraph $G(\Phi)$ has $\mathbb{C}$ as set of vertices and an ordered pair (u, v) is an arc of multiplicity m if v is a root of multiplicity m of the polynomial $\Phi(u, y)$. If $\Phi(u, y)$ is the zero polynomial for some $u \in \mathbb{C}$ the multiplicity of all arcs (u, v), $v \in \mathbb{C}$, is taken to be 1, and the vertex u is called a *source universal vertex*. Note that the source universal vertices are the roots of the polynomial $A(x) = \gcd(a_0(x), \ldots, a_d(x))$. Analogously, a vertex v is called a *sink universal vertex* if $\Phi(x, v)$ is the zero polynomial; the sink universal vertices are the roots of the $B(y) = \gcd(b_0(y), \ldots, b_e(y))$. In the following we assume that $G(\Phi)$ has no universal vertices, or, equivalently that the polynomials $A(x)$ and $B(y)$ are constant. If $\Phi(x, y)$ is constant then $G(\Phi)$ is the complete or the null digraph on $\mathbb{C}$ depending on whether the constant is zero or not. Therefore, we can also assume from now on that $e, d \geq 1$ and that $a_d(x)$ and $b_e(y)$ are non zero polynomials.

The structure of $G(\Phi)$ is given by the structure of its components. To study the components of $G(\Phi)$, it is useful to put aside some special cases.

Consider the discriminant

$$D(x) = \text{Resultant}(\Phi(x,y),\ \Phi'_y(x,y),\ y),$$

where $\Phi'_y(x,y)$ denotes the partial derivative of $\Phi(x,y)$ with respect to y. We have that $D(x) = 0$ if and only if $\Phi(x,y)$ has a multiple factor of positive degree in y, say $\Phi(x,y) = \Phi_1(x,y)^k \Phi_2(x,y)$ with $k \geq 2$. In this case, the digraphs $G(\Phi)$ and $G(\text{rad}\,\Phi)$ differ only in the multiplicity of the arcs corresponding to the factor $\Phi_1(x,y)$. Therefore, we can assume that $D(x)$ is not the zero polynomial. Analogously, we can assume that

$$E(y) = \text{Resultant}(\Phi(x,y),\ \Phi'_x(x,y),\ x),$$

is not the zero polynomial.

Note that all vertices have outdegree at most d and a vertex u has outdegree $< d$ if and only if $a_d(u) = 0$. Analogously, all vertices have indegree at most e and a vertex u has indegree $< e$ if and only if $b_e(u) = 0$. The roots of $a_d(x)$ are called *out-defective vertices* and the roots of $b_e(y)$ are called *in-defective vertices*.

Let $D(x) \neq 0$ and $A(x) = 1$. The leading coefficients of $\Phi(x,y)$ and $\Phi'_y(x,y)$ as polynomials in the indeterminate y are $a_d(x)$ and $da_d(x)$ respectively, so $a_d(x)$ is a factor of $D(x)$. Thus, the out-defective vertices are roots of $D(x)$. If u is the origin of a multiple arc, then $\Phi(u,y)$ has a multiple root. Therefore, $D(u) = 0$. Conversely, if $D(u) = 0$, then either $a_d(u) = 0$ or $\Phi(u,y)$ has a multiple root, i.e. u is an out-defective vertex or it is the origin of a multiple arc. We conclude that the roots of $D(x)$ are the out-defective vertices and the origins of multiple arcs. Analogously, the roots of $E(y)$ are the in-defective vertices and the ends of multiple arcs.

The vertices with a loop are the roots of $L(x) = \Phi(x,x)$. There is a loop at each vertex if and only if $L(x)$ is the zero polynomial, which means that $\Phi(x,y)$ admits a factorization $\Phi(x,y) = (y-x)^k \Phi_1(x,y)$ with $k \geq 1$ and $\Phi_1(x,y)$ not divisible by $y-x$. The structure of $G(\Phi)$ is then completely determined by the structure of $G(\Phi_1)$, and thus it is not a restriction to assume that $L(x)$ is not the zero polynomial.

A polynomial $\Phi(x,y)$ is *standard* if it is non constant ($G(\Phi)$ is neither the complete or the null graph), $A(x)B(y)$ is constant (there are not universal vertices), $D(x), E(x) \neq 0$ (the polynomial $\Phi(x,y)$ is a radical polynomial), and $L(x)$ is not the zero polynomial ($G(\Phi)$ has no loops at every vertex). For a standard polynomial $\Phi(x,y)$, the roots of $S(x) = L(x)D(x)E(x)$ are called *singular vertices*. They are the vertices with a loop, vertices which are origin or end of multiple arcs, and defective vertices. A *singular component* (resp. *singular strong component*) is a component (resp. strong component) which contains some singular vertex. Note that only a finite number of singular components (strong components) exists. We denote by $G(\Phi)^*$ the digraph obtained from $G(\Phi)$ by removing all its singular components.

Note that different polynomials $\Phi(x,y)$ can give isomorphic digraphs $G(\Phi)$, as stated in the following lemma of straightforward proof.

Lemma 1. *Let $\Phi(x,y) \in \mathbb{C}[x,y]$ and $a,b,c \in \mathbb{C}$ with $a,c \neq 0$. If $\Psi(x,y) = c\Phi(ax+b, ay+b)$, then the mapping $u \mapsto au + b$ is an isomorphism from $G(\Psi)$ to $G(\Phi)$.*

3. Finite strong components

Our immediate goal is to show that every finite strongly connected d-regular digraph can be seen as a strong component of $G(\Phi)$ for some appropriate $\Phi(x,y)$. We need the following Lemma:

Lemma 2. *Every d-regular digraph admits a 1-factorization.*
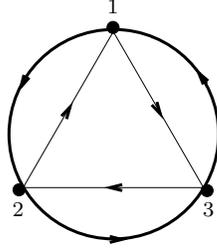
FIGURE 1. A factorization of the digraph K_3

Lemma 2 can be proved by using that every regular graph of even degree admits a 2-factorization [15]. See also [8] for a detailed proof.

Theorem 3. *Let $D = (V, E)$ be a finite strongly connected d-regular digraph of order $n \geq 2$ with $V \subset \mathbb{C}$. Then there exists a polynomial $\Phi(x, y)$ such that D is a strong component of $G(\Phi)$.*

Proof. Lemma 2 ensures that D admits a 1-factorization. Let $F_1, \ldots, F_d$ be the set of arcs of the 1-factors. For each $i \in [d] = \{1, \ldots, d\}$ let $L_i(x)$ be the interpolation polynomial such that $L_i(u) = v$ for each $(u, v) \in F_i$. In this way, the vertices adjacent from $u \in V$ are $L_1(u), \ldots, L_d(u)$. Define $\Phi(x, y) = (y - L_1(x)) \cdots (y - L_d(x))$. In $G(\Phi)$, the vertex u is also adjacent to $L_1(u), \ldots, L_d(u)$. Therefore $D = \vec{G}(\Phi, u)$. $\square$

In the proof of Theorem 3, the polynomials $L_i(x)$ can have degree $n - 1$, so $\Phi(x, y)$ can have degree d in y and degree $d(n - 1)$ in x. If the given digraph D has order $n \geq 3$, then $d(n - 1) > d$ and the component $G(\Phi, u)$ is infinite, while the strong component $\vec{G}(\Phi, u) = D$ is finite. For instance, take $D = K_3$, the complete symmetric digraph of order 3, and choose $1, 2$ and 3 as the vertices of D. The digraph D is 2-regular and admits the factorization F_1, F_2 where $F_1 = \{(1, 2), (2, 3), (3, 1)\}$ and $F_2 = \{(1, 3), (3, 2), (2, 1)\}$. Figure 1 shows the factorization of K_3; the arcs of F_1 are the thick ones.

The polynomial of degree 2 such that $L_1(1) = 2$, $L_1(2) = 3$ and $L_1(3) = 1$ is $L_1(x) = -\frac{3}{2}x^2 + \frac{11}{2}x - 2$, and the polynomial of degree 2 such that $L_2(1) = 3$, $L_2(3) = 2$, $L_2(2) = 1$ is $L_2(x) = \frac{3}{2}x^2 - \frac{13}{2}x + 8$. If $\Phi(x, y) = (y - L_1(x))(y - L_2(x))$, then $\vec{G}(\Phi, 1)$ is $D = K_3$. Nevertheless, the component $G(\Phi, 1)$ is infinite. In the Figure 2 vertices adjacent to 1, 2 and 3 which are in $G(\Phi, 1)$ but not in $\vec{G}(\Phi, 1)$ are shown.
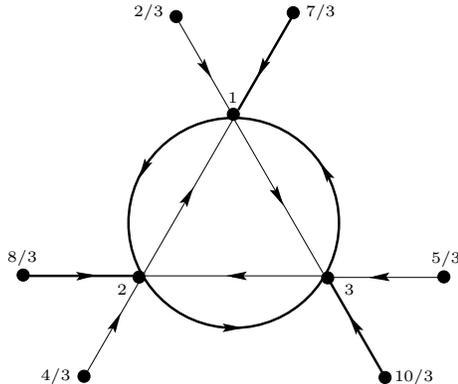
FIGURE 2. Part of $G(\Phi, 1)$

By Theorem 3 the condition on a finite strongly connected d-regular digraph of being isomorphic to a strong component of $G(\Phi)$ for some polynomial $\Phi(x,y)$ is not restrictive at all. A d-regular digraph D is said to be *polynomial* if, for some standard polynomial $\Phi(x,y)$, the digraph D is isomorphic to $G(\Phi)$ or to a non singular component of $G(\Phi)$.

4. Cayley digraphs

Cayley digraphs are relevant structures in different contexts such as modeling interconnection networks [11, 14], tessellations of the sphere and of the Euclidean Plane [1] and in combinatorial group theory [17]. Recall that, given a group Γ and a finite set $S \subseteq \Gamma$ with $1 \notin S$, the *Cayley digraph* $\mathrm{Cay}(\Gamma, S)$ is defined by taking the elements in Γ as vertices and an ordered pair (u, v) is an arc if $v = su$ for some $s \in S$. A Cayley digraph $\mathrm{Cay}(\Gamma, S)$ is connected if and only if S is a generating set of Γ (this is the reason why the condition of S being a generating system is often included in the definition). If $s^{-1} \in S$ for all $s \in S$, then $\mathrm{Cay}(\Gamma, S)$ is a graph. Cayley digraphs are known to be vertex transitive. This implies that all components of a Cayley digraph are isomorphic and also that all strong components are isomorphic.

Theorem 4. (i) *Let $\Phi(x,y)$ be a standard polynomial. If $\Phi(x,y) = f(y - x)$ for some univariate polynomial $f(s) \in \mathbb{C}[s]$, then $G(\Phi)$ is a Cayley digraph on $(\mathbb{C}, +)$.*
  (ii) *Let $D = \mathrm{Cay}(\mathbb{C}, S)$ be a Cayley digraph on $(\mathbb{C}, +)$. Then there exists a univariate polynomial $f(s) \in \mathbb{C}[s]$ such that D is isomorphic to $G(\Phi)$ where $\Phi(x,y) = f(y - x)$ is a standard polynomial.*

Proof. (i) Let $s_1, \ldots, s_d$ be the roots of $f(s)$. Then $\Phi(x,y) = f(y-x) = c(y-x-s_1)\cdots(y-x-s_d)$ with $c \neq 0$. In $G(\Phi)$ a vertex u is adjacent to the vertices $u + s_1, \ldots, u + s_d$. Because $\Phi(x,y)$ is standard, $s_i \neq 0$ for all i and $s_i \neq s_j$ for $i \neq j$. If $S = \{s_1, \ldots, s_d\}$, then we get $G(\Phi) = \mathrm{Cay}(\mathbb{C}, S)$.

(ii) Given $\mathrm{Cay}(\mathbb{C}, S)$ with $S = \{s_1, \ldots, s_d\}$, consider the polynomial $f(s) = (s - s_1) \cdots (s - s_d)$ and take $\Phi(x,y) = f(y-x)$. The polynomial $\Phi(x,y)$ is standard and $\mathrm{Cay}(\mathbb{C}, S) = G(\Phi)$. $\qquad\square$

Note that for a polynomial $\Phi(x,y) = f(y-x)$ as in Theorem 4 the components are always infinite. For instance, if $\Phi(x,y) = (y-x)^4 - 1 = (y-x-1)(y-x+1)(y-x-i)(y-x+i)$ then $G(\Phi)$ has no singular vertices and it is isomorphic to $\mathrm{Cay}(\mathbb{C}, \{1, -1, i, -i\})$. In this example, components and strong components coincide and all of them are isomorphic to $G(\Phi, 0)$, the grid of integer coordinates.

Next theorem is the corresponding to Theorem 4 for Cayley digraphs on the multiplicative group of $\mathbb{C}$. As usual, $\mathbb{C}^*$ denotes $\mathbb{C} \setminus \{0\}$.

Theorem 5. (i) *Let $\Phi(x,y)$ be an homogeneous standard polynomial. Then $G(\Phi)^*$ is a Cayley digraph on $(\mathbb{C}^*, \cdot)$.*
  (ii) *Let $\mathrm{Cay}(\mathbb{C}^*, S)$ be a Cayley digraph on $(\mathbb{C}^*, \cdot)$. Then, there exists an homogeneous standard polynomial $\Phi(x,y)$ such that $\mathrm{Cay}(\mathbb{C}^*, S) = G(\Phi)^*$.*

Proof. (i) Let $\Phi(x,y)$ be an homogeneous standard polynomial of total degree d. Note that $\Phi(x,y)$ being standard, it must be also of partial degree d in both indeterminates. We have $\Phi(x, sx) = x^d f(s)$ where $f(s)$ is a univariate polynomial in s of degree d. Let $s_1, \ldots, s_d$ be the roots of $f(s)$. Then, $\Phi(x, s_i x) = 0$ for $1 \leq i \leq d$ and $\Phi(x,y) = c(y - s_1 x) \cdots (y - s_d x)$ for some $c \neq 0$. As $\Phi(x,y)$ is standard, $s_i \neq 1$ and $s_i \neq 0$ for all i, and $s_i \neq s_j$ for $i \neq j$. Each vertex $u \in \mathbb{C}^*$ is adjacent to the d vertices $s_1 u, \ldots, s_d u$. Therefore, if $S = \{s_1, \ldots, s_d\}$, we have $G(\Phi)^* = \mathrm{Cay}(\mathbb{C}^*, S)$.

(ii) Given a Cayley digraph $\mathrm{Cay}(\mathbb{C}^*, S)$ on the multiplicative group on $(\mathbb{C}^*, \cdot)$, where $S = \{s_1, \ldots, s_d\}$, then $\Phi(x, y) = (y - s_1 x) \cdots (y - s_d x)$ is a standard polynomial and $G(\Phi)^* = \mathrm{Cay}(\mathbb{C}^*, S)$. $\qquad\square$

Note that if $\Phi(x, y)$ is an homogeneous standard polynomial of total degree d, then $G(\Phi)$ has 0 as the unique singular vertex, and $(0, 0)$ is a loop of multiplicity d.

As a Corollary of Theorems 4 and 5 we have:

Corollary 6. *Cayley digraphs on the additive and multiplicative groups of $\mathbb{C}$ are polynomial.*

A *circulant* digraph is a strongly connected Cayley digraph on a finite cyclic group. It is not a restriction to take the group U_n of the n-th roots of the unity as the cyclic group of order n. Then, a circulant digraph is a Cayley digraph of the form $\mathrm{Cay}(U_n, S)$, where S is a generating set of U_n. Now, $\mathrm{Cay}(U_n, S)$ is the component of 1 in $\mathrm{Cay}(\mathbb{C}^*, S)$, so we conclude

Corollary 7. *Circulant digraphs are polynomial.*

For instance, if ω is a primitive n-root of unity and we define $\Phi(x, y) = \prod_{i=1}^{n-1}(y - \omega^i x)$, the components of $G(\Phi)^*$ are complete graphs K_n. If ω is a primitive $2d$-root of unity and $\Phi(x, y) = \prod_{i=1}^{d}(y - \omega^{2i-1} x)$, then the components of $G(\Phi)^*$ are complete bipartite graphs $K_{d,d}$.

The n-prisms are a family of Cayley digraphs over non cyclic groups that are also polynomial. The *n-prism* is the Cayley graph

$$\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, \{(1, 0), (n - 1, 0), (0, 1)\}).$$

For instance, the 3-dimensional cube is the 4-prism. The n-prism can be obtained by the polynomial $\Phi(x, y) = (y - \omega x)(y - \omega^{n-1} x)(xy - 2)$, where ω is a n-th primitive root of the unity.

5. POLYNOMIALS OF PARTIAL DEGREE ONE

If $\Phi(x, y) = a_1(x)y + a_0(x)$ is a polynomial of partial degree one in y, then the vertices v such that there exists a directed path from u to v are the vertices u_n defined by $u_0 = u$ and $u_{n+1} = -a_0(u_n)/a_1(u_n)$ for all $n \geq 0$. Thus, the structure of $G(\Phi)$ is closely related to the dynamical system defined by the rational function $f(x) = -a_0(x)/a_1(x)$. The iteration of rational functions was studied by G. Julia [12] as early as 1918 and P. Fatou [9] in 1922, and there is a huge literature on the topic (see, for instance [13]). In this section we give a complete description of the digraphs $G(\Phi)$ when $\Phi(x, y)$ is a polynomial of partial degree one in both indeterminates.

Note that a component of a digraph of indegree and outdegree equal to one is isomorphic either to a directed n-cycle $\vec{C}_n = \mathrm{Cay}(\mathbb{Z}_n, \{1\})$ for some n or to an infinite path $\vec{P} = \mathrm{Cay}(\mathbb{Z}, \{1\})$. We shall see that in any case, all components of $G(\Phi)^*$ are isomorphic. This result is applied to exhibit examples of Cayley digraphs on non commutative groups that are polynomial digraphs. First, we characterize the standard polynomials of partial degree 1.

Lemma 8. *A polynomial $\Phi(x, y) = (cx + d)y - (ax + b)$ is standard if and only if $ad - bc \neq 0$ and it is not divisible by $y - x$.*

Proof.

Assume that $\Phi(x, y) = (cx + d)y - (ax + b) = (cy - a)x + dy - b$ is standard. Then the polynomials $D(x) = cx + d$ and $E(x) = cy - a$ are not the zero polynomials. First, consider the case $c = 0$. Then $d \neq 0$ and $a \neq 0$, so $ad - bc = ad \neq 0$. Second, assume $a = 0$. Then $c \neq 0$. If $b = 0$, then $B(y) = \gcd(cy - a, dy - b) = \gcd(cy, dy) \neq 1$, a contradiction. Thus, $b \neq 0$ and $ad - bc = -bc \neq 0$. Finally, let $ac \neq 0$. By dividing $cx + d$ by $ax + b$, the remainder

is $-bc/a + d = (ad - bc)/a$. As $A(x) = \gcd(cx + d, ax + b) = 1$, this remainder must be not zero. Therefore $ad - bc \neq 0$. The condition of not being divisible by $y - x$ ensures that $L(x)$ is not the zero polynomial.

Conversely, assume that $ad - bc \neq 0$. Then c and d can not be simultaneously zero, so $D(x) = cx + d$ is not the zero polynomial. Analogously, $E(x) = cy - a$ is not the zero polynomial. If $A(x) = \gcd(cx + d, ax + b)$ is non constant, then $c = a\lambda$ and $d = b\lambda$, which implies $ad - bc = 0$, a contradiction. Analogously, $B(x) = \gcd(cy - a, dy - b)$ non constant implies $ad - bc = 0$. Finally, if $L(x) = cx^2 + (d - a)x - b$ is the zero polynomial, then $b = c = 0$ and $d = a$. Therefore $\Phi(x, y) = d(y - x)$ is divisible by $y - x$. $\qquad\square$

For standard polynomials $\Phi(x, y) = (cx + d)y - (ax + b)$ of partial degree one, the structure of $G(\Phi)^*$ is –as observed above– closely related to the properties of the maps $f(z) = (az + b)/(cz + d)$ with $ad - bc \neq 0$ (or, equivalently, with $ad - bc = 1$). These maps, called *linear fractional transformations* [10] or *Möebius transformations* [4], apply the complex plane minus $-d/c$ to the complex plane minus a/c. They are examples of conformal maps and form a group $M(\mathbb{C})$ under composition. It is useful to represent the group $M(\mathbb{C})$ as a quotient of the group $S_2(\mathbb{C})$ of square matrices of order 2 with determinant 1 as follows. The mapping

$$
\begin{array}{ccc}
S_2(\mathbb{C}) & \longrightarrow & M(\mathbb{C}) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & f(z) = \dfrac{az + b}{cz + d}
\end{array}
$$

is an surjective group homomorphism and its kernel is $\{1, -1\}$. Therefore $M(\mathbb{C}) \simeq S_2(\mathbb{C})/\{+1, -1\}$. Given $f \in M(\mathbb{C})$ defined by $f(z) = (az + b)/(cz + d)$ we denote $A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $[A_f]$ the class of A_f in $S_2(\mathbb{C})/\{+1, -1\}$.

Theorem 9. *If $\Phi(x, y) = (cx + d)y - (ax + b)$ is a standard polynomial, then all components of $G(\Phi)^*$ are isomorphic.*

Proof. Let ℓ_1, ℓ_2 be the roots of $L(x) = \Phi(x, x) = cx^2 + (d - a)x - b$. Note that if $c = 0$ all vertices have outdegree 1; otherwise, $-d/c$ is the unique out-defective vertex. Let $V = \mathbb{C}$ if $c = 0$ and $V = \mathbb{C} \setminus \{-d/c\}$ if $c \neq 0$. A vertex u in V is adjacent to the vertex $f(u) = (au + b)/(cu + d)$. Consider the function $f(x) = (ax + b)/(cx + d)$ defined on V. Assume that there exist components of $G(\Phi)^*$ which are directed cycles and let $n \geq 2$ be the minimum of the lengths of these cycles. Then n is the minimum positive integer such that there exists a vertex u in a non-singular component such that $f^n(u) = u$, or equivalently,

$$
u = f^n(u) = \frac{a_n u + b_n}{c_n u + d_n} \quad \text{where} \quad \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = A_f^n.
$$

Thus, the solutions of $f^n(x) = x$ are the roots of $F(x) = c_n x^2 + (d_n - a_n)x - b_n$. But ℓ_1, ℓ_2 and u are three roots of $F(x)$ (if $\ell_1 = \ell_2$, then ℓ_1 has multiplicity 2), so $F(x)$ is the zero polynomial and $f^n(v) = v$ for all v. This implies that all components of $G(\Phi)^*$ are isomorphic to $\vec{C}_n$. $\qquad\square$

From a computational point of view, conditions on a, b, c and d for the components of $G(\Phi)^*$ being directed n-cycles are easily obtained. The polynomial $F(x)$ in the above proof is of the form $F(x) = c_n x^2 + (d_n - a_n)x - b_n = F_n(a, b, c, d)L(x) = F_n(a, b, c, d)(cx^2 + (d - a)x - b)$. Therefore, $F_n(a, b, c, d) = c_n/c = b_n/b = (d_n - a_n)/(d - a)$. Now, for each divisor k of n, the polynomial $F_k(a, b, c, d)$ must be a factor of $F_n(a, b, c, d)$. By dividing $F_n(a, b, c, d)$ by all the factors corresponding to digraphs $G(\Phi)^*$ with directed k-cycles as components (k divisor of n), we obtain the condition $\vec{C}_n(a, b, c, d) = 0$ for the components of $G(\Phi)^*$ to be directed cycles of length n. In the table 1 the conditions $\vec{C}_n(a, b, c, d) = 0$ for n from 2 to 10 are given.

TABLE 1. Conditions for the components of $G(\Phi)^*$ being directed n-cycles.

n	$\vec{C}_n(a,b,c,d)$
2	$a + d$
3	$a^2 + bc + ad + d^2$
4	$a^2 + 2bc + d^2$
5	$a^4 + 3a^2bc + b^2c^2 + a^3d + 4abbd + a^2d^2 + 3bcd^2 + ad^3 + d^4$
6	$3bc + a^2 - ad + d^2$
7	$8ca^3bd + 9ca^2bd^2 + 6c^2a^2b^2 + 9c^2ab^2d + a^6 + 8cabd^3 + 5a^4bc + c^3b^3$
	$\quad + 6c^2b^2d^2 + 5d^4cb + da^5 + d^2a^4 + d^3a^3 + d^4a^2 + d^5a + d^6$
8	$2c^2b^2 + a^4 + d^4 + 4a^2bc + 4dabc + 4cbd^2$
9	$c^3b^3 + 9c^2b^2d^2 + 15c^2ab^2d + 9c^2a^2b^2 + 6ca^3bd + 6d^4cb + 3ca^2bd^2$
	$\quad + 6a^4bc + 6cabd^3 + d^6 + a^6 + d^3a^3$
10	$5c^2b^2 - da^3 + d^2a^2 + 5cbd^2 + 5a^2bc - d^3a + d^4 + a^4$

The above proposition can be interpreted in the sense that if a linear fractional transformation $f(z) = (az+b)/(cz+d)$ generates a cyclic group of order n, then all its orbits but the fixed points (the loops in the graph) have n points (and therefore the components of $G(\Phi)^*$, where $\Phi(x,y) = (cx+d)y - (ax+b)$, are directed n-cycles).

The following theorem is a characterization of polynomials of partial degree one which give directed n-cycles. By a *primitive n-root* of a complex number z we mean a complex r such that n is the minimum positive integer with $r^n = z$.

Theorem 10. *Let $\Phi(x,y) = (cx+d)y - (ax+b)$ be a standard polynomial and let r be a square root of $(a+d)^2 - 4$. Then the components of $G(\Phi)^*$ are directed n-cycles if and only if $a+d \notin \{+2, -2\}$ and $(a+d \pm r)/2$ are both n-th roots of 1 or both n-roots of -1, and at least one of them primitive.*

Proof.   Let $f \in M(\mathbb{C})$ defined by $f(z) = (az+b)/(cz+d)$ with $ad - bc = 1$. We have the following equivalences:

The components of $G(\Phi)^*$ are isomorphic to $\vec{C}_n$

$\Leftrightarrow f^n$ is the identity mapping

$\Leftrightarrow$ the order of $[A_f]$ is n

$\Leftrightarrow A_f$ has two distinct eigenvalues which are n-th roots of 1 or -1 and at least one of them primitive. The characteristic polynomial of A_f is $\lambda^2 - (a+d)\lambda + 1$ and the discriminant $(a+d)^2 - 4$. Therefore A_f has two distinct eigenvalues if and only if $a+d \neq \pm 2$. In this case the eigenvalues are $(a+d \pm r)/2$ and $[A_f]$ is of order n if they are n-th roots of 1 or -1 and at least one primitive. $\square$

For instance, consider the polynomial $\Phi(x,y) = (x-1+\sqrt{3})y - (x-2+\sqrt{3})$. We have the values $a = c = 1$, $b = -2 + \sqrt{3}$ and $d = -1 + \sqrt{3}$. They satisfy the 6 cycle-condition $3bc + a^2 - ad + d^2 = 0$ of table 1. On the other hand, we can use the Theorem 10. The matrix associated to the corresponding f is $A_f = \begin{pmatrix} 1 & -2+\sqrt{3} \\ 1 & -1+\sqrt{3} \end{pmatrix}$. The eigenvalues of A_f are $\lambda = (+\sqrt{3} \pm i)/2$ which are primitive 6-roots of -1. Therefore $[A_f]$ has order 6 and all components of $G(\Phi)^*$ are directed 6-cycles.

Taking into account that $M(\mathbb{C})$ is a non commutative group, we can find polynomial Cayley digraphs on non abelian groups. The following theorem opens the way.

Theorem 11. *Let Γ be a subgroup of $M(\mathbb{C})$ generated by a set $S = \{f_1, \ldots, f_d\}$ of d linear fractional transformations. Then $\mathrm{Cay}(\Gamma, S)$ is polynomial.*

Proof. Let $f \in \Gamma$ be a transformation defined by $f(z) = (az + b)/(cz + d)$. We associate to f the polynomial $\Phi_f(x, y) = (cx + d)y - (ax + b)$. The group Γ is generated by a finite set, so it is a countable set. Therefore, the set of $u \in \mathbb{C}$ such that there exists $f \in \Gamma$ with u belonging to a singular component of $G(\Phi_f)$ is also a countable set. Thus, we can choose $u \in \mathbb{C}$ such that $\vec{G}(\Phi_f, u)$ is not a singular component of $G(\Phi_f)$ for all $f \in \Gamma$.

Let $\Phi(x, y) = \Phi_{f_1}(x, y) \cdots \Phi_{f_d}(x, y)$. The mapping $\mathrm{Cay}(\Gamma, S) \to \vec{G}(\Phi, u)$ defined by $f \mapsto f(u)$ is a digraph isomorphism. Indeed: it is injective, because $f(u) = u$ implies that f is the identity (otherwise u would be a loop vertex in $G(\Phi_f)$). It is surjective, because if v is a vertex in $\vec{G}(\Phi, u)$, then there exists a path $u = u_0, \ldots, u_\ell = v$. Now, each arc is of the form $(u_j, f_{i_j}(u_j))$ for some $f_{i_j} \in S$. If $f = f_{i_\ell} \cdots f_{i_1}$, we have $f(u) = v$. Finally, it preserves adjacencies: (g, h) is an arc in $\mathrm{Cay}(\Gamma, S) \Leftrightarrow g = f_i h$ for some $f_i \in S \Leftrightarrow g(u) = f_i(h(u))$ for some $i \Leftrightarrow \Phi_{f_i}(h(u), g(u)) = 0$ for some $i \Leftrightarrow \Phi(x, y) = 0$. □ The finite subgroups

of $M(\mathbb{C})$ are determined ([10], Chapter VI). In particular, dihedral groups and the groups of symmetries of regular polyhedra are finite subgroups of $M(\mathbb{C})$. Therefore, we have:

Corollary 12. *Cayley digraphs on dihedral groups and on the groups of symmetries of regular polyhedra are polynomial.*

For instance, the dihedral group $D_{2n} = \langle f, t \mid f^n = t^2 = 1, \ tftf = 1 \rangle$ is the subgroup of $M(\mathbb{C})$ generated by $f(z) = \omega z$ and $t(z) = 2/z$, where ω is a primitive n-root of unity. Then, the Cayley digraph $\mathrm{Cay}(D_{2n}, \{f, t\})$ is obtained by the polynomial $\Phi(x, y) = (y - \omega x)(xy - 2)$.

6. Symmetric polynomials of degree two

In this section we give a method for analyzing the components of $G(\Phi)$ for a standard symmetric polynomial $\Phi(x, y)$ of total degree two. The method implies long but routine calculations, so we skip them and give only results.

From Theorem 9 we can assume that the standard polynomial is not of partial degree one, so it is of the form $\Phi(x, y) = x^2 + y^2 + axy + b(x + y) + c$. Since $G(\Phi)^*$ is a 2-regular graph, a component of $G(\Phi)^*$ is isomorphic to a (undirected) cycle $C_n = \mathrm{Cay}(\mathbb{Z}_n, \{1, -1\})$ or to the double ray graph $R = \mathrm{Cay}(\mathbb{Z}, \{1, -1\})$. Denote $\ldots, v_{-n}, \ldots, v_{-1}, v_0, v_1, \ldots, v_n, \ldots$ the vertices of $G(\Phi, v_0)$ with v_i and v_{i+1} adjacent vertices. The polynomial of degree two $\Phi(v_{n-1}, y)$ has two roots, namely v_{n-2} and v_n. As the sum of the two roots is minus de coefficient of y, we have $v_{n-2} + v_n = -(av_{n-1} + b)$ or, equivalently,

$$(1) \qquad\qquad v_n + av_{n-1} + v_{n-2} = -b.$$

The two roots of $\Phi(v_0, y)$ are v_{-1} and v_1. The solutions of the recurrence (1) with initial values v_0 and v_1 determine the vertices with positive subscripts and analogously for the negative subscripts taking v_0 and v_{-1} as initial values. Then, to determine the structure of $G(\Phi)$ the method is the following. First, to solve the recurrence (1), wich give v_n in terms of two initial values v_0 and v_1. Second, to find the singular vertices. There are not defective vertices, so only vertices with loops and multiple arcs should be calculated. In any case, there exist at most two loops and two origin of multiple arcs. For each loop-vertex ℓ, the solution of (1) for the initial values $v_0 = v_1 = \ell$ gives the vertices in $G(\Phi, \ell)$. Analogously, for each multiple arc (m, m_1), the solution of (1) for the initial conditions $v_0 = m$ and $v_1 = m_1$ gives the vertices in $G(\Phi, m)$. The explicit form of the solutions allows to decide if some of these singular components coincide and if they are finite or not. Finally, for v_0 in a non singular component, the solutions of (1) for the initial values v_0 and v_1 and v_0 and v_{-1} gives the vertices in the non singular component $G(\Phi, v_0)$.

FIGURE 3. Structure of $G(\Phi)$ for $a = -2$
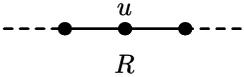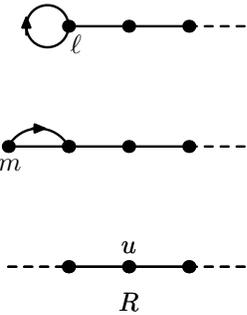
$b = 0,\ (c \neq 0)$	$b \neq 0$
	$\ell = -c/(2b),\ m = \left(b^2 - 4c\right)/(8b)$

The characteristic equation of the second order linear recurrence (1) is $\lambda^2 + a\lambda + 1 = 0$, and the discriminant is $\Delta(a) = a^2 - 4$. To solve the recurrence three cases have to be considered: $a = -2$, $a = 2$, and $a^2 - 4 \neq 0$. We sumarize the discussion in each case.

First consider the case $a = -2$, see Figure 3. We have $\Phi(x, y) = (x - y)^2 + b(x + y) + c$. As $\Phi(x, y)$ is standard, b and c cannot be simultaneously zero. For $b = 0$ there are no singular components. For $b \neq 0$ there exist two singular components, both infinite, one containing the vertex-loop $\ell = -c/(2b)$ and one containing the origin of a double arc $m = (b^2 - 4c)/(8b)$. In both cases ($b = 0$ and $b \neq 0$) all non singular components are isomorphic to R.

For $a \neq -2$, it follows from Lemma 1 that if $\Psi(x, y) = \Phi(x - b/(a + 2), y - b/(a + 2)) = x^2 + y^2 + axy + (c - b^2/(a + 2))$, then $G(\Phi)$ is isomorphic to $G(\Psi)$. Therefore we can assume (and we do) without loss of generality that the polynomial $\Phi(x, y)$ is of the form $\Phi(x, y) = x^2 + y^2 + axy + c$.

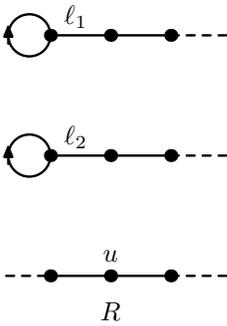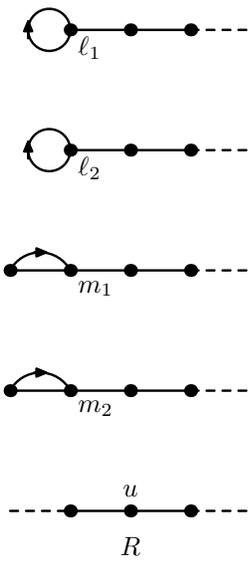FIGURE 4. Structure of $G(\Phi)$ for $a = 2$

$c \neq 0$
$\ell_1 = \sqrt{-c}/2,\ \ell_2 = -\ell_1$

FIGURE 5.   Structure of $G(\Phi)$ for $a^2 - 4 \neq 0$, $a \neq 2\cos\frac{2\pi k}{n}$, $\gcd(k,n) = 1$

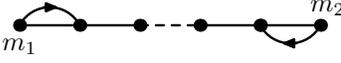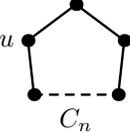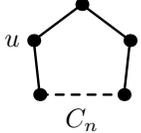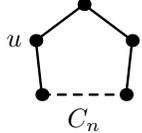| $c = 0$	$c \neq 0$
$\ell = 0$	$\ell_1 = \sqrt{-c/(a+2)}$, $\ell_2 = -\ell_1$, $m_1 = 2\sqrt{c/(a^2-4)}$, $m_2 = -m_1$

Consider now the case $a = 2$, see Figure 4. Now the polynomial is $\Phi(x,y) = (x+y)^2 + c$. As $\Phi(x,y)$ is a standard polynomial, necessarily $c \neq 0$. There exist two non-singular components, both infinite, one containing a loop at the vertex $\ell_1 = \sqrt{-c}/2$, where $\sqrt{-c}$ is one of the two square roots of $-c$, and the other containing a loop at the vertex $\ell_2 = -\ell_1$. The non singular components are isomorphic to R.

Finally, consider the case $a^2 - 4 \neq 0$, see Figures 5 and 6. If $c = 0$, there exists only one singular component, which contains only the vertex $\ell = 0$ with a double loop. If $c \neq 0$, there exist two loop-vertices $\ell_1 = \sqrt{-c/(a+2)}$ and $\ell_2 = -\ell_1$ and two origin of double arcs, $m_1 = 2\sqrt{c/(a^2-4)}$ and $m_2 = -m_1$. The number and the finitness of the components depends on the values of a or, equivalently, on the values of the two distinct roots ω_1 and $\omega_2 = 1/\omega_1$ of $\lambda^2 + a\lambda + 1$. If ω_i are primitive n-th roots of the unity for some positive integer n, then the components of $G(\Phi)$ are finite, otherwise they are infinite. Now, ω_i is a primitive n-th root of the unity if and only if a is of the form $a = 2\cos(2\pi k/n)$ for some k with $\gcd(k,n) = 1$.

The following theorem sumarizes the form of the non singular components.

Theorem 13. *Let $\Phi(x,y) = x^2 + y^2 + axy + b(x + y) + c$ be a standard polynomial. If $a \neq \pm 2$ and $a = 2\cos(2\pi k/n)$ for some positive integers n and k with $\gcd(k,n) = 1$, then all components of $G(\Phi)^*$ are isomorphic to C_n, the cycle of length n. Otherwise, they are isomorphic to the double ray graph R.*

FIGURE 6.   Structure of $G(\Phi)$ for $a^2 - 4 \neq 0$, $a = 2\cos\frac{2\pi k}{n}$, $\gcd(k, n) = 1$

7. A CONJECTURE

If $\Phi(x, y)$ is an homogeneous standard polynomial, then $G(\Phi)^*$ is a Cayley digraph (Theorem 5). Therefore if a component of $G(\Phi)^*$ is finite, all of them are finite and isomorphic. In Section 5 we have seen that if $\Phi(x, y)$ is a polynomial of partial degree one, then if a component of $G(\Phi)^*$ is a directed n-cycle, then all of them are directed n-cycles. Also, for a symmetric polynomials $\Phi(x, y)$ of partial and total degree 2, if a component of $G(\Phi)^*$ is a n-cycle, then all of them are n-cycles. All these examples suggest the following conjecture:

Conjecture 14. *Let $\Phi(x, y)$ be a standard polynomial of partial degree d and H a d-regular digraph isomorphic to a component of $G(\Phi)^*$. Then, all components of $G(\Phi)^*$ are isomorphic to H.*

In [3] more evidence of Conjecture 14 is given. For instance, it is shown that if for a symmetric polynomial $\Phi(x, y)$ of partial degree two $G(\Phi)^*$ has a component which is a n-cycle (for small values of n) then all of them are n-cycles. Also, if $\Phi(x, y)$ is a polynomial such that $G(\Phi)^*$ has a component isomorphic to K_n ($2 \leq n \leq 6$) then all of them are isomorphic to K_n.

8. ACKNOWLEDGEMENTS

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